On \( n \)-contractive fuzzy logics: first results

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Abstract

In order to reach a deeper understanding of the structure of fuzzy logics, some very general new logics are defined. Namely, we consider the extensions of MTL by adding the generalized contraction and excluded middle laws introduced in [4], and we enrich this family by means of the axiom of weak cancellation and the \( \Omega \) operator defined in [18]. The algebraic counterpart of these logics is studied characterizing the subdirectly irreducible, the semisimple and the simple algebras. Finally, some important algebraic and logical properties of the considered logics are discussed: local finiteness, finite embedding property, finite model property, decidability and standard completeness.

Keywords: Algebraic Logic, Fuzzy logics, Generalized contraction, Generalized excluded middle, Left-continuous t-norms, MTL-algebras, Non-classical logics, Residuated lattices, Standard completeness, Substructural logics, Varieties, Weak cancellation.

1 Introduction

The research on formal systems for fuzzy logic has been growing rapidly during the last years. The origin of this development can be traced back to Hájek’s works (see [12]) when he defined the basic fuzzy logic BL in order to capture the common fragment of the three main fuzzy logics known at that time: Lukasiewicz logic, Product logic and Gödel logic. These three logics were proved to be standard complete, i.e. complete with respect to the semantics where the set of truth values is the real unit interval \([0, 1]\), the conjunction is interpreted by a continuous t-norm (Lukasiewicz t-norm, the product t-norm and the minimum t-norm, respectively) and the implication is interpreted by the residuum of the t-norm. In [5] it was proved that BL is, in fact, complete with respect to the semantics given by all continuous t-norms and their residua. Nevertheless, the necessary and sufficient condition for a t-norm to be residuated is not the continuity, but only the left-continuity. For this reason it makes perfect sense to consider a more general fuzzy logic system whose semantic completeness would be the class of all left-continuous t-norm and their residua. This logic, MTL, was introduced by Esteva and Godo in [7] and its standard completeness was proved in [14]. Therefore, if we understand fuzzy logic systems as those that are complete with respect to some class of t-norms and their residua, then MTL becomes the weakest fuzzy logic and the research on fuzzy logic systems becomes research on extensions of MTL. Moreover, since it is an algebraizable logic whose equivalent algebraic semantics is the variety of all MTL-algebras, then there is a one-to-one correspondence between axiomatic extensions of MTL and subvarieties of MTL-algebras.
Some of them are already known (see for instance [15, 9, 6, 10, 11, 20, 21, 22, 18, 13]) but a general description of the structure of all these extensions is still far from being known. In this paper we make some new steps in this direction by considering some very general varieties of MTL-algebras, namely the varieties of $n$-contractive MTL-algebras. Some of them were already introduced in [4].

After some necessary general preliminaries in Section 2 about axiomatic extensions of MTL and their algebraization, we consider in Section 3 some equations introduced by Kowalski and Ono in [16] to define $n$-contractive fuzzy logics. Section 4.1 deals with the algebraic counterpart of these logics, the $n$-contractive MTL-algebras; subdirectly irreducible, semisimple and simple algebras are characterized. In Section 4.2 we add some other logics to the hierarchy of $n$-contractive fuzzy logics by means of the weak cancellation law and the $\Omega$ operator and we show that all of them are finitely axiomatizable. Finally, Section 4.3 is a discussion of some relevant logical and algebraic properties of the considered logics, namely local finiteness, finite embedding property, finite model property, de-cidability and standard completeness. Most of the problems are solved, but some are still open.\(^1\)

## 2 Axiomatic extensions of MTL and their algebraization

In [7] Esteva and Godo define MTL (Monoidal T-norm based Logic) as the sentential logic in the language $\mathcal{L} = \{\&, -, \land, \lor, 0\}$ of type $(2, 2, 2, 0)$ given by a Hilbert-style calculus with the inference rule of Modus Ponens and the following axioms (using implication as the least binding connective):

\begin{align*}
(A1) & \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \\
(A2) & \quad \varphi \land \psi \rightarrow \varphi \\
(A3) & \quad \varphi \land \psi \rightarrow \psi \land \varphi \\
(A4) & \quad \varphi \land \psi \\
(A5) & \quad \varphi \land \psi \rightarrow \varphi \land \psi \\
(A6) & \quad \varphi \land \psi \\
(A7a) & \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \\
(A7b) & \quad (\varphi \land \psi \\
(A8) & \quad ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \chi) \rightarrow (\psi \rightarrow \chi)) \\
(A9) & \quad \overline{0} \rightarrow \varphi
\end{align*}

