Egoroff’s theorem on Sugeno fuzzy measure spaces

Jun Li†
Department of Applied Mathematics, Southeast University
Nanjing 210096, People’s Republic of China
Masami Yasuda
Department of Mathematics & Informatics, Faculty of Science
Chiba University, Chiba 263-8522, Japan

Abstract
In this paper, the well-known Egoroff’s theorem in classical measure theory is established on Sugeno fuzzy measure spaces. Taylor’s theorem, which concerns almost everywhere convergence of measurable function sequence in classical measure theory, is also generalized. The converse problem of the theorems are discussed, and a necessary and sufficient condition for the Egoroff’s theorem is obtained on semicontinuous fuzzy measure space with $S$-compactness.

Keywords: Non-additive set function; fuzzy measure; Egoroff’s theorem

1 Introduction
Egoroff’s theorem is one of the most important convergence theorem in classical measure theory. Wang [9] first generalized the well-known theorem to fuzzy measure spaces under the autocontinuity from above condition. The more research on the theorem were made by Wang and Klir [11], Li et al. [1] and Li [3]. In these discussions, the fuzzy measures are considered in the sense of Ralescu ([6]), that is, the continuity from below and continuity from above are required.

In this paper, we further investigate the convergence of measurable function sequence on fuzzy measure spaces. Here the fuzzy measure is considered in the sense of Sugeno, i.e., it is a nonnegative monotone set function and vanishes at $\emptyset$ ([7]). Egoroff’s theorem and Taylor’s theorem ([8]) are generalized to Sugeno fuzzy measure spaces by using the strongly order continuity and the property (S) of set functions. We also discuss the converse problem of the Egoroff’s theorem by using strongly order continuity of set function and obtain a necessary condition that Egoroff’s theorem holds on Sugeno fuzzy measure space. These are further improvements and generalizations of the related results in Li et al. [1]. Finally, we obtain an encouraging result: a necessary and sufficient condition that Egoroff’s theorem remain true on lower semicontinuous fuzzy measure space with $S$-compactness is that the lower semicontinuous fuzzy measure be strongly order continuous.

2 Preliminaries
Let $X$ be a non-empty set, $\mathcal{F}$ be a $\sigma$-algebra of subsets of $X$, and let $N$ denote the set of all positive integers. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

A Sugeno fuzzy measure (cf. [4], [7]) is a set function $\mu : \mathcal{F} \rightarrow [0, +\infty]$ with the following properties:

\begin{equation}
(1) \quad \mu(\emptyset) = 0;
\end{equation}
(2) \( A \subseteq B \) implies \( \mu(A) \leq \mu(B) \) (monotonicity).

If, moreover, \( \mu \) satisfies:

(3) \( A_1 \subseteq A_2 \subseteq \cdots \) implies \( \lim_{n \to \infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n) \) (continuity from below);

(4) \( A_1 \supseteq A_2 \supseteq \cdots \), and there exists \( n_0 \) with \( \mu(A_{n_0}) < +\infty \) imply \( \lim_{n \to \infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n) \) (continuity from above),

then \( \mu \) is called continuous fuzzy measure \([6]\).

A set function \( \mu \) is called a lower semicontinuous fuzzy measure \([11]\), if it satisfies the conditions (1) – (3).

**Definition 1** ([2]) \( \mu \) is called strongly order continuous, if \( \lim_{n \to +\infty} \mu(A_n) = 0 \) whenever \( \{A_n\}_n \subset \mathcal{F} \), \( A_n \not\subseteq B \) and \( \mu(B) = 0 \).

**Definition 2** ([10]) \( \mu \) is called to have property \((S)\), if for any \( \{A_n\}_n \) with \( \lim_{n \to +\infty} \mu(A_n) = 0 \), there exists a subsequence \( \{A_{n_i}\}_{i} \) of \( \{A_n\}_n \) such that \( \mu(\limsup A_{n_i}) = 0 \).

