A Comparison of Andness/Orness Indicators

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Abstract

The goal of this paper is to present various indicators of andness/orness, and to analyze their properties, similarities and differences.

Keywords: Andness, orness, aggregation operators.

1 Introduction

The idea that observable properties of human reasoning in the area of system evaluation could be modeled using a function that supports a continuous transition from conjunction to disjunction was proposed in 1973 [4]. This aggregation function includes conjunctive and disjunctive properties and we call it the Generalized Conjunction/Disjunction (GCD) [11], and symbolically denote $y = x_1 \Diamond \cdots \Diamond x_n$, $x_i \in I = [0,1]$, $i = 1,...,n$, $y \in I$. Since GCD is located between the extreme cases of conjunction and disjunction the initial proposal [4] included two complementary parameters of such aggregators: the conjunction degree (andness) and the disjunction degree (orness). The andness ($\alpha \in I$) indicates the similarity between GCD and conjunction, and the orness ($\omega \in I$) indicates the similarity between GCD and disjunction; α and ω are complementary: $\alpha + \omega = 1$. The parameters $\alpha = 1, \omega = 0$ denote the full conjunction and $\alpha = 0, \omega = 1$ denote the full disjunction. The location of GCD with respect to conjunction and disjunction was defined in [4] as follows:

$$
\begin{aligned}
x_1 \Diamond \cdots \Diamond x_n &= \omega(x_1 \vee \cdots \vee x_n) + (1-\omega)(x_1 \wedge \cdots \wedge x_n) \\
&= (1-\alpha)(x_1 \vee \cdots \vee x_n) + \alpha(x_1 \wedge \cdots \wedge x_n) \\
&= \omega(x_1 \vee \cdots \vee x_n) + \alpha(x_1 \wedge \cdots \wedge x_n)
\end{aligned}
$$

If $\alpha > 0.5 > \omega$ then $x_1 \Diamond \cdots \Diamond x_n$ is called the partial conjunction and symbolically denoted $x_1 \Delta \cdots \Delta x_n$ [10]. If $\alpha < 0.5 < \omega$ then $x_1 \Diamond \cdots \Diamond x_n$ is called the partial disjunction and symbolically denoted $x_1 \nabla \cdots \nabla x_n$. If $\alpha = 0.5 = \omega$ then $x_1 \Diamond \cdots \Diamond x_n$ is called the neutrality function, denoted $x_1 \ominus \cdots \ominus x_n$ and implemented as the arithmetic mean. Consequently,

$$
x_1 \Diamond \cdots \Diamond x_n = \begin{cases}
(x_1 \vee \cdots \vee x_n), & \alpha = 0, \omega = 1 \\
x_1 \nabla \cdots \nabla x_n \quad , & 0 < \alpha < 0.5 < \omega < 1 \\
x_1 \ominus \cdots \ominus x_n \quad , & \alpha = 0.5 = \omega \\
x_1 \Delta \cdots \Delta x_n \quad , & 1 > \alpha > 0.5 > \omega > 0 \\
(x_1 \wedge \cdots \wedge x_n), & \alpha = 1, \omega = 0
\end{cases}
$$

GCD can be implemented in various ways. During the intuitive evaluation and comparison of complex alternatives human reasoning simultaneously combines two almost orthogonal components: a formal logic component, and a semantic component [11]. The formal logic component is expressed by the position of a logic aggregator in the interval between conjunction and disjunction. The semantic component is the concept or relative importance that is always present in human decision-making. Human decisions are based on combining degrees of truth expressed by continuous logic variables, but all variables don't have the same weight in decision models: some are more important, and some are less important. No realistic model can be built unless we are able to properly (and independently) adjust both the andness and the relative importance of inputs of logic aggregators. This was the reason why the first implementation of GCD proposed in [4] was based on weighted power means:

$$
y = (W_1 x_1^r + ... + W_n x_n^r)^{1/r},
$$
$$
W_i > 0, \quad i = 1,...,n, \quad \sum_1^n W_i = 1, \quad -\infty \le r \le +\infty
$$

From the standpoint of GCD, all means can be interpreted as logic functions. Those means that have a continuously adjustable parameter enabling the transition from $\min(x_1,...,x_n)$ to $\max(x_1,...,x_n)$ are candidates for implementing the GCD aggregator. Such means are presented in [2,3,15,20]. A detailed presentation of mathematical aspects of the whole area of aggregation operators can be found in [18].

From its introduction in 1973 the continuous transfer from conjunction to disjunction was used to build decision models in many areas (e.g. system evaluation [13], extended Boolean queries and retrieval [21], fuzzy decision models [22,23], etc.) In all cases it is necessary to have appropriate definitions of andness/orness, and research in this area is still active [12,17].

If we have alternative definitions of andness/orness it is natural to ask several questions:

- Do we actually need a variety of definitions of andness/orness?
- How to compare alternative definitions?
- Is there the "best andness/orness"?
- Which definition should we use in a given application and why?

This paper investigates techniques for classification and comparison of andness/orness indicators and presents results that contribute to answering the above questions.