Other usual connectives are defined by:

\begin{align*}
\varphi \lor \psi & \quad : \quad ((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)) \\
\varphi \leftrightarrow \psi & \quad : \quad (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi) \\
\neg \varphi & \quad : \quad \varphi \rightarrow 0 \\
\overline{1} & \quad : \quad \neg 0
\end{align*}

We denote by $Fm_{\mathcal{L}}$ the set of $\mathcal{L}$-formulas (built using a countable set of variables). If $\Gamma \subseteq Fm_{\mathcal{L}}$, we write $\Gamma \vdash_{MTL} \varphi$ if, and only if, $\varphi$ is derivable from $\Gamma$ in the given calculus. We write $\vdash_{MTL} \varphi$ instead of $\emptyset \vdash_{MTL} \varphi$.

**Definition 1** (cf. [7]). Let $\mathcal{A} = \langle A, \ast, -, \land, \lor, 0, 1 \rangle$ be an algebra of type $\langle 2, 2, 2, 2, 0 \rangle$. $\mathcal{A}$ is an MTL-algebra iff it is a bounded integral commutative residuated lattice satisfying the prelinearity equation: $(x \rightarrow y) \lor (y \rightarrow x) \approx 1$. The negation operation is defined as $\neg a = a \rightarrow 0$.

If the lattice order is total we will say that $\mathcal{A}$ is an MTL-chain. The MTL-chains defined over the real unit interval $[0, 1]$ (with the usual order) are those where $\ast$ is a left-continuous t-norm and they are called standard MTL-chains. $[0, 1]_\ast$ will denote the standard chain given by a left-continuous t-norm $\ast$. The class of all MTL-algebras is a variety and it is denoted by MTL. The $\overline{0}$-free subreducts of MTL-algebras are called prelinear semihoops and they are defined in [8].

**Definition 2.** Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and a class $\mathcal{K}$ of MTL-algebras, we define:

$\Gamma \vdash_{\mathcal{K}} \varphi$ iff for all $\mathcal{A} \in \mathcal{K}$ and for all evaluation $v$ in $\mathcal{A}$, we have $v(\varphi) = 1^A$ whenever $v(\psi) = 1^A$ for every $\psi \in \Gamma$.

**Theorem 3.** If $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, then $\Gamma \vdash_{MTL} \varphi$ iff $\Gamma \vdash_{MTL} \varphi$.

\(^1\)In this short note, due to the obvious lack of space, our results are presented without proof, but the full version of this work will be available in a forthcoming paper.
Moreover, it is not difficult to prove that MTL is an algebraizable logic in the sense of Blok and Pigozzi (see [2]) and MTL is its equivalent algebraic semantics. This implies much more than the algebraic completeness. In particular, there is an order-reversing isomorphism between axiomatic extensions of BL and sub-varieties of MTL:

- If $\Sigma \subseteq Fm_L$ and $L$ is the extension of MTL obtained by adding the formulae of $\Sigma$ as schemata, then the equivalent algebraic semantics of $L$ is the subvariety of MTL axiomatized by the equations $\{ \varphi \approx 1 : \varphi \in \Sigma \}$. We denote this variety by $L$ and we call its members $L$-algebras.

- Let $L \subseteq MTL$ be the subvariety axiomatized by a set of equations $\Lambda$. Then the logic associated to $L$ is the axiomatic extension $L$ of MTL given by the axiom schemata $\{ \varphi \rightarrow \psi : \varphi \approx \psi \in \Lambda \}$.

