THREE-MEMBER COMMITTEE LOOKING FOR A SPECIALIST WITH TWO HIGH ABILITIES

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ABSTRACT. An optimal stopping problem belonging to 3-player 2-choice n-stage sequential game is studied. A three-player committee is looking for a specialist with two kinds of high abilities. There are n applicants. The committee interviews each applicant one-by-one and it decides it's R(=reject) or A(=accept) by simple majority over each member's choice of R/A. Payoff to each player is the minimum of the two kinds of abilities of the applicant accepted by the committee. The solution without using computer is given.

1 Statement and Formulation of the Problem. A 3-player(=member) committee has players I, II and III. The committee wants to employ one specialist among n applicants. It interviews applicants sequentially one-by-one. Facing each applicant player I(II, III) evaluates the management ability at $X_1(Y_1, Z_1)$ and computer ability at $X_2(Y_2, Z_2)$. Evaluation by the players are made independently and each player chooses, based on his evaluation, either one of R and A. The committee's choice is made by simple majority. If the committee rejects the first $n-1$ applicants, then it should accept the n-th applicant. Denote

$$\xi = x_1 \wedge x_2, \eta = y_1 \wedge y_2, \zeta = z_1 \wedge z_2. \tag{1.1}$$

If the committee accepts an applicant with talents evaluated at $\mathbf{x}, \mathbf{y}, \mathbf{z}$ by I, II, III, resp., then the game stops and each player is paid ξ, η, ζ to I, II, III, resp.. If the committee rejects an applicant, then the next applicant is interviewed and the game continues. Each player of the committee aims to maximize the expected payoff he can get.

The two different kinds of talents (management and computer abilities) for each applicant, are bivariate r.v.s, *i.i.d.* with pdf

$$h(x_1, x_2) = 1 + \gamma(1-2x_1)(1-2x_2), \ \forall (x_1, x_2) \in [0,1]^2, |\gamma| \leq 1 \tag{1.2}$$

for player I. For II and III, pdfs are $h(y_1, y_2)$ and $h(z_1, z_2)$ respectively, with the same γ. If $X_1(X_2)$ for I is the evaluation of ability of management (foreign language), then γ will be $0 \leq \gamma \leq 1$. If X_2 is the evaluation of the computer ability, then γ may be $-1 \leq \gamma \leq 0$.

The bivariate pdf (1.2) is one of the simplest pdf that has the identical uniform marginal and correlated component variables. The correlation coefficient is equal to $\gamma/3$.

Denote the state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\mathbf{x} = (x_1, x_2)$, *etc.*, to mean that ① the first $j-1$ applicants were rejected by the committee, ② the j-th applicant is currently evaluated at $\mathbf{x}, \mathbf{y}, \mathbf{z}$, by I, II, III resp. and ③ $n-j$ applicants remain un-interviewed if the j-th is rejected by the committee. The state is illustrated by Figure 1.

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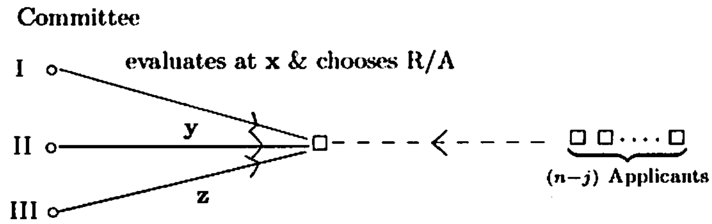

Figure 1. State $(j, \mathbf{x}, \mathbf{y}, \mathbf{z})$

We define u_j = Expected payoff, player I can get, if I is in state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z})$ and all players play optimally hereafter.

Define v_j, for II, and w_j, for III, similarly. Moreover we introduce a number

$$c \equiv E_{\mathbf{x}}(\xi) = 2\int_0^1 dx_1 \int_0^{x_1} x_2 h(x_1, x_2)dx_2 = \frac{1}{3} + \frac{1}{30}\gamma$$

where is in $[3/10, 11/30]$ for $\forall \gamma \in [-1, 1]$.