2 A Classification of Andness/Orness

The first step in the comparison of andness/orness indicators is to identify and classify their features. We propose the following list of letter-coded features:

L = local indicator that has a specific value in each point of the input space $[0,1]^n$.

G = global indicator that is an overall aggregated value that characterizes GCD in all points of the input space $[0,1]^n$.

M = mean value indicator, obtained as the mean value of a local andness/orness indicator.

D = direct indicator, obtained by processing directly the GCD function in all points of $[0,1]^n$.

I = indirect indicator, obtained not from the GCD function, but indirectly from some of its related features (e.g. from the properties of the generator function of quasi-arithmetic means).

S = statistical indicator, e.g. various forms of distribution of local andness/orness inside $[0,1]^n$.

N = an indicator that is a function of the number of variables *n*.

C = a constant indicator, independent of n (this property is opposite of **N**).

Using these features we can characterize andness/orness indicators using their type code defined as **X/Y/Z** where $\mathbf{X} \in \{\mathbf{L,G,M}\}$, $\mathbf{Y} \in \{\mathbf{D,I,S}\}$, $\mathbf{Z} \in \{\mathbf{N,C}\}$. For example, **G/D/N** denotes a global indicator that is directly derived form the GCD aggregator and is a function of the number of variables *n*.

3 A Brief History of Andness/Orness

The oldest andness and orness indicators (initially called the conjunction degree and the disjunction degree), were type **L/D/N**, introduced in 1973 as the following pointwise *local andness/orness* [4]:

$$\alpha_x(x_1,\dots,x_n) = \frac{(x_1 \vee \cdots \vee x_n) - (x_1 \Diamond \cdots \Diamond x_n)}{(x_1 \vee \cdots \vee x_n) - (x_1 \wedge \cdots \wedge x_n)}$$

$$\omega_x(x_1,\dots,x_n) = \frac{(x_1 \Diamond \cdots \Diamond x_n) - (x_1 \wedge \cdots \wedge x_n)}{(x_1 \vee \cdots \vee x_n) - (x_1 \wedge \cdots \wedge x_n)}$$

$$0 \le \alpha_x(x_1,\dots,x_n) \le 1, \quad 0 \le \omega_x(x_1,\dots,x_n) \le 1,$$

$$x_1 \vee \cdots \vee x_n \ne x_1 \wedge \cdots \wedge x_n, \text{ i.e. } x_i \ne x_j,$$

$$i \in \{1,\dots,n\}, \quad j \in \{1,\dots,n\}$$

Taking into account that decision makers design decision models by selecting a desired level of andness/orness, it was clear that the pointwise local andness has limited practical interest. Consequently, an indicator of type **M/D/N** ("*mean local andness/orness*") was also proposed in [4]:

$$\overline{\alpha}_x = \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 \alpha_x(x_1,\dots,x_n)dx_n, \quad 0 \le \overline{\alpha}_x \le 1$$

$$\overline{\omega}_x = \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 \omega_x(x_1,\dots,x_n)dx_n, \quad 0 \le \overline{\omega}_x \le 1$$

$$\overline{\alpha}_x + \overline{\omega}_x = 1$$

Of course, practical decision makers have limited ability to select appropriate levels of andness/orness [9]. According to Miller's "magical number 7 plus or minus 2" [19], a system of 7 levels of andness/orness for weighted power means, shown in Table 1, was also proposed in [4].

Table 1. A system of 7 levels of andness/orness

$\overline{\alpha}_x$	$\overline{\omega}_x$	Name of aggregator	r
0	1	Full disjunction	$+\infty$
1/6	5/6	Strong partial disjunction	11.56
1/3	2/3	Weak partial disjunction	3.35
1/2	1/2	Neutrality (arithmetic mean)	1
2/3	1/3	Weak partial conjunction	-0.60
5/6	1/6	Strong partial conjunction	-6.09
1	0	Full conjunction	$-\infty$

Analytic computation of $\overline{\alpha}_x$ and $\overline{\omega}_x$ for various values of *n* is not a trivial task, and accurate numerical computation of parameters of GCD function for desired values of andness/orness was also a difficult problem for weak processors of early 1970's. These problems caused the development of a **G/D/N** indicator called *global andness/orness* [12]. This indicator was based on the mean values of conjunction and disjunction derived in [5]:

$$\overline{x_1 \wedge ... \wedge x_n} = \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 (x_1 \wedge ... \wedge x_n) dx_n = \frac{1}{n+1}$$

$$\overline{x_1 \vee ... \vee x_n} = \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 (x_1 \vee ... \vee x_n) dx_n = \frac{n}{n+1}$$

Using these results, the global andness/orness was introduced in 1974 as follows [6]:

$$\alpha_g = \frac{\overline{x_1 \vee ... \vee x_n} - \overline{x_1 \Diamond ... \Diamond x_2}}{\overline{x_1 \vee ... \vee x_n} - \overline{x_1 \wedge ... \wedge x_n}} = \frac{n-(n+1)(\overline{x_1 \Diamond ... \Diamond x_n})}{n-1}$$

$$\omega_g = \frac{\overline{x_1 \Diamond ... \Diamond x_n} - \overline{x_1 \wedge ... \wedge x_n}}{\overline{x_1 \vee ... \vee x_n} - \overline{x_1 \wedge ... \wedge x_n}} = \frac{(n+1)(\overline{x_1 \Diamond ... \Diamond x_n})-1}{n-1}$$

$$\overline{x_1 \Diamond ... \Diamond x_n} = \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 (x_1 \Diamond ... \Diamond x_n) dx_n$$

$$\alpha_g \in I, \quad \omega_g \in I, \quad \alpha_g + \omega_g = 1$$

This indicator was used in [7] for building decision criteria with 17 discrete levels of andness/orness, and it is still in use [9,11,13].

Ordered weighted averaging (OWA) aggregator was introduced in 1988 [22]. The aggregator type is **G/D/N**. It is based on positional weights $v_1,...,v_n$ (associated with a rank, rather than an input), and the andness and orness are defined as follows:

$$x_1 \Diamond ... \Diamond x_n = \sum_{i=1}^{n} v_i x_i, \; x_1 \geq ... \geq x_n, \quad (v_1,...,v_n) \in [0,1]^n,$$

$$\sum_{i=1}^{n} v_i = 1; \qquad \alpha_{OWA} = \frac{1}{n-1} \sum_{i=2}^{n} (i-1) v_i \, ,$$

$$\omega_{OWA} = \frac{1}{n-1} \sum_{i=1}^{n-1} (n-i) v_i = 1 - \alpha_{OWA}$$

OWA andness/orness is consistent with the global andness/orness. If n=2 then $v_1 = \omega_{OWA}$, $v_2 = \alpha_{OWA}$, and $x_1 \Diamond x_2 = \omega_{OWA} x_1 + \alpha_{OWA} x_2$. If n>2 the desired level of andness can be achieved for multiple combinations of weights.

A similar approach, based on constant positional weights, was proposed in 1991 [8]. To explain the idea, suppose that we have three sorted input values: $1 \geq x_1 \geq x_2 \geq x_3 \geq 0$. Using x_1, x_2, x_3 we compute three new values y_1, y_2, y_3 as follows: $y_1 = \omega x_1 + \alpha x_2$, $y_2 = \omega x_1 + \alpha x_3$, $y_3 = \omega x_2 + \alpha x_3$. In this way we have $y_1 \geq y_2 \geq y_3$ and reduce the interval of values: $y_1 - y_3 < x_1 - x_3$. The new values replace the old values ($x_1 = y_1$; $x_2 = y_2$; $x_3 = y_3$) and by iterating this process it quickly converges to a mean value y that can also be computed as follows:

$$y = (\omega^2 x_1 + \alpha\omega x_2 + \alpha^2 x_3)/(\omega^2 + \alpha\omega + \alpha^2)$$

We call such aggregators the iterative OWA (ItOWA), [11]. Since α (or ω) is selected as a constant value the ItOWA aggregator type is **G/I/C**. The positional weights can be computed analytically avoiding the iterative numerical process [11]. This concept can be expanded to cases with any number of variables. The ItOWA andness/orness is not the same as the OWA andness/orness, except in special cases $\alpha \in \{0, 1/2, 1\}$. In the OWA case the andness and orness are computed from positional weights, and in the ItOWA case the positional weights are computed from desired andness/orness.

Recently, the oldest concept of local andness/orness was reactivated by Fernández Salido and Murakami [14]. Their *orness distribution function* is equivalent to the local orness and their *orness average value* is equivalent to the mean local orness. They proved that ω_{OWA} equals the mean local andness:

$$\int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 \frac{\sum_{i=1}^{n} v_i s_i - (x_1 \wedge ... \wedge x_n)}{(x_1 \vee ... \vee x_n) - (x_1 \wedge ... \wedge x_n)} dx_n = \omega_{OWA}$$

Here s_i denotes the i^{th} largest of the x_i. OWA aggregator has the feature $\omega_{OWA} = \overline{\omega}_x = \omega_g$.

The local andness $\alpha_x(x_1,...,x_n) \in I$ can be characterized using its probability density function defined as $p_\alpha(z) = dP_\alpha(z)/dz$, $P_\alpha(z) = \Pr(\alpha_x < z)$, and exemplified in Fig. 1. From this distribution we can compute all its parameters including $\overline{\alpha}_x$ and $\overline{\omega}_x$.

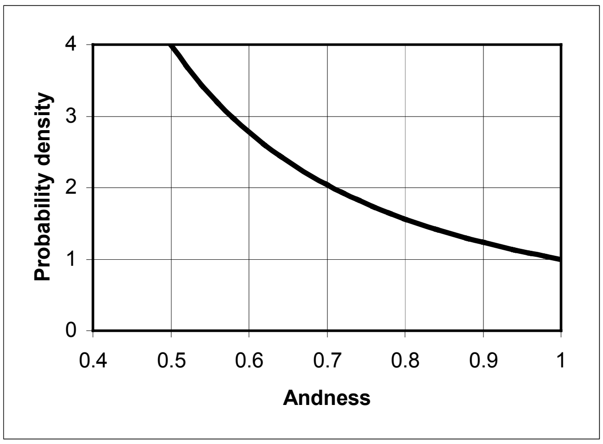

Figure 1. The probability density function for the local andness of the harmonic mean of two variables

In decision making practice it is suitable to use constant andness/orness because of its simplicity. Such indicators can be defined for quasi-arithmetic means (QAM) using their generator function $F: I \to R$:

$$y = F^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} F(x_i)\right)$$

For finite, strictly monotonic convex and concave generator functions $F(x),\ x \in I$ the *generator function andness/orness* is proposed in [12] as follows:

$$\alpha_F = \frac{\int_0^1 F(x)dx - F(0)}{F(1) - F(0)}, \quad \omega_F = \frac{F(1) - \int_0^1 F(x)dx}{F(1) - F(0)},$$
$$|F(x)| < +\infty, \quad F'(x) \neq 0$$

Restrictions in applying these formulas are also presented in [12]. If $F'(x)F''(x) < 0,\ x \in I$, QAM is a model of partial conjunction ($\alpha_F > \omega_F$), and if $F'(x)F''(x) > 0,\ x \in I$, then QAM is a model of partial disjunction ($\alpha_F < \omega_F$).

The most recent definition of the orness of QAM on interval $[a,b]$ is proposed by Liu [17]:

$$\omega_L = (b-a)^{-1}\left[F^{-1}\left((b-a)^{-1}\int_a^b F(x)dx\right) - a\right]$$

In this Section we have not included all existing concepts of andness/orness (see e.g. [1, 12]), but we showed a chronology of main events and the fact that this area of research is active for more than thirty years.

4 Global vs. Mean Local Andness/Orness

Local andness and orness are indicators defined at each point of $[0,1]^n$. In a general case, the computation of $\bar{\alpha}_x$ and $\bar{\omega}_x$ can be rather complex. However, in some special cases these indicators can be easily derived, and let us first show one such method.

Let us consider the case where the GCD is implemented as the harmonic mean of two variables: $y = x_1 \Diamond x_2 = 2x_1x_2/(x_1+x_2)$. Then at points where $x_1 < x_2$ we have $\alpha(x_1,x_2) = x_2/(x_1+x_2)$; along the line $x_2 = ax_1,\ a > 1$, the local andness is constant, $\alpha(x_1,x_2) = a/(a+1)$, as illustrated in Fig. 2. Similarly, if $x_1 > x_2$ then $\alpha(x_1,x_2) = x_1/(x_1+x_2)$, and along the line $x_2 = x_1/a,\ a > 1$, the local andness is $\alpha(x_1,x_2) = a/(a+1)$. Therefore, the minimum andness $\alpha = 1/2$ is along the line $x_1 = x_2$, and along coordinate axes we have the maximum andness $\alpha = 1$. The region $a/(a+1) \le \alpha(x_1,x_2) \le 1$ consists of two right-angled triangles next to axes with total area $1/a$. Hence, the probability that the andness in a random point is less than $a/(a+1)$ is $\Pr(\alpha < a/(a+1)) = 1 - 1/a$. After the substitution $z = a/(a+1)$ we have the probability distribution function $P_\alpha(z)$ and its density $p_\alpha(z)$ as follows (Fig. 1):

$$P_\alpha(z) = \Pr(\alpha < z) = \frac{2z-1}{z},\ p_\alpha(z) = \frac{dP_\alpha(z)}{dz} = \frac{1}{z^2}, 0.5 \le z \le 1$$

The corresponding mean local andness of the harmonic mean is:

$$\bar{\alpha}_x[H] = \int_{0.5}^{1} z p_\alpha(z)dz = \int_{0.5}^{1} \frac{dz}{z} = \ln(2) = 0.693$$

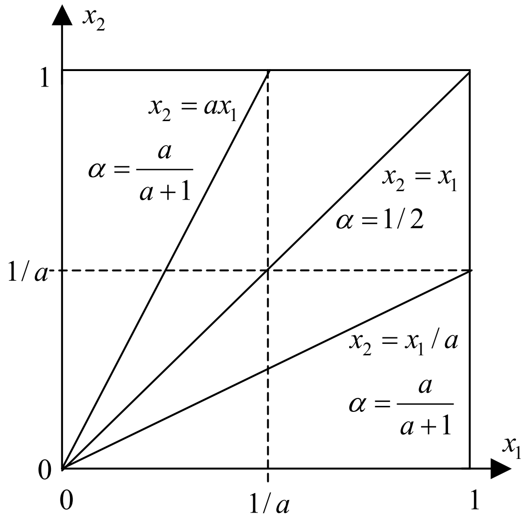

Figure 2. Local andness of the harmonic mean

The presented approach can also be applied in the case of geometric mean:

$$x_1 \Diamond x_2 = \sqrt{x_1x_2}, \quad \alpha(x_1,x_2) = \begin{cases} \sqrt{x_2}\big/\left(\sqrt{x_1}+\sqrt{x_2}\right) & x_2 \ge x_1 \\ \sqrt{x_1}\big/\left(\sqrt{x_1}+\sqrt{x_2}\right) & x_2 \le x_1 \end{cases}$$

$$\alpha(x_1,x_2) = \frac{\sqrt{a}}{\sqrt{a}+1}, \quad x_2 = \begin{cases} ax_1 & a \ge 1, \quad x_2 \ge x_1 \\ x_1/a & a \ge 1, \quad x_2 \le x_1 \end{cases}$$

$$\Pr\left(\alpha < \frac{\sqrt{a}}{\sqrt{a}+1}\right) = 1 - \frac{1}{a}, \quad z = \frac{\sqrt{a}}{\sqrt{a}+1}, \quad \frac{1}{a} = \left(\frac{1-z}{z}\right)^2$$

$$P_\alpha(z) = \Pr(\alpha < z) = \frac{2z-1}{z^2}, \quad p_\alpha(z) = \frac{dP_\alpha(z)}{dz} = \frac{2-2z}{z^3},$$
$$0.5 \le z \le 1; \qquad p_\alpha(0.5) = 8, \quad p_\alpha(1) = 0$$

$$\bar{\alpha}_x[G] = \int_{0.5}^{1} z p_\alpha(z)dz = \int_{0.5}^{1} \frac{2-2z}{z^2}dz = 2 - \ln(4) = 0.614$$

According to [12] the global andness of harmonic and geometric means can be computed analytically, and they are greater than the mean local andness: $\alpha_g[G] = 2/3 = 0.667, \quad \alpha_g[H] = \ln(16) - 2 = 0.773$.

In a general case of weighted power means we have

$$x_1 \Diamond x_2 = (0.5x_1^r + 0.5x_2^r)^{1/r}$$

$$\alpha(x_1, x_2) = \frac{a - (0.5 + 0.5a^r)^{1/r}}{a-1},$$

$$x_2 = \begin{cases} ax_1 & a \ge 1, \quad x_2 \ge x_1 \\ x_1 / a & a \ge 1, \quad x_2 \le x_1 \end{cases}$$

$$\Pr(\alpha < z) = 1 - \frac{1}{a}, \quad z = \frac{a - (0.5 + 0.5a^r)^{1/r}}{a-1}$$

The analytic computation of a from the equation $a - (0.5 + 0.5a^r)^{1/r} - z(a-1) = 0$ is not possible for any value of r, and this reduces the applicability of the presented technique. Of course, numerical solutions are always possible and relatively simple.

In the case of power means there is a substantial difference (frequently more than 10%) between the global andness and the mean local andness, as shown in Figures 3 and 4. For power means the mean local and global andness/orness indicators in the range $0.1 \le \bar{\omega}_x \le 0.9$ and $0.1 \le \omega_g \le 0.9$ can be rather accurately computed from each other as follows:

$$\bar{\omega}_x = 1.1297\omega_g^3 - 1.7108\omega_g^2 + 1.51\omega_g + 0.0243$$

$$\omega_g = -1.8238\bar{\omega}_x^3 + 2.7527\bar{\omega}_x^2 + 0.0337\bar{\omega}_x + 0.0303$$

From these formulas, if $\omega_g = 1/3$, $\omega_g = 2/3$ (geometric mean) we get $\bar{\omega}_x = 0.395$, $\bar{\alpha}_x = 0.605$. If $\bar{\omega}_x = 0.307$, $\bar{\alpha}_x = 0.693$ (harmonic mean) we get $\omega_g = 0.231$, $\alpha_g = 0.769$.

The local andness distribution in Fig. 1 shows that the harmonic mean includes all values of andness from 0.5 (along the line $x_1 = x_2 = ... = x_n$) to 1 (along the lines $x_i = 0,\ i \in \{1,...,n\}$). Consequently, in a given point we usually find andness that differs from the desired mean value. To reduce unexpected local andness values it is useful to use the following *mean interval local andness*:

$$\bar{\alpha}_x(a,b) = \frac{1}{(b-a)^n} \int_a^b dx_1 \int_a^b dx_2 ... \int_a^b \alpha_x(x_1,...,x_n) dx_n,$$

$$\bar{\omega}_x(a,b) = \frac{1}{(b-a)^n} \int_a^b dx_1 \int_a^b dx_2 ... \int_a^b \omega_x(x_1,...,x_n) dx_n,$$

$$0 \le \bar{\alpha}_x(a,b) \le 1, 0 \le \bar{\omega}_x(a,b) \le 1, \bar{\alpha}_x(a,b) + \bar{\omega}_x(a,b) = 1$$

This approach assumes the uniform distribution of input preferences within the interval [*a,b*]. In professional system evaluation practice evaluated systems usually satisfy at least 50% or requirements and a reasonable interval for definition of the mean interval andness/orness would be [*a,b*]=[0.5, 1].

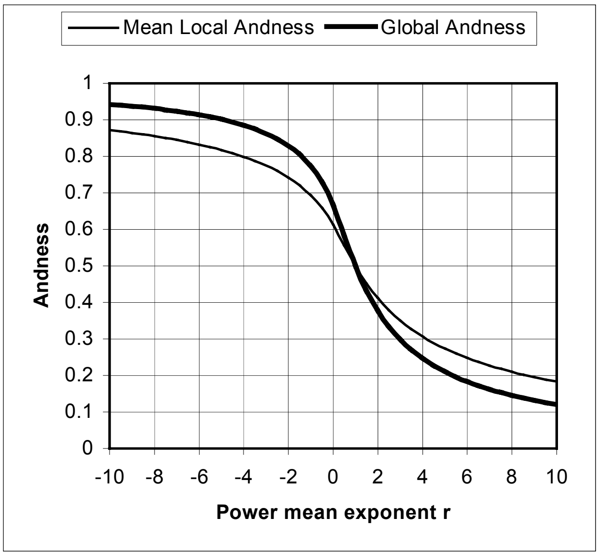

Figure 3. Andness of power means

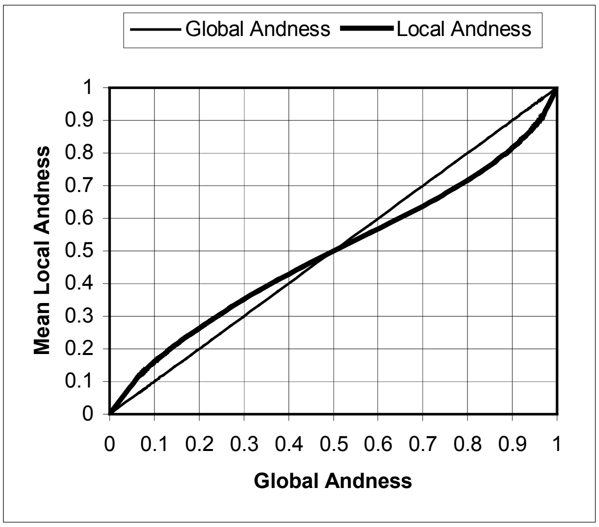

Figure 4. Mean local vs. global andness of power means

In system evaluation practice evaluators must select desired andness/orness at each step of building preference aggregation structures [13]. Therefore, the dependence on the number of variables is not a desirable property of andness/orness indicators. For example, in the case of geometric mean the global andness and orness are [6]:

$$\alpha_g(n) = \frac{n}{n-1}\left[1 - \left(\frac{n}{n+1}\right)^{n-1}\right], \lim_{n\to\infty} \alpha_g(n) = 1 - \frac{1}{e} = 0.632$$

$$\omega_g(n) = \frac{n}{n-1}\left[\left(\frac{n}{n+1}\right)^{n-1} - \frac{1}{n}\right], \quad \lim_{n\to\infty} \omega_g(n) = \frac{1}{e} = 0.368$$

The variations of andness/orness for various values of n can be illustrated using the relative andness and orness: $\alpha_g(n)/\alpha_g(+\infty)$, $\omega_g(n)/\omega_g(+\infty)$ (Fig. 5).

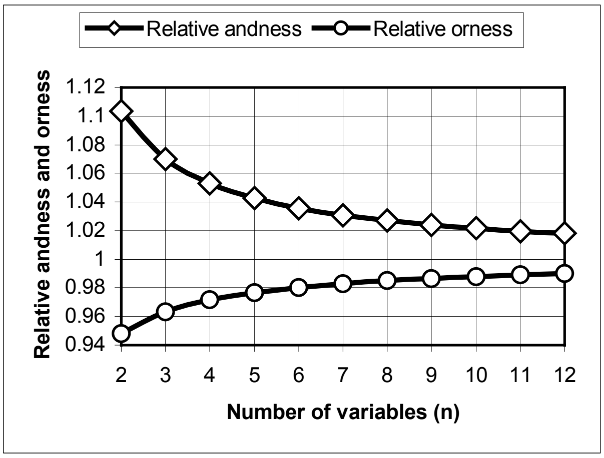

Figure 5. Relative global andness and orness of the geometric mean as functions of n

From the standpoint of applications it is desirable that the andness and orness are constant, and not functions of n. However, variations of andness and orness in the most frequently used range $2 \le n \le 5$ are less than 6% and all parameters of aggregators can be numerically computed for any desired level of andness/orness. Thus, the dependence of andness/orness on the number of variables causes no problems in practical system evaluation studies. Of course, using the *andness/orness limit* that corresponds to $n \to \infty$ is one of ways to eliminate the dependence on n, yielding indicators of type **G/D/C.**

5 Generator function orness for QAM

Similarly to differences between ω_g and $\overline{\omega}_x$ there are also differences between ω_F and ω_L that we can illustrate using power means. If $F(x) = x^r, r \ge 1$, then $F(0) = 0,\ F(1) = 1,\ F'(x) > 0, F''(x) > 0,\ x \in I$, $\alpha_F = \int_0^1 x^r dx = 1/(r+1)$ and $\omega_F = r/(r+1)$. Larsen used these formulas for $r \ge 1$ as an approximation of andness/orness of AIWA aggregators [16]. If $r < 1$, these formulas should not be used because the condition $-\infty < F(x) < +\infty,\ x \in I$ is not satisfied for r<0, and the boundary conditions [12] are not satisfied. In the case of Liu orness [17], if a=0, b=1, and $F(x) = x^r, r > 0$ then $\omega_L = 1/(r+1)^{1/r}$. Hence, $\omega_L = \alpha_F^{1/r} = (1-\omega_F)^{1/r}$ and $\omega_F / \omega_L = r(r+1)^{1/r-1}$; $\omega_F = \omega_L$ if $r = +\infty$ or r=1/2; otherwise, $\omega_F > \omega_L$.

6 Symmetric global andness/orness

The global orness $\omega_g \in [0,1]$ is defined as a relative position of $\overline{x_1 \Diamond \ldots \Diamond x_n}$ inside the interval $[\overline{x_1 \wedge \ldots \wedge x_n}, \overline{x_1 \vee \ldots \vee x_n}] = [1/(n+1), n/(n+1)]$.

Let us now define a *symmetric global orness* Ω as a position of $\overline{x_1 \Diamond \ldots \Diamond x_n}$ with respect to the arithmetic mean $\overline{(x_1 + \ldots + x_n)/n} = 1/2$, as follows:

$$\Omega = 2\frac{\overline{x_1 \Diamond \ldots \Diamond x_n} - \overline{(x_1 + \ldots + x_n)/n}}{\overline{x_1 \vee \ldots \vee x_n} - \overline{x_1 \wedge \ldots \wedge x_n}} = \frac{(n+1)[2(\overline{x_1 \Diamond \ldots \Diamond x_n}) - 1]}{n-1}$$

The special cases are:

$$\Omega = \begin{cases} 1 \ , & x_1 \Diamond \ldots \Diamond x_n = x_1 \vee \ldots \vee x_n \\ 0 \ , & x_1 \Diamond \ldots \Diamond x_n = (x_1 + \ldots + x_n)/n \\ -1, & x_1 \Diamond \ldots \Diamond x_n = x_1 \wedge \ldots \wedge x_n \end{cases}$$

Now $\Omega \in [-1,1]$; the interval $-1 < \Omega < 0$ denotes the partial conjunction, and the interval $0 < \Omega < 1$ denotes the partial disjunction. For example,

$$x_1 \Diamond \ldots \Diamond x_n = (x_1 \cdots x_n)^{1/n},\ \overline{x_1 \Diamond \ldots \Diamond x_n} = [n/(n+1)]^n$$

$$\Omega_{geo}(n) = \frac{n+1}{n-1}\left[2\left(\frac{n}{n+1}\right)^n - 1\right], \quad \Omega_{geo}(2) = -1/3.$$

The global symmetric andness A can be obtained from the corresponding global symmetric orness Ω as $\mathrm{A} = -\Omega$ (e.g. $\mathrm{A}_{geo}(2) = -\Omega_{geo}(2) = 1/3$).

The global andness and orness satisfy $\alpha_g + \omega_g = 1$ and the symmetric global andness and orness satisfy $\mathrm{A} + \Omega = 0$. From the definitions of ω_g and Ω we have $\dfrac{(n-1)\omega_g + 1}{n+1} = \overline{x_1 \Diamond \ldots \Diamond x_n} = \dfrac{(n-1)\Omega}{2(n+1)} + \dfrac{1}{2}$

Consequently, Ω and ω_g are related as follows:

$$\Omega = 2\omega_g - 1, \qquad \omega_g = (\Omega + 1)/2.$$

The properties of the symmetric global orness can be fine tuned using an exponent p>0 as follows:

$$\Omega^{(p)} = \mathrm{sgn}\left(\overline{x_1 \Diamond \ldots \Diamond x_n} - 1/2\right) \cdot \left|\frac{(n+1)[2(\overline{x_1 \Diamond \ldots \Diamond x_n}) - 1]}{n-1}\right|^p$$

The original definition is now a special case obtained for the parameter $p = 1$: $\Omega = \Omega^{(1)}$. Since $\forall \delta > 0,\quad \Omega^{(1+\delta)} \le \Omega^{(1)} \le \Omega^{(1-\delta)}$, and

$$\Omega^{(p)} = \begin{cases} 1 \ , & x_1 \Diamond \ldots \Diamond x_n = x_1 \vee \ldots \vee x_n \\ 0 \ , & x_1 \Diamond \ldots \Diamond x_n = (x_1 + \ldots + x_n)/n, \quad p > 0 \\ -1, & x_1 \Diamond \ldots \Diamond x_n = x_1 \wedge \ldots \wedge x_n \end{cases}$$

it follows that the power p can be used for tuning the properties of the symmetric global orness for partial conjunction and partial disjunction. By decreasing the value of p we can increase the sensitivity of $\Omega^{(p)}$ to $\overline{x_1 \Diamond ... \Diamond x_n}$.

In the case of power means the power can be computed from the desired value of global or symmetric global orness using this approximation:

$$r = (-0.742 + 3.363\omega_g - 4.729\omega_g^2 + 3.937\omega_g^3)/[(1-\omega_g)\omega_g]$$

$$\omega_g = (\Omega+1)/2, \quad 0 \le \Omega \le 1$$

For power means we have $\omega_F = r/(1+r)$, and $\Omega_F = 2\omega_g - 1 = (r-1)/(r+1)$. Liu's version of generating function orness for power means yields

$$\omega_L = 1/(1+r)^{1/r}, \quad \Omega_L = 2/(1+r)^{1/r} - 1, \quad r \ge 1$$

For any $\Omega \in [0,1]$ we can compute the corresponding value of r and insert it in formulas for Ω_F and Ω_L. This yields $\Omega_F(\Omega)$ and $\Omega_L(\Omega)$ that differ, as shown in Fig. 4. If we want to reduce their differences we can perform exponential tuning as shown in Figures 7, 8: $[\Omega_F(\Omega)]^{1.3761} \approx \Omega \approx [\Omega_L(\Omega)]^{0.68}$.

7 Conclusions

There are three main categories of general andness/orness indicators: *global*, *mean local*, and (for QAM) the *generator function* indicators. For aggregation operators based on power means these three indicators are different, and the differences are not always negligible. Fortunately, they can be easily analytically expressed and tuned according to needs. For some aggregators (like OWA) the global indicators and mean local indicators are equal. Generally, desirable properties of andness/orness indicators include the equality of global and mean local indicators, and independence of the number of variables.

We analyzed the differences between individual andness/orness indicators, emphasizing the aggregators based on power means. The reasons for interest in power means are purely practical: quasi-independence of formal logic and semantic components, and the ability to create a next level of complex aggregators, such as the partial absorption function. These features are indispensable for modeling observable properties of human reasoning and for building professional system evaluation criteria.

We have identified eight characteristic features of andness/orness indicators and proposed 3-letter codes for their classification and comparison.

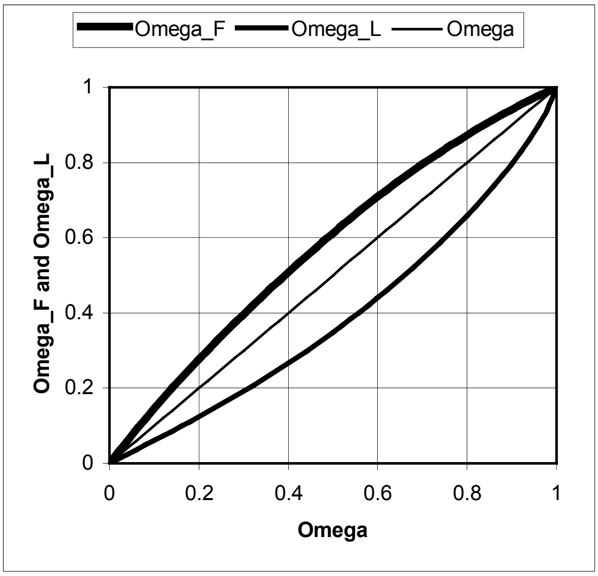

Figure 6. Ω_F and Ω_L before tuning

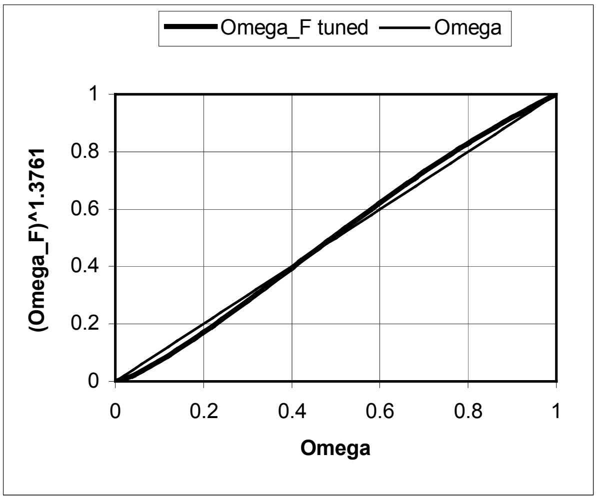

Figure 7. Ω_F tuned to be similar to Ω

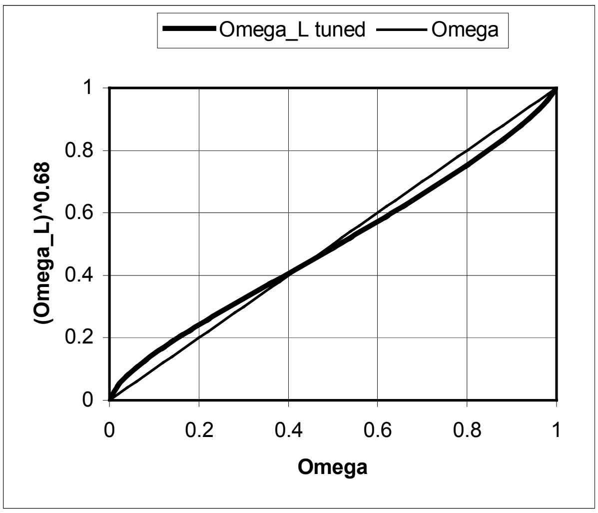

Figure 8. Ω_L tuned to be similar to Ω

Human precision in the process of selecting andness/orness and the appropriate levels of relative importance (weights) is rather limited. Fortunately, decision makers can be trained to use any kind of andness indicator to successfully build complex criteria. The preferred types of indicators are **G/D/C** or **M/D/C,** but other indicators are also useful. Thus, the future research should be focused on finding those aggregators and andness indicators that best fit the observable properties of human reasoning.

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