In the study of these subvarieties the chains play a crucial role due to the next result:

**Theorem 4 ([7]).** Each MTL-algebra is isomorphic to a subdirect product of MTL-chains.

This implies that MTL is complete with respect the semantics given by MTL-chains. The same kind of result is true for every axiomatic extension of MTL. In some cases, it is also possible to restrict the semantics to the algebras defined in the real unit interval by a left-continuous t-norm and its residuum, obtaining the so-called standard completeness results. If a logic $L$ is an axiomatic extension of MTL, we say that $L$ enjoys (finite) strong standard completeness if, and only if, for every (finite) set of formulas $T \subseteq Fm_L$ and every formula $\varphi$, $T \vdash_L \varphi$ iff $T \models_A \varphi$ for every standard L-algebra $A$. We will call this property (F)SSC, for short. We say that L enjoys the standard completeness (SC, for short) if, and only if, the equivalence is true for $T = \emptyset$.

Tables 1 and 2 collect several axiom schemata and important axiomatic extensions of MTL that are defined by adding them to the Hilbert-style calculus given above for MTL, and Table 3 collects the standard completeness properties that they satisfy.

### Table 1: Some usual axiom schemata in fuzzy logics.

<table>
<thead>
<tr>
<th>Axiom schema</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg \neg \varphi \rightarrow \varphi$</td>
<td>(Inv)</td>
</tr>
<tr>
<td>$\neg \varphi \lor ( (\varphi \rightarrow \varphi &amp; \psi) \rightarrow \psi)$</td>
<td>(C)</td>
</tr>
<tr>
<td>$\varphi \rightarrow \varphi &amp; \varphi$</td>
<td>($C_2$)</td>
</tr>
<tr>
<td>$\varphi \lor \neg \varphi$</td>
<td>($S_2$)</td>
</tr>
<tr>
<td>$\varphi \land \psi \rightarrow \varphi &amp; (\varphi \rightarrow \psi)$</td>
<td>(Div)</td>
</tr>
<tr>
<td>$(\varphi &amp; \psi \rightarrow 0) \lor (\varphi \land \psi \rightarrow \varphi &amp; \psi)$</td>
<td>(WNM)</td>
</tr>
</tbody>
</table>

### Table 2: Some important axiomatic extensions of MTL.

<table>
<thead>
<tr>
<th>Logic</th>
<th>Axiom schemata</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMTL</td>
<td>(Inv)</td>
</tr>
<tr>
<td>WNM</td>
<td>(WNM)</td>
</tr>
<tr>
<td>NM</td>
<td>(Inv), (WNM)</td>
</tr>
<tr>
<td>BL</td>
<td>(Div)</td>
</tr>
<tr>
<td>L</td>
<td>(Div), (Inv)</td>
</tr>
<tr>
<td>H</td>
<td>(Div), (C)</td>
</tr>
<tr>
<td>G</td>
<td>($C_2$)</td>
</tr>
<tr>
<td>CPC</td>
<td>($S_2$)</td>
</tr>
</tbody>
</table>

### Table 3: Standard completeness properties.

<table>
<thead>
<tr>
<th>Logic</th>
<th>SC</th>
<th>FSSC</th>
<th>SSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>IMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>WNM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>NM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>BL</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>L</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>H</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>G</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>CPC</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

However, we are not interested only in the standard completeness properties of the logics. We will also consider several other properties which are interesting both from the algebraic and from the logical point of view.

**Definition 5.** A class $\mathcal{K}$ of algebras is locally finite (LF, for short) if, and only if, for every $A \in \mathcal{K}$ and for every finite set $B \subseteq A$,
the subalgebra generated by \( B, \langle B \rangle_A \), is also finite.

**Definition 6.** A class \( \mathcal{K} \) of algebras has the finite embeddability property (FEP, for short) if, and only if, every finite partial subalgebra of some member of \( \mathcal{K} \) can be embedded in some finite algebra of \( \mathcal{K} \).

**Definition 7.** A class \( \mathcal{K} \) of algebras of the same type has the strong finite model property (SFMP, for short) if, and only if, every quasiequation that fails to hold in every algebra of \( \mathcal{K} \) can be refuted in some finite member of \( \mathcal{K} \).

**Definition 8.** A class \( \mathcal{K} \) of algebras of the same type has the finite model property (FMP, for short) if, and only if, every equation that fails to hold in every algebra of \( \mathcal{K} \) can be refuted in some finite member of \( \mathcal{K} \).

A variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members. In [3] it is proved that for classes of algebras of finite type closed under finite products (hence, in particular, for varieties of MTL-algebras) the FEP and the SFMP are equivalent. Moreover, it is clear that for every class of algebras \( