Solutions to Part I of Game Theory

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Solutions to Section I.1

1. To make your opponent take the last chip, you must leave a pile of size 1. So 1 is a P-position, and then 2, 3, and 4 are N-positions. Then 5 is a P-position, etc. The P-positions are 1, 5, 9, 13,. . . , i.e. the numbers equal to 1 mod 4.

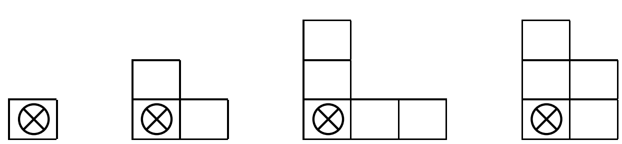
2.(a) The target positions are now 0, 7, 14, 21, etc.; i.e. anything divisible by 7.
(b) With 31 chips, you should remove 3, leaving 28.

3.(a) The target sums are 3, 10, 17, 24, and 31. If you start by choosing 3 and your opponent chooses 4, and this repeats four times, then the sum is 28, but there are no 3's left. You must choose 1 or 2 and he can then make the sum 31 and so win.
(b) Start with 5. If your opponent chooses 5 to get in the target series, you choose 2, and repeat 2 every time he chooses 5. When the sum is 26, it is his turn and there are no 5's left, so you will win. But if he ever departs from the target series, you can enter the series and win.

4.(a) The P-positions are the even numbers, $\{0, 2, 4, \ldots\}$.
(b) The P-positions are $\{0, 2, 4, 9, 11, 13, 18, 20, 22, \ldots\}$, the nonnegative integers equal to 0, 2, or 4 mod 9.
(c) The P-positions are $\{0, 3, 6, 9, \ldots\}$, the nonnegative integers divisible by 3.
(d) In (a), 100 is a P-position. In (b), 100 is an N-position since $100 = 1$ mod 9. It can be put into a P-position by subtracting 1 (or 6). In (c), 100 is an N-position. It can be put into a P-position by subtracting 1 (or 4 or 16 or 64).

5. The P-positions are those (m, n) with both m and n odd integers. If m and n are both odd, then any move will require putting an odd number of chips in two boxes; one of the two boxes would contain an even number of chips. If one of m and n is even, then we can empty the other box and put an odd number of chips in each box.

6. (a) In solving such problems, it is advisable to start the investigation with simpler positions and work up to the more difficult ones. Here are the simplest P-positions.



The last position shows that chomping at (3,1) is a winning move for the first player.

(b) The proof uses an argument, called "strategy stealing" that is useful in other problems as well. Consider removing the upper right corner. If this is a winning move, we are done. If not, then the second player has a winning reply. But whatever that reply is, the first player could have used it instead of the move he chose. (He could "steal" the second player's move.) Thus, in either case, the first player has a winning first move.

7. (a) Write the integer n in binary, $44 = (101100)_2$. A strategy that wins if it can be started out is to remove the smallest power of 2 in this expansion, in this case $4 = (100)_2$. Then the next player must leave a position for which this strategy can be continued. Another optimal first move for the first player is to remove $12 = (1100)_2$ chips. The only initial values of n for which the second player can win are the powers of 2: $n = 1, 2, 4, 8, 16, \ldots$.

(b) A strategy that wins is to remove the smallest Fibonacci number in the Zeckendorf expansion of n, (if possible). To see this, we note two things. First, if you do this, your opponent will be unable to take the smallest Fibnonacci number of the Zeckendorf expansion of the result, because it is greater than twice what you took. Second, if your opponent takes less than the smallest Fibonacci number in the Zeckendorf expension, you can again follow this strategy.

To prove this last sentence, suppose your opponent cannot take the smallest Fibonacci number in the Zeckendorf expansion of n. Let F_{n_0} represent this number, and suppose he takes $x < F_{n_0}$. The difference has a Zeckendorf expansion, $F_{n_0} - x = F_{n_1} + \cdots + F_{n_k}$, where F_{n_k} is the smallest. We must show $F_{n_k} \le 2x$, i.e. that you can take F_{n_k}. We do this by contradiction. Suppose $2x < F_{n_k}$. Then x is less than the next lower Fibonacci number. This implies that when x is replaced by its Zeckendorf expansion, $x = F_{n_{k+1}} + \cdots + F_{n_\ell}$, we have

$$F_{n_0} = F_{n_1} + \cdots + F_{n_k} + x = F_{n_1} + \cdots + F_{n_k} + F_{n_{k+1}} + \cdots + F_{n_\ell}$$

which gives a second Zeckendorf expansions of F_{n_0}. This contradicts unicity.