Let \( F \) be the class of all finite real-valued measurable functions on measurable space \((X, \mathcal{F})\), and let \( f, f_n \in F \) \((n \in \mathbb{N})\). We say that \( \{f_n\}_n \) converges almost everywhere to \( f \) on \( X \), and denote it by \( f_n \overset{a.e.}{\longrightarrow} f \), if there is subset \( E \subseteq X \) such that \( \mu(E) = 0 \) and \( f_n \) converges to \( f \) on \( X \setminus E \); \( \{f_n\}_n \) converges almost uniformly to \( f \) on \( X \), and denote it by \( f_n \overset{a.u.}{\longrightarrow} f \), if for any \( \epsilon > 0 \) there is a subset \( E_\varepsilon \in \mathcal{F} \) such that \( \mu(X - E_\varepsilon) < \epsilon \) and \( f_n \) converges to \( f \) uniformly on \( E_\varepsilon \).

### 3 Egoroff’s theorems

Now we generalize Egoroff’s theorem and Taylor’s theorem in classical measure theory to Sugeno fuzzy measure spaces.

**Theorem 1** (Egoroff’s theorem) Let \( \mu \) be a Sugeno fuzzy measure. If \( \mu \) is strongly order continuous and has property \((S)\), then

\[
f_n \overset{a.e.}{\longrightarrow} f \iff f_n \overset{a.u.}{\longrightarrow} f.
\]

**Proof.** Assume that \( \mu \) is strongly order continuous and has property \((S)\). Let \( E \) be the set of these points \( x \) in \( X \) at which \( \{f_n(x)\}_n \) do not converge to \( f(x) \). Then \( \mu(E) = 0 \) and \( \{f_n\}_n \) converges to \( f \) everywhere on \( X \setminus E \). If we denote

\[
E^{(m)}_n = \bigcap_{i=n}^{+\infty} \left\{ x \in X : |f_i(x) - f(x)| < \frac{1}{m} \right\}
\]

for any \( m \geq 1 \), then \( E^{(m)}_n \) is increasing in \( n \) for each fixed \( m \), and we get

\[
X \setminus E = \bigcap_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} E^{(m)}_n.
\]

Since for any fixed \( m \geq 1 \), \( X \setminus E \subseteq \bigcup_{n=1}^{+\infty} E^{(m)}_n \), we have

\[
X \setminus E^{(m)}_n \subseteq \bigcap_{n=1}^{+\infty} (X \setminus E^{(m)}_n).
\]

Noting that \( \bigcap_{n=1}^{+\infty} (X \setminus E^{(m)}_n) \subseteq E \) for any fixed \( m \geq 1 \), therefore \( \mu(\bigcap_{n=1}^{+\infty} (X \setminus E^{(m)}_n)) = 0 \) \((m = 1, 2, \ldots)\). By using the strongly order continuity of \( \mu \), we have

\[
\lim_{n \to +\infty} \mu(X \setminus E^{(m)}_n) = 0. \quad \forall m \geq 1
\]

Thus, there exists a subsequence \( \{X \setminus E^{(m)}_{n(m)}\}_m \) of \( \{X \setminus E^{(m)}_n : n, m \geq 1\} \) satisfying

\[
\mu(X \setminus E^{(m)}_{n(m)}) \leq \frac{1}{m}, \quad \forall m \geq 1
\]

and therefore

\[
\lim_{n \to +\infty} \mu(X \setminus E^{(m)}_n) = 0.
\]

By applying the property \((S)\) of \( \mu \) to the sequence \( \{X \setminus E^{(m)}_{n(m)}\}_m \), there exists a subsequence \( \{X \setminus E^{(m)}_{n(m_i)}\}_i \) of \( \{X \setminus E^{(m)}_{n(m)}\}_m \) such that

\[
\mu \left( \lim_{i \to +\infty} (X \setminus E^{(m_i)}_{n(m_i)}) \right) = 0.
\]

and \( m_1 < m_2 < \ldots \).
On the other hand, since
\[
\left( \bigcup_{i=k}^{+\infty} (X \setminus E^{(m_i)}_{n(m_i)}) \right) \setminus \lim_{i \to +\infty} (X \setminus E^{(m_i)}_{n(m_i)})
\]
therefore, by using the strongly order continuity of \( \mu \), we have
\[
\lim_{k \to +\infty} \mu \left( \bigcup_{i=k}^{+\infty} (X \setminus E^{(m_i)}_{n(m_i)}) \right) = 0.
\]
For any \( \epsilon > 0 \), we take \( k_0 \) such that
\[
\mu \left( \bigcup_{i=k_0}^{+\infty} (X \setminus E^{(m_i)}_{n(m_i)}) \right) < \epsilon,
\]
that is,
\[
\mu \left( X \setminus \bigcap_{i=k_0}^{+\infty} E^{(m_i)}_{n(m_i)} \right) < \epsilon.
\]
Put \( E_\epsilon = \bigcap_{i=k_0}^{+\infty} E^{(m_i)}_{n(m_i)} \), then \( \mu(X \setminus E_\epsilon) < \epsilon \). Now, we just need to prove that \( \{f_n\} \) converges to \( f \) on \( E_\epsilon \) uniformly. Since
\[
E_\epsilon = \bigcap_{i=k_0}^{+\infty} \bigcap_{j=n(m_i)}^{+\infty} \{ x \in X : |f_j(x) - f(x)| < \frac{1}{m_i} \},
\]
therefore, for any fixed \( i \geq k_0 \), \( E_\epsilon \subset \bigcap_{j=n(m_i)}^{+\infty} \{ x \in X : |f_j(x) - f(x)| < \frac{1}{m_i} \} \). For any given \( \sigma > 0 \), we take \( i_0 \) \(( \geq k_0 \)) such that \( \frac{1}{m_{i_0}} < \sigma \). Thus, as \( j > n(m_{i_0}) \), for any \( x \in X \),
\[
|f_j(x) - f(x)| < \frac{1}{m_{i_0}} < \sigma.
\]
This shows that \( \{f_n\} \) converges to \( f \) on \( X \) uniformly. The proof of the theorem is thereby completed. \( \Box \)

In the following, we give a generalization of Taylor's theorem ([8]) concerning convergence almost everywhere in classical measure theory.

**Theorem 2** (Taylor's theorem) Let \( \mu \) be a Sugeno fuzzy measure. If \( \mu \) is strongly order continuous and has property (S), and \( f_n \xrightarrow{a.e.} f \), then there exists a sequence \( \{\delta_n\} \) of positive numbers, such that \( \delta_n \downarrow 0 \) as \( n \to \infty \), and
\[
\frac{|f_n(x) \setminus f(x)|}{\delta_n} \xrightarrow{a.e.} 0 \quad (n \to \infty).
\]

**Proof.** Using conclusion of Theorem 1 (Egoroff's theorem), it is similar to the proof of Theorem 1 in [8]. \( \Box \)

The following corollary gives an alternative form of Egoroff's theorem on Sugeno fuzzy measure spaces.

**Theorem 3** Under the conditions given in Theorem 2, there exists a sequence \( \{\delta_n\} \) of positive numbers, such that \( \delta_n \downarrow 0 \) as \( n \to \infty \),
\[
\frac{|f_n(x) \setminus f(x)|}{\delta_n} \xrightarrow{a.e.} 0 \quad (n \to \infty).
\]

**Proof.** This follows immediately on applying Theorem 1 to the measurable function sequence
\[
\varphi_n(x) = \frac{|f_n(x) \setminus f(x)|}{\delta_n}
\]
where \( \{\delta_n\} \) satisfy the conditions of Theorem 2. \( \Box \)

Theorem 3 is an apparently stronger form of Theorem 1.

**Remark 1** In Theorem 1, 2 and 3, the continuity from below and the continuity from above of set functions are not required.

**Example 1** Let \( X = [0, 1] \), \( \mathcal{F} \) denote \( \sigma \)-algebra on \( X \), and let \( m \) be \( \sigma \)-additive measure on \( \mathcal{F} \) and \( \mu(X) = 1 \). Let \( \mu : \mathcal{F} \to [0, 1] \) be defined by
\[
\mu(E) = \begin{cases} 
\frac{m(E)}{2} & \text{if } m(E) \leq \frac{1}{2} \\
1 & \text{if } m(E) > \frac{1}{2} \text{ and } E \neq X \\
1 & \text{if } E = X.
\end{cases}
\]
It is not too difficult to verify that set function \( \mu \) is monotone and strongly order continuous and has property (S). But \( \mu \) is neither continuous from below nor continuous from above.
A set function $\mu : \mathcal{F} \rightarrow [0, +\infty]$ is called order-continuous if $\lim_{n \to +\infty} \mu(E_n) = 0$ whenever $E_n \downarrow \emptyset$; exhaustive if $\lim_{n \to +\infty} \mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_n$ (cf. [5]). $\mu$ is called to have pseudometric generating property, if for each $\epsilon > 0$ there is $\delta > 0$ such that for any $E, F \in \mathcal{F}$, $\mu(E) \lor \mu(F) < \delta$ implies $\mu(E \cup F) < \epsilon$ (cf. [1]).

**Proposition 1** If $\mu$ is strongly order continuous, then it is order-continuous.

**Proposition 2** ([5]) If $\mu$ is a monotone set function with order-continuity, then it is exhaustive.

**Proposition 3** ([1]) Let $\mu$ be a continuous fuzzy measure, then $\mu$ is exhaustive if and only if $\mu$ is order-continuous.

**Proposition 4** ([1]) Let $\mu$ be a continuous fuzzy measure. If $\mu$ have pseudometric generating property, then it has property (S).

**Proposition 5** Let $\mu$ be a continuous fuzzy measure. If $\mu$ is order continuous and have pseudometric generating property, then it is strongly order continuous and has property (S).

**Proof.** It can be easily obtained that $\mu$ is strongly order continuous, and it follows from Proposition 4, that $\mu$ has property (S). \(\square\)

As special result of Theorem 1 and Proposition 5, we can obtained the following corollary immediately:

**Corollary 1** (Li et al.[2, Theorem 1]) Let $\mu$ be a continuous fuzzy measure with order continuity and pseudometric generating property. Then, for any $f \in \mathcal{F}$ and $\{f_n\}_n \subset \mathcal{F}$,

$$f_n \overset{a.e.}{\longrightarrow} f \implies f_n \overset{a.u.}{\longrightarrow} f.$$

**Remark 2** An strongly order continuous monotone set function with property (S) may not possess pseudometric generating property. Therefore Theorem 1 is an improvement and generalization of the related result in Li et al. [1].

### Example 2

Let $X = \{a, b\}$ and $\mathcal{F} = \wp(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E \neq X \end{cases}$$

Then $\mu$ is continuous fuzzy measure and it is obvious that $\mu$ is strongly order continuous and has property (S). But $\mu$ has not pseudometric generating property.

In fact, $\mu(\{a\}) = \mu(\{b\}) = 0$, but $\mu(\{a\} \cup \{b\}) = 1 \neq 0$.

In the following we discuss the converse problem of the Egoroff's theorem for monotone set function.

**Theorem 4** Let $\mu$ be a Sugeno fuzzy measure. If for any $f, f_n \in \mathcal{F}$ ($n \in \mathbb{N}$), $f_n \overset{a.e.}{\longrightarrow} f$ implies $f_n \overset{a.u.}{\longrightarrow} f$, then $\mu$ is strongly order continuous and hence order continuous.

**Proof.** For any decreasing set sequence $\{E_n\}_n$ with $E_n \downarrow E$ and $\mu(E) = 0$, we define a measurable function sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin E_n \\ 1 & \text{if } x \in E_n \end{cases}$$

for any $n \geq 1$. It is easy to see that $f_n \overset{a.c.}{\longrightarrow} 0$. If $f_n \overset{a.u.}{\longrightarrow} 0$, then from Theorem 6.11 in [11], we can get for any $\sigma > 0$, $\lim_{n \to +\infty} \mu(\{x : |f_n(x)| \geq \sigma\}) = 0$. Therefore

$$\lim_{n \to +\infty} \mu(E_n) = \lim_{n \to +\infty} \mu(\{x : f_n(x) \geq \frac{1}{2}\}) = 0.$$

This shows $\mu$ is strongly order continuous and hence order continuous. \(\square\)

As special result of Theorem 4, Proposition 1 and Proposition 2, we have the following corollary:

**Corollary 2** (Li et al.[2, Theorem 8(3)]) Let $\mu$ be a continuous fuzzy measure. If for any $f, f_n \in \mathcal{F}$ ($n \in \mathbb{N}$), $f_n \overset{a.e.}{\longrightarrow} f$ implies $f_n \overset{a.u.}{\longrightarrow} f$, then $\mu$ is exhaustive.

**Remark 3** A order continuous (or exhaustive) fuzzy measure may not be strongly order continuous. Therefore Theorem 4 generalize the related result in Li et al. [1].
Example 3 Let $X = [0, +\infty)$, $\mathcal{F}$ be the class of all Lebesgue measurable sets on $X$, and $m$ be Lebesgue’s measure. Put

$$
\mu(E) = \begin{cases} 
0 & \text{if } 0 \not\in E \\
m(E) & \text{if } 0 \in E
\end{cases}
$$

Then $\mu$ is an order-continuous (hence exhaustive) fuzzy measure and has property (S). However $\mu$ is not strongly order continuous. In fact, if we take $E_n = [0, \frac{1}{n}) \cup (n, +\infty)$, $n = 1, 2, \cdots$, then $E_n \cap \{0\}$ and $\mu(\{0\}) = 0$. But $\mu(E_n) = \infty$, $n = 1, 2, \cdots$.

The following corollary is a direct result of Theorem 1 and 4:

Corollary 3 Let $\mu$ be a Sugeno fuzzy measure with property (S). Then, for any $f, f_n \in \mathcal{F}$ ($n \in N$),

$$
f_n \overset{a.c.}{\Rightarrow} f \implies f_n \overset{a.u.}{\to} f
$$

if and only if $\mu$ is strongly order continuous.

A measurable space $(X, \mathcal{F})$ is called $S$-compact, if for any sequence of sets in $\mathcal{F}$ there exists some convergent subsequence ([10], [11]). Any countable measurable space is $S$-compact ([10], [11]).

Proposition 6 Let $\mu$ be a Sugeno fuzzy measure on $S$-compact space $(X, \mathcal{F})$. If $\mu$ is continuous from below, then it has property (S).

Proof. Suppose $\{A_n\}_n \subset \mathcal{F}$ and $\lim_{n \to +\infty} \mu(A_n) = 0$. Since $(X, \mathcal{F})$ $S$-compact, there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\limsup A_{k_i} = \liminf A_{k_i}$, that is, $\bigcap_{m=1}^{+\infty} \bigcup_{i=m}^{+\infty} A_{k_i} = \bigcup_{m=1}^{+\infty} \bigcap_{i=m}^{+\infty} A_{k_i} \subseteq \lim_{m \to +\infty} \mu \left( \bigcap_{i=m}^{+\infty} A_{k_i} \right) \leq \lim_{m \to +\infty} \mu(\bigcup_{i=m}^{+\infty} A_{k_i}) = 0$.

Combining Theorem 1, Theorem 4 and Proposition 6, we can obtain the following result:

Theorem 5 Let $(X, \mathcal{F})$ be $S$-compact space (especially, $X$ is countable) and $\mu$ be a lower semicontinuous fuzzy measure. Then, for any $f, f_n \in \mathcal{F}$ ($n \in N$),

$$
f_n \overset{a.c.}{\Rightarrow} f \implies f_n \overset{a.u.}{\to} f
$$

if and only if $\mu$ is strongly order continuous.

References