The Optimality Equation of our 3-player 2-choice n-stage game is

$$(1.3) \qquad (u_j, v_j, w_j) = E_{\mathbf{x},\mathbf{y},\mathbf{z}}[\text{Optimal payoffs facing } \mathbf{M}_j(x, y, z)],$$

$$(j \in [1, n], u_n = v_n = w_n == E_{\mathbf{x}}(\xi) = c),$$

where the payoff matrix is represented by

$$(1.4) \qquad \mathbf{M}_j(\mathbf{x},\mathbf{y},\mathbf{z}) \begin{cases} \xrightarrow{\text{R by I}} \mathbf{M}_{j,R}(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \xrightarrow{\text{A by I}} \mathbf{M}_{j,A}(\mathbf{x},\mathbf{y},\mathbf{z}) \end{cases}$$

(1.5) $\mathbf{M}_{jR}(\mathbf{x},\mathbf{y},\mathbf{z}) =$

	R by III	A by III
R by II	$u, \ v, \ w$	$u, \ v, \ w$
A by II	$u, \ v, \ w$	$\xi, \ \eta, \ \zeta$

(1.6) $\mathbf{M}_{jA}(\mathbf{x},\mathbf{y},\mathbf{z}) =$

$u, \ v, \ w$	$\xi, \ \eta, \ \zeta$
$\xi, \ \eta, \ \zeta$	$\xi, \ \eta, \ \zeta$

because of the simple majority rule.

(In each cell, the subscript $j+1$ of $u_{j+1}, v_{j+1}, w_{j+1}$ is omitted. We use this convention hereafter too, when needed.)

Related problems are investigated in Ref.[2, 4, 5] . Ref.[1] studies a two player game. Ref.[3, 4, 5] are concerned with 3-or-more player games.

2 Solution to the Problem.

Lemma 1 *For* I *in state* $(j, \mathbf{x}, \mathbf{y}, \mathbf{z})$, R (A) *dominates* A (R), *if* $u_{j+1} > (<)\xi$. *By symmetry, for* II (III), u_{j+1} *and* ξ *are replaced by* v_{j+1} *and* η (w_{j+1} *and* ζ).

Proof. Eq.(1.5) minus Eq.(1.6) is

$$\mathbf{M}_R - \mathbf{M}_A = \begin{array}{c|ccc|ccc|} & & \text{R by III} & & & \text{A by III} & \\ \hline \text{R by II} & 0, & 0, & 0 & u-\xi, & v-\eta, & w-\zeta \\ \hline \text{A by II} & u-\xi, & v-\eta, & w-\zeta & 0, & 0, & 0 \\ \hline \end{array}$$

and we look at the signs of 4 components for I. Hence the lemma follows □

Lemma 2

(2.1) $$f(u) \equiv E_{\mathbf{x}} I(\xi > u) = (\overline{u})^2 (1 + \gamma u^2), \quad \forall u \in [0, 1].$$

This function is decreasing with values $f(0) = 1, f\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{16}\gamma$, *and* $f(1) = 0$. *Moreover* $f(u)$ *is convex if* $0 < \gamma \leq 1$.

(2.2) $$g(u) \equiv E_{\mathbf{x}} \left[\xi I(\xi > u)\right] = c - u^2 + \frac{2}{3}u^3 + \gamma u^3 \left(\frac{2}{3} - \frac{3}{2}u + \frac{4}{5}u^2\right)$$

is decreasing with values $g(0) = c, g\left(\frac{1}{2}\right) = \frac{1}{6} + \frac{23}{480}\gamma$, *and* $g(1) = 0$.

Proof.

$$\begin{aligned} f(u) &= \int_u^1 \int_u^1 \{1 + \gamma(1 - 2x_1)(1 - 2x_2)\} \, dx_1 dx_2 \\ &= (\overline{u})^2 + \gamma \left[\int_0^1 (1 - 2x_1) dx_1\right]^2 = (\overline{u})^2 \left[1 + \gamma(-u\overline{u})^2\right] \end{aligned}$$

i.e., Eq.(2.1). Moreover we obtain

$$f''(u) = 2\gamma \left[\gamma^{-1} + (1 - 6u\overline{u})\right] > 0, \quad \forall u \in [0, 1], \text{ if } 0 < \gamma \leq 1.$$