For $n = 43 = 34 + 8 + 1$, the strategy requires that we take 1 chip. (Another optimal initial move is to remove 9 chips, leaving 34, since twice 9 is still smaller than 34.) The only initial values of n for which the second player can win are the Fibonacci numbers themselves: $n = 1, 2, 3, 5, 8, \ldots$.

8. (a) If the first player puts an S in the first square, the second player can win by putting an S in the last square. Then no matter what letter the first player puts in either empty square, the second player can complete an SOS.

(b) Player I can win by placing an S in the central square. Then if Player II plays on the left, say, without allowing I to win immediately, Player I plays an S in the last square. Now neither player can play on the right. But after Player II and then I play innoculously on the left, Player II must play on the right and lose.

(c) Call a square **x-rated** if no matter which letter a player places in the square, the other player can win immediately. It is not hard to show that the *only* way to make an x-rated square is to have it and another x-rated square between two S's as in (a). Thus, x-rated squares come in pairs. So, if n is even (like 2000) and if neither player makes an error allowing the opponent to win in one move, then after an even number of moves only x-rated squares will remain. It will then be Player I's turn and he must fill an x-rated square and so lose. However, Player II must make sure there is at least one x-rated pair. But this is easy to do if n is large say greater than 14. Just play an S in a square with at least three or four empty spots on either side. On your next move you will be able to make an x-rated pair on one side or the other. Generally, Player I wins if n is odd, and Player II wins if n is even.

(d) The case $n = 14$ is special. Player I begins by playing an O at position 7. Then if Player II plays an S at position 11, Player I plays an O at position 13, say, and then Player II cannot play an S at position 8 because Player I could win immediately with an S at position 6. The position is actually drawn. Player I can prevent Player II from making any x-rated squares.

Solutions to Section I.2

1.(a) $27 \oplus 17 = 10$.

(b) If $38 \oplus x = 25$, then $x = 38 \oplus 25 = 49$.

2.(a) The unique winning move is to remove 4 chips from the pile of 12 leaving 8.

(b) There are three winning moves; removing 8 chips from the pile of 17 or the pile of 19, or the pile of 23.

(c) Exactly the same answer as for (a) and (b).

3. We may identify a coin on a square labelled n with a nim pile of size n and a move of that coin to the left to a square labelled k as removing $n - k$ chips from the nim pile. Since the coins do not interact, this is exactly nim. The next player wins the displayed diagram by moving the coin on square 9 to square 0 (or moving the coin on square 10 to square 3, or by moving the coin on square 14 to square 7).

4.(a) Suppose there is an H in place n.
(1) Turning this H to T without turning over another coin corresponds to completely removing a pile of n chips.
(2) Turning this H to T and some T in place k to H, where $k < n$, corresponds to removing $n - k$ chips from a pile of n.
(3) Turning this H to T and some H in place k to T, where $k < n$, corresponds to removing two piles of sizes n and k. But this is equivalent to removing $n - k$ chips from the pile of size n, thus creating two piles of size k, which effectively cancel because $k \oplus k = 0$.

(b) Since $2 \oplus 5 \oplus 9 \oplus 10 \oplus 12 = 8$, we must reduce the 9, 10 or 12 by 8. One method is to turn the H in place 9 to T and the T in place 1 to H. Another would be to turn the H in place 10 to T and the H in place 2 to T.

5. The player who moves first wins. A row with n spaces between the checkers corresponds to a nim pile with n chips. So the given position corresponds to a nim position with piles of sizes 4, 2, 3, 5, 3, 6, 2, and 1. The nim sum of these numbers is 6. You can win, for example, by moving the checker in the sixth row six squares toward the other, making the nim sum 0. Now if the opponent moves away from you in some row, you can move in the same row to keep the nim sum the same. (Such a move is called reversible.) If he moves toward you in some row, the nim sum is no longer 0, so you can find some row such that moving toward him reduces the nim sum to 0. In this way, the game will eventually end and you will be the winner.

6. Any move from $(x_1, x_2, x_3, \ldots, x_n)$ in staircase nim changes exactly one of the numbers, $x_1, x_3, \ldots, x_k$. Moreover, any nim move from $(x_1, x_3, \ldots, x_k)$ can be achieved as a staircase nim move from $(x_1, x_2, x_3, \ldots, x_n)$ by reducing one of the numbers, $x_1, x_3, \ldots, x_k$. Therefore a winning strategy is to keep the odd numbered stairs as a P-position in nim.

7. (a) When expanded in base 2 and added without carry modulo 3, we find 2212. To change the first (most significant) column to a 0, we must reduce two numbers that are 8 or greater. We may change the 10 to a 5 and the 13 to a 5.

$$\begin{array}{rr@{\qquad}rr} 4 = & 100_2 & 4 = & 100_2 \\ 8 = & 1000_2 & 8 = & 1000_2 \\ 8 = & 1000_2 & 8 = & 1000_2 \\ 9 = & 1001_2 & 9 = & 1001_2 \\ 10 = & 1010_2 & 5 = & 101_2 \\ 13 = & \underline{1101_2} & 5 = & \underline{101_2} \\ \text{sum mod } 3 = & 2212 \;\longrightarrow & & 0000 \end{array}$$

(b) Let $x_i = \sum_{j=0}^{m} x_{ij} 2^j$ be the base 2 expansion of x_i, where each x_{ij} is either 0 or 1 and m is sufficiently large. Let $\mathcal{P}$ be the set of all $(x_1, \ldots, x_n)$ such that for all j, $s_j \equiv \sum_{j=0}^{m} x_{ij} = 0 \pmod{k+1}$. (We refer to the vector s as the nim$_k$-sum of the x's. Note $0 \le s_j \le k$ for all j.) We show that $\mathcal{P}$ is the set of P-positions by following the proof of Theorem 1.

(1) *All terminal positions are in* $\mathcal{P}$. This is clear since $(0, \ldots, 0)$ is the only terminal position.

(2) *Every move from a position in* $\mathcal{P}$ *is to a position not in* $\mathcal{P}$. Suppose that $s_j = 0$ for all j, and that at most k of the x_i are reduced. Find the leftmost column j that is changed by one of these changes. If only one x_i had a 1 in position j, then s_j would be changed to k. If two x_i, then s_j would be changed to $k-1$, etc. But at most k changes are made, so that s_j is changed into a number between 1 and k. Thus the move cannot be in $\mathcal{P}$.

(3) *From each position not in* $\mathcal{P}$*, there is a move to a position in* $\mathcal{P}$. The difficulty of finding a winning move is to select which piles of chips to reduce. The problem of finding how many chips to remove from each of the selected piles is easy and there are usually many solutions. The algorithm below finds which piles to select.

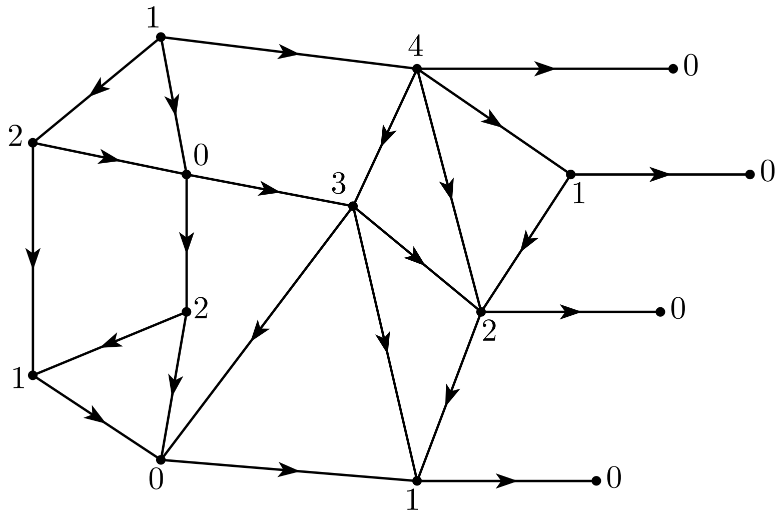
Find the leftmost column j with a nonzero s_j, and select any $t = s_j$ of the x_i with $x_{ij} = 1$. If $t = k$, you are done.

Let s' denote the nim$_k$-sum of the remaining x's, and find the leftmost column $j' < j$ such that $1 \le s'_{j'} < k - t$. If there is no such j', you are done and the t selected x's may be used. Otherwise, select any $t' = s'_m$ of the remaining x's with a 1 in position j' of their binary expansion. Then set $t = t + t'$, $j = j'$, and repeat this paragraph.

(c) Move as you would in normal Nim$_k$ until you would move to a position with all piles of size 1. Then move to leave 1 mod $k+1$ piles instead of 0 mod $k+1$ piles.

Solutions to Section I.3

1.



The Sprague-Grundy function.

2. The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	0	1	2	3	2	0	1	0	1	2	3	...

Then pattern for the first 7 nonnegative integers repeats forever. We have

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 2 \bmod 7 \\ 1 & \text{if } x = 1 \text{ or } 3 \bmod 7 \\ 2 & \text{if } x = 4 \text{ or } 6 \bmod 7 \\ 3 & \text{if } x = 5 \bmod 7. \end{cases}$$

3. The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...
$g(x)$	0	0	1	0	2	1	3	0	4	2	5	1	6	3	7	0	8	4	...

One may describe this function recursively as follows.

$$g(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ g((x-1)/2) & \text{if } x \text{ is odd.} \end{cases}$$

One may find $g(x)$ as follows. Take $x+1$ and factor out 2 as many times as possible (i.e. write $x+1 = 2^n y$ where y is an odd number). Then $g(x) = (y-1)/2$.

One may also write it as

$$g(x) = \begin{cases} 0 & \text{if } x = 2^n - 1 \\ 1 & \text{if } x = 2^n 3 - 1 \\ 2 & \text{if } x = 2^n 5 - 1 \\ \vdots & \vdots \\ k & \text{if } x = 2^n(2k+1) - 1 \\ \vdots & \vdots \end{cases} \qquad \text{for } n = 0, 1, 2, \ldots$$

4. (a) The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5	...

It may be represented mathematically as

$$g(x) = k + 1 \qquad \text{where } 2^k \text{ is the largest power of 2 dividing } x$$

(b) The Sprague-Grundy finction for Aliquot is simply 1 less than the Sprague-Grundy function for Dim$^+$; so for $x \geq 1$, $g(x) = k$ where 2^k is the largest power of 2 dividing x.

5. The Sprague-Grundy function is

7	8	6	9	0	1	4	5
6	7	8	1	9	10	3	4
5	3	4	0	6	8	10	1
4	5	3	2	7	6	9	0
3	4	5	6	2	0	1	9
2	0	1	5	3	4	8	6
1	2	0	4	5	3	7	8
0	1	2	3	4	5	6	7

For larger boards, the entries seem to become chaotic, but Wythoff found that the zero entries have coordinates $(0,0), (1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), \ldots$ with differences $0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$, the first number in each pair being the smallest number that hasn't yet appeared. He also showed that the nth pair is $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$, for $n = 0, 1, 2, \ldots$, where τ is the golden ratio $(1 + \sqrt{5})/2$.

6. (a) The Sprague-Grundy values are

5ω	$5\omega+1$	$5\omega+2$	$5\omega+3$	$5\omega+4$	$5\omega+5$	$5\omega+6$	
4ω	$4\omega+1$	$4\omega+2$	$4\omega+3$	$4\omega+4$	$4\omega+5$	$4\omega+6$	
3ω	$3\omega+1$	$3\omega+2$	$3\omega+3$	$3\omega+4$	$3\omega+5$	$3\omega+6$	
2ω	$2\omega+1$	$2\omega+2$	$2\omega+3$	$2\omega+4$	$2\omega+5$	$2\omega+6$	
ω	$\omega+1$	$\omega+2$	$\omega+3$	$\omega+4$	$\omega+5$	$\omega+6$	
0	1	2	3	4	5	6	

(b) The nim-sum of these transfinite Sprague-Grundy values follows the rule:

$$(x_1\omega + y_1) \oplus (x_2\omega + y_2) = (x_1 \oplus x_2)\omega + (y_1 \oplus y_2).$$

Therefore the Sprague-Grundy value of the given position is

$$(4\omega) \oplus (2\omega + 1) \oplus (\omega + 2) \oplus (5) = 7\omega + 6.$$

Since this is not zero, the position is an N-position. It can be moved to a P-position by moving the counter at 4ω down to $3\omega + 6$. There is no upper bound to how long the game can last, but every game ends in a finite number of moves.

(c) Yes.

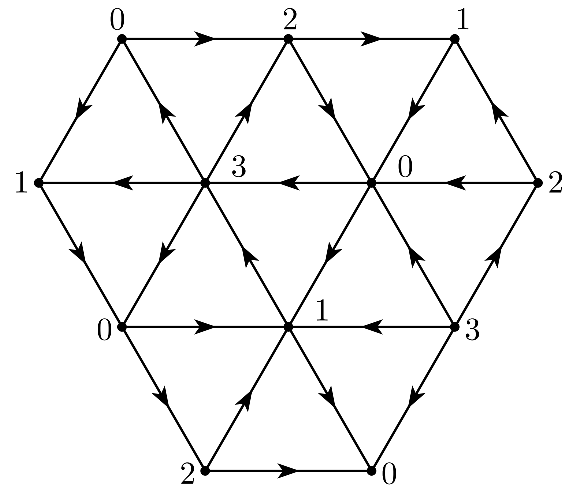
7. Suppose S consists of n numbers. Then no Sprague-Grundy value can be greater than n since the set $\{g(x-y) : y \in S\}$ contains at most n numbers. Let x_n be the largest of the numbers in S. There are exactly $(n+1)^{x_n}$ sequences of length x_n consisting of the integers from 0 to n. Therefore, when by time $(n+1)^{x_n} + 1$ there will have been two identical such sequences in the Sprague-Grundy sequence. From the second time on, the Sprague-Grundy sequence will proceed exactly the same as it did the first time.

8. We have $g(x) = \text{mex}\{g(x-y) : y \in S\}$, and $g^+(x) = \text{mex}\{0, \{g^+(x-y) : y \in S\}\}$. We will show $g^+(x) = g(x-1) + 1$ for $x \geq 1$ by induction on x. It is easily seen to be true for small values of x. Suppose it is true for all $x < z$. Then,

$$\begin{aligned} g^+(z) &= \text{mex}\{0, \{g^+(z-y) : y \in S\}\} = \text{mex}\{0, \{1 + g(x-y-1) : y \in S\}\} \\ &= 1 + \text{mex}\{g(x-y-1) : y \in S\} = 1 + g(x-1). \end{aligned}$$

9. (a) The Sprague-Grundy function does not exist for this graph. However, the terminal vertex is a P-position and the other vertex is an N-position.

(b) The Sprague-Grundy function exists here. However, backward induction does not succeed in finding it. The terminal vertex has SG-value 0, the vertex above it has SG-value 2, and of the two vertices above, the one on the left has SG-value 1 and the one on the right has SG-value 0. Those vertices of SG-value 0 are P-positions and the others are N-positions.



(c) The node at the bottom right, call it α, obviously has Sprague-Grundy value 0. But every other node can move to some node whose Sprague-Grundy value we don't know. Here is how we make progress. Consider the node, call it β, at the middle of the southwest edge. It can move to only two positions. But neither of these positions can have Sprague-Grundy value 0 since they can both move to α. So β must have Sprague-Grundy value 0. Continuing in a similar manner, we find:

Solutions to Section I.4

1. Remove any even number, or 1 chip if it is the whole pile. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	1	0	2	2	3	3	4	4	5	5	6	6	7	7	8	...

The general rule is $g(0) = 0$, g(1)=1, $g(2) = 1$, g(3)=0, and $g(x) = \lfloor x/2 \rfloor$ for $x \geq 4$, where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x, sometimes called the floor of x.

2. Remove any multiple of 3 if it is not the whole pile, or the whole pile if it contains 2 (mod 3) chips. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	0	1	0	1	2	1	2	3	2	3	4	3	4	5	4	5	...

The general rule is $g(0) = 0$ and for $k \geq 0$,

$$\begin{aligned} g(3k+1) &= k \\ g(3k+2) &= k+1 \\ g(3k+3) &= k \end{aligned}$$

3. There are three piles of sizes 18, 17, and 7 chips. The first pile uses the rules of Exercise 1, the second pile uses the rules of Exercise 2, and the third pile uses the rules of nim. The respective SG-values are 9, 6, and 7, with nim-sum $(1001)_2 \hat{+} (0110)_2 \hat{+} (0111)_2 = (1000)_2 = 8$. The can be put into a position of nim-sum 0 by moving the first pile to a position of SG-value 1. This can be done by removing 16 chips from the pile of 18, leaving 2, which has SG-value 1.

4. (a) The given position represents 2 piles of sizes 1 and 11. From Table 4.1, the SG-values are 1 and 6, whose nim-sum is 7. Since the nim-sum is not zero, this is an N-position.

(b) We must change the SG-value 6 to SG-value 1. This may be done by knocking down pin number 6 (or pin number 10), leaving a position corresponding to 3 piles of sizes 1, 3, and 7, with SG-values 1, 3, and 2 respectively. This is a P-position since the nim-sum is 0.

5. Remove one chip and split if desired, or two chips without splitting. (a) The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	...

We have $g(x) = x \pmod 4$, $0 \le g(x) \le 3$. This is periodic of period 4.

(b) Since 15 has SG-value 3, the moves to a P-position are those that remove 1 chip and split into two piles the nim-sum of whose SG-values is 0. For example, the move to two piles of sizes 1 and 13 is a winning move.

6. Remove two or more chips and split if desired, or one chip if it the whole pile. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	10	...

The general rule is

$$g(3k) = 2k$$
$$g(3k+1) = 2k$$
$$g(3k+2) = 2k+1$$

for $k \ge 0$, except for $g(1) = 1$.

7. Remove any number of chips equal to 1 (mod 3) and split if desired. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	0	1	2	3	2	3	4	5	4	5	6	7	6	7	8	...

The general rule is, for $k \ge 0$,

$$g(4k) = 2k$$
$$g(4k+1) = 2k+1$$
$$g(4k+2) = 2k$$
$$g(4k+3) = 2k+1.$$

8. (a) The loops divide the plane into regions. A move in a region with n dots divides that region into two regions with a and b dots, where $a+b$ is less than n but where a and b are otherwise arbitrary. We claim that a region with n dots has SG-value n, i.e. $g(n) = n$. (This may be seen by induction: Clearly $g(0) = 0$ since 0 is terminal. If $g(k) = k$ for all $k < n$, then $g(n) \ge n$ since all SG-values less than n can be reached in one move without splitting the region into two. But if a region of n dots is split into regions of size a and b with $a + b < k$, then since $a \oplus b \le a + b$, n cannot be obtained as the SG-value of a follower of n.) Thus a region of n dots corresponds to a nim pile of n chips.

(b) The given position corresponds to nim with three piles of sizes 3, 4 and 5. Since the nim-sum is 2, this is an N-position. An optimal move must reduce the 3 to a 1. This is achieved by drawing a loop through exactly two of the three free dots at the bottom of the figure.

9. (a) A loop in this game takes away 1 or 2 dots from a region and splits the region into two parts one of which may be empty of dots. This is exactly the same as the rules for Kayles.

(b) Using Table 4.1, $g(5) \oplus g(4) \oplus g(3) = 4 \oplus 1 \oplus 3 = 6$ so this is an N-position. An optimal move is to draw a closed loop through a dot from the innermost 5 dots such that exactly three dots stay inside the loop.

10. (a)

1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	0	2	1	0	2	1	0	2	1	3

(b) The SG-values of 5, 8, and 13 are 2, 2, and 3 respectively. The winning first moves are (1) splitting 5 into 2 and 3, (2) splitting 8 into 2 and 6, and (3) splitting 13 into 5 and 8.

11. (a) $g(S_n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even, } n \geq 2. \end{cases}$

(b) When played on a line with n edges, the rules of the game are: (1) You may remove one chip if it is the whole pile, or (2) you may remove two chips from any pile and if desired split that pile into two parts. In the notation of Winning Ways, Chapter 4, this game is called .37 (or .6 if one counts vertices rather than edges). The Sprague-Grundy values up to $n = 10342$ have been computed without finding any periodic pattern. It is generally believed that none exists. Here are the first few values.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$g(L_n)$	0	1	2	0	1	2	3	1	2	3	4	0	...

(c) $g(C_n) = \begin{cases} 0 & \text{if } g(L_{n-2}) > 0 \\ 1 & \text{if } g(L_{n-2}) = 0 \end{cases}$. Because of (b), there seems to be no periodicity in the appearance of the 1's. But we can say that $g(C_n) = 0$ if n is even.

(d) Let $DS_{m,n}$ denote the stars S_m and S_n joined by an (additional) edge (so that $DS_{0,n} = S_{n+1}$, and $DS_{1,1} = L_3$). For $n \geq 0$, $g(DS_{0,n}) = g(DS_{n,0}) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$. For $m \geq 1$ and $n \geq 1$, $g(m, n) = \begin{cases} 0 & \text{if } m + n \text{ is even} \\ 3 & \text{if } m + n \text{ is odd} \end{cases}$.

(e) The first player wins the square lattice (i) by taking the central vertex and reducing the position to C_8 with Sprague-Grundy value 0 from (c). The second player wins the tic-tac-toe board by playing symmetrically about the center of the graph. To generalize to larger centrally symmetric graphs, we need to define the symmetry for an arbitrary graph, (V, E). Here is one way.

Suppose there exists a one-to-one map, g, of V onto V such that
(1) (graph preserving) $\{v_1, v_2\} \in E$ implies $\{g(v_1), g(v_2)\} \in E$
(2) (pairing) $u = g(v)$ implies $v = g(u)$
(3) (no fixed vertex) $v \neq g(v)$ for all $v \in V$
(4) (no fixed edge) $\{v_1, v_2\} \in E$ implies $\{v_1, v_2\} \neq \{g(v_2), g(v_1)\}$.

The second player wins such symmetrically paired graphs without fixed vertices or fixed edges, by playing symmetrically. Can the second player always win if we allow exactly one fixed edge in the mapping?

Solutions to Section I.5

1. (a) In Turning Turtles, the positions are labelled starting at 1, so the heads are in positions 3, 5, 6 and 9. The position has SG-value $3 \oplus 5 \oplus 6 \oplus 9 = 9$, so a winning move is to turn over the coin at position 9.
(b) In Twins, the labelling starts at 0, so the heads at in positions 2, 4, 5 and 8. The position has SG-value $2 \oplus 4 \oplus 5 \oplus 8 = 11$. . A winning move is to turn over the coins at positions 3 and 8.
(c) For the subtraction set $S = \{1, 3, 4\}$, the Sprague-Grundy sequence is

position x :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14 ...
$g(x)$:	0	1	0	1	2	3	2	0	1	0	1	2	3	2	0 ...

The labelling starts at 0 so the heads are in positions 2, 4. 5 and 8, with a combined SG-value $0 \oplus 2 \oplus 3 \oplus 1 = 0$. This is a P-position.
(d) The labelling starts at 0 so the heads are in positions 2, 4, 5 and 8. In nim, this has SG-value $2 \oplus 4 \oplus 5 \oplus 8 = 11$. It can be moved to a position of SG-value 0 by turning over the coins at 3 and 8. Since this leaves an even number of heads, it is a P-position in Mock Turtles. The Mock Turtle did not need to be turned over.

2. (a) The maximum number of moves the game can last is n.
(b) Let T_n denote the maximum number of moves the game can last. This satisfies the recursion, $T_n = T_{n-1} + T_{n-2} + 1$ for $n > 2$ with initial values $T_1 = 1$ and $T_2 = 2$. We see that $T_n + 1$ is just the Fibonacci sequence, $2, 3, 5, 8, 13, 21 \ldots$. So T_n is the sequence $1, 2, 4, 7, 12, 20, \ldots$.
(c) This time T_n satisfies the recursion, $T_n = T_{n-1} + \cdots + T_1 + 1$ with initial condition $T_1 = 1$. So T_n is the sequence, $1, 2, 4, 8, 16, 32, \ldots$.

3. (a) Suppose we start the labelling from 0. Then a single heads in positions 0 or 1 is a terminal position and so receives SG-value 0. Continuing as in Mock Turtles, we find

position x :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14 ...
$g(x)$:	0	0	1	2	4	7	8	11	13	14	16	19	21	22	25 ...

This is just the SG-sequence for Mock Turtles moved over two positions.
(b) To get nim out of this, we should have started labelling the positions of the coins from -2. The first two coins on the left are dummies. It doesn't matter whether they are heads or tails. The third coin on the left is the Mock Turtle. The P-positions in Triplets are exactly the P-positions in Mock Turtles when the first two coins on the left are ignored.

4. The SG-sequence for Rulerette is easily found to be

position x :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
$g(x)$:	0	1	0	2	0	1	0	4	0	1	0	2	0	1	0	8 ...

g(x) is half of the SG-value of x for Ruler except for x odd when $g(x) = 0$.

5. This becomes an impatient subtraction game mentioned in Exercise 3.8. The Sprague-Grundy function, $g^+(x)$ of this game is just $g(x-1)+1$, where $g(x)$ is the Sprague-Grundy function of the subtraction game.

6. (a) We have $6 \otimes 21 = 6 \otimes (16 \oplus 5) = (6 \otimes 16) + (6 \otimes 5) = 96 \oplus 8 = 104$.
(b) We have $25 \otimes 40 = (16 \oplus 9) \otimes (32 \oplus 8) = (16 \otimes 32) \oplus (16 \otimes 18) \oplus (9 \otimes 32) \oplus (9 \otimes 8)$.
Then using $16 \otimes 32 = 16 \otimes 16 \otimes 2 = 24 \otimes 2 = (16 \oplus 8) \otimes 2 = 32 \oplus 12 = 44$,
and $9 \otimes 32 = 9 \otimes 16 \otimes 2 = 13 \otimes 16 = 224$,
we have $25 \otimes 40 = 44 \oplus 128 \oplus 224 \oplus 5 = 79$.
(c) $1 \oslash 14 = 13$, so $15 \oslash 14 = 15 \otimes 13 = 12$.
(d) Since $14 \otimes 14 = 8$, we have $\sqrt{8} = 14$.
(e) $x^2 \oplus x \oplus 6$ is the same as $x \otimes (x \oplus 1) = 6$. Looking as Table 5.2, we see this occurs for $x = 14$ or $x = 15$.

7. (a) Suppose there exists a move in Turning Corners from (v_1, v_2) into a position of SG-value u. Then there is a $u_1 < v_1$ and a $u_2 < v_2$ such that $(u_1 \otimes u_2) \oplus (v_1 \otimes u_2) \oplus (u_1 \otimes v_2) = u$. Since $u_1 < g_1(x)$, there exists a move in G_1 to an SG-value u_1, turning over the coins, say, at positions $x_1, x_2, \ldots, x_m, x$, where all $x_i < x$. Similarly there exists a move in G_2 to an SG-value u_2 turning over coins, say, at positions $y_1, y_2, \ldots, y_n, y$, where all $y_j < y$. This implies

$$\begin{aligned} g_1(x_1) \oplus g_1(x_2) \oplus \cdots \oplus g_1(x_m) &= u_1 \qquad \text{and} \\ g_2(y_1) \oplus g_2(y_2) \oplus \cdots \oplus g_2(y_m) &= u_2. \end{aligned} \tag{1}$$

Then the move, $\{x_1, \ldots, x_m, x\} \times \{y_1, \ldots, y_n, y\}$ in $G_1 \times G_2$ results in SG-value

$$\begin{aligned} &\Big(\sum^{*}\sum^{*} g_1(x_i) \otimes g_2(y_j)\Big) \oplus \Big(\sum^{*} g_1(x_i) \otimes g_2(y)\Big) \oplus \Big(\sum^{*} g_1(x) \otimes g_2(y_j)\Big) \\ &\quad = ((g_1(x_1) \oplus \cdots \oplus g_1(x_m)) \otimes (g_2(y_1) \oplus \cdots \oplus g_2(y_n))) \\ &\qquad \oplus (g_1(x) \otimes (g_2(y_1) \oplus \cdots \oplus g_2(y_n)) \\ &\qquad \oplus (g_1(x_1) \oplus \cdots \oplus g_1(x_m)) \otimes g_2(y)) \\ &\quad = (u_1 \otimes u_2) \oplus (v_1 \otimes u_2) \oplus (u_1 \otimes v_2) = u \end{aligned} \tag{2}$$

where $\sum^{*}$ represents nim-sum. Conversely, for any move, $\{x_1, \ldots, x_m, x\} \times \{y_1, \ldots, y_n, y\}$, in $G_1 \times G_2$, we find u_1 and u_2 from (1). Then the same equation (2) shows that the corresponding move in Turning Corners has the same SG-value.
(b) We may conclude that the mex of the SG-values of the followers of (x, y) in $G_1 \times G_2$ is the same as the mex of the SG-values of the followers of (v_1, v_2) in Turning corners, implying $g_1(x) \otimes g_2(y) = v_1 \otimes v_2$.

8. (a) The table is

	1	2	1	4	1	2	1	8
1	1	2	1	4	1	2	1	8
2	2	3	2	8	2	3	2	12
4	4	8	4	6	4	8	4	11
7	7	9	7	4	10	9	7	15
8	8	12	8	11	8	12	8	13

(b) The given position has SG-value $2 \oplus 13 = 15$. A winning move must change the SG-value 13 to 2. In Turning corners, the move from (8,8) that that changes the SG-value 13 to 2 is the move with north west corner at (3,3). A move in Mock Turtles that changes $x = 5$ with $g_1(x) = 8$ into a position of SG-value 2, is the move that turns over 5, 2 and 1. A move in Ruler that change $y = 8$ with $g_2(8) = 9$ into a position with SG-value 2 is the move that turns over 8, 7, and 6. Therefore a winning move in the given position is $\{1, 2, 5\} \times \{6, 7, 8\}$. This gives

T	H	T	T	T	H	H	H
T	T	T	T	T	H	H	H
T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	T
T	T	T	T	T	H	H	T

which has SG-value 0.

9. (a) Since the game is symmetric, the SG-value of heads at (i, j) is the same as the SG-value of heads at (j, i). This implies that the SG-value of the initial position is 0. It is a P-position for all n. A simple winning strategy is to play symmetrically. If your opponent makes a move with (i, j) as the south east coin, you make the symmetric move at (j, i). Such a play keeps the game symmetric without heads along the diagonal. This holds true in any tartan game that is the square of some coin turning game.
(b) The SG-values of off-diagonal elements cancel, so the SG-value of the game is the sum of the SG-values on the diagonal. For $n = 1, 2, \ldots$, these are $1, 2, 3, 5, 4, 7, 6, 11, 10, 9, 8, 14, 15, 12, 13, \ldots$. One can show that this hits all positive integers without repeating, and is never 0. So this is a first player win. However there doesn't seem to be a simple winning strategy.

10. The SG-sequence for G_1 is

position x :	1	2	3	4	5	6	7	8	9	10	...	98	99	100
$g(x)$:	0	1	2	3	4	0	1	2	3	4	...	2	3	4.

For G_2, it is

position x :	1	2	3	4	5	6	7	8	...	97	98	99	100
$g(x)$:	1	2	1	4	1	2	1	8	...	1	2	1	4.

The coin at (100,100) has SG-value $4 \otimes 4 = 6$ and the coin at (4,1) has SG-value $3 \otimes 1 = 3$. You cam win by turning over the 8 coins at positions (x, y) with $x = 98, 100$ and $y = 97, 98, 99, 100$.

This works in any two-dimensional game which is the product of the two same one-domensional games.

Solutions to Section I.6

1. The SG-value of the three-leaf clover is 2. The SG-value of the girl is 3. The SG-value of the dog is 2. And the SG-value of the tree is 5. So there exists a winning move on the tree that reduces the SG-value to 3. The unique winning move is to hack the left branch of the rightmost branch completely away.