On the other hand

$$\begin{aligned} g(u) &= \int_u^1 dx_1 \int_u^{x_1} x_2 h(x_1, x_2) dx_2 + \int_u^1 dx_2 \int_u^{x_2} x_1 h(x_1, x_2) dx_1 \\ &= 2 \int_u^1 dx_1 \int_u^{x_1} x_2 \{1 + \gamma(1 - 2x_1)(1 - 2x_2)\} \, dx_2 \end{aligned}$$

After a bit of calculations, we have

$$2 \int_u^1 (1 - 2x_1) dx_1 \int_u^{x_1} x_2 (1 - 2x_2) dx_2 = \frac{1}{30} + \frac{2}{3}u^3 - \frac{3}{2}u^4 + \frac{4}{5}u^5$$

and so Eq.(2.2) follows.

Both of $f(u)$ and $g(u)$ are decreasing, because of their definitions. □

It is evident that

(2.3) $$1 > f(u) > g(u) > 0, \quad \forall u \in (0, 1)$$

by the definitions of $f(u)$ and $g(u)$.

Theorem 1 *Optimal expected payoff to* I *satisfies the recursion*

(2.4) $$u_j = Q(u_{j+1}), \quad \forall j \in [1, n-1], u_n = c,$$

where

(2.5) $$Q(u) = u \left[1 - 3(f(u))^2 + 2(f(u))^3\right] + (f(u))^2 (c - 2g(u)) + 2f(u)g(u).$$

In the r.h.s., $f(u)$ *and* $g(u)$ *are given by* (2.1) *and* (2.2) *resp., in* Lemma 2. $Q(u)$ *is a continuous function with values* $Q(0) = c$ *and* $Q(1) = 1$.

Proof. From Lemmas 1 and 2 and Eqs (1.1)~(1.6), the optimal payoff to player I is the sum of eight terms

$$u\,[I(\xi<u,\eta<v,\zeta<w)+I(\xi<u,\eta<v,\zeta>w)+I(\xi<u,\eta>v,\zeta<w)$$
$$+I(\xi>u,\eta<v,\zeta<w)]+\xi\,[I(\xi<u,\eta>v,\zeta>w)+I(\xi>u,\eta<v,\zeta>w)$$
$$+I(\xi>u,\eta>v,\zeta<w)+I(\xi>u,\eta>v,\zeta>w)]\,.$$

Remember that the u, v, w, here, are $u_{j+1}, v_{j+1}, w_{j+1}$, resp.

Committee's decision is R (A) in the first (second) 4 events.

Taking $E_{\mathbf{x},\mathbf{y},\mathbf{z}}$ of the r.v.s, we get

$$u\left\{\overline{f(u)}\;\overline{f(v)}\;\overline{f(w)}+\overline{f(u)}\;\overline{f(v)}f(w)+\overline{f(u)}f(v)\overline{f(w)}+f(u)\overline{f(v)}\;\overline{f(w)}\right\}$$
$$+\left\{(c-g(u))f(v)f(w)+g(u)\overline{f(v)}f(w)+g(u)f(v)\overline{f(w)}+g(u)f(v)f(w)\right\}$$

From symmetry among the three players, we can take $u=v=w$. Then the above expression becomes

$$u\left\{\overline{f(u)}^3+3f(u)\overline{f(u)}^2\right\}+\left\{c(f(u))^2+2g(u)f(u)\overline{f(u)}\right\}$$
$$=u\overline{f(u)}^2\left\{\overline{f(u)}+3f(u)\right\}+c(f(u))^2+2g(u)f(u)\overline{f(u)}$$
$$=u\left\{1-3(f(u))^2+2(f(u))^3\right\}+(f(u))^2(c-2g(u))+2f(u)g(u).$$

The last expression is $Q(u)$ given by (2.5). □

Theorem 2

(2.6) $$u_1>u_2>\cdots>u_n=c$$

Proof. We have by Theorem 1,

$$\begin{aligned} u_j-u_{j+1} &= Q(u_{j+1})-u_{j+1} \\ &= \left[u\left\{-3(f(u))^2+2(f(u))^3\right\}+(f(u))^2(c-2g(u))+2f(u)g(u)\right]_{u=u_{j+1}} \\ &= f(u_{j+1})\left[u\left\{-3f(u)+2(f(u))^2\right\}+f(u)(c-2g(u))+2g(u)\right]_{u=u_{j+1}}. \end{aligned}$$

We want to prove that the inside of $[\cdots]$, *i.e.*,

(2.7) $$m(u)\equiv 2g(u)+f(u)(-3u+c-2g(u))+2u(f(u))^2$$

is positive for $\forall u\in(0,1)$. We find that its proof is un-expectedly intractable. Clearly $m(0)=c$ and $m(1)=0$.

Suppose that $m(u_0)=0$ for some $u_0\in(0,1)$. Then, from (2.4) and (2.7), it must hold

(2.8) $$(\overline{u_0})^2(1+\gamma u_0^2)=f(u_0)=\frac{3}{4}+(4u_0)^{-1}\left[2g-c-\sqrt{(3u_0-c+2g)^2-16u_0g}\right]$$
$$=\frac{3}{4}+(4u_0)^{-1}\left[2g-c-\sqrt{(2g-c)^2+(c-3u_0)^2-(4u_0g+c^2)}\right]$$

The inside of the square root becomes negative for $u_0=(1/3)\,c$, since

$$(2g-c)^2-\left(\frac{4}{3}cg+c^2\right)=4g\left(g-\frac{4}{3}c\right)<0.$$

So, Eq.(2.8) doesn't hold true.

Hence $m(u)\neq 0, \forall\ u\in(0,1)$, and since $m(0)=c$, and $m(1)=0$, we find that $m(u)>0, \forall\ u\in(0,1)$. □

3 The Case $\gamma = 0$. Consider the special case $\gamma = 0$. Then from (2.1), (2.2) and (2.4) we obtain

$$c_0 = \frac{1}{3}, f_0(u) = \bar{u}^2, g_0(u) = \frac{1}{3} - u^2 + \frac{2}{3}u^3,$$

and

$$(3.1) Q_0(u) = u\left[1 - 3(f_0(u))^2 + 2(f_0(u))^3\right] + (f_0(u))^2\left(\frac{1}{3} - 2g_0(u)\right) + 2f_0(u)g_0(u)$$
$$= u + \bar{u}^2\left(\frac{2}{3} - 2u^2 + \frac{4}{3}u^3\right) + \bar{u}^4\left(-\frac{1}{3} - 3u + 2u^2 - \frac{4}{3}u^3\right) + 2u\bar{u}^6$$

Therefore

$$(3.2) \quad u_{n-1} = Q_0(1/3) = \frac{1}{3} + \frac{4}{9}\cdot\frac{40}{81} + \text{two more terms}$$
$$\approx \frac{1}{3} + 0.21948 - 0.22923 + 0.05853 \approx 0.38211$$

and

$$u_{n-2} = Q_0(0.38211) \approx 0.3821 + 0.17145 - 0.18397 + 0.04253 \approx 0.4121.$$

We have a conjecture that the sequence (2.6) is concavely decreasing.

An example.

There are 3-player committee and 3-applicants. The common optimal strategy for each player is

"Choose A (R), if $x_1 \wedge x_2 > (<) u_1 = 0.4121$ in the first stage"

[For player II (III) $x_1 \wedge x_2$ is replaced by $y_1 \wedge y_2 (z_1 \wedge z_2)$.]

If the committee rejects the 1st applicant, then it interviews the 2nd applicant, and

"Choose A (R), if his $x_1 \wedge x_2 > (<) u_2 = 0.3821$ in the second stage"

If the committee rejects the 2nd applicant, then the committee should accept the last applicant. Each player's expected payoff is $u_3 = 1/3$.

4 Final Remark. The present author cannot use computer by some inevitable private reasons. If we can use computer, it would be interesting to make the table of

	u_n	u_{n-1}	u_{n-2}	$\cdots$
$\gamma = -1$	3/10	0.3420	0.3685	$\cdots$
0	1/3	0.3821	0.4121	$\cdots$
1	11/30	0.4284	0.4620	$\cdots$

(The numbers above are obtained, from (2.1)~(2.5), by using a small calculator.)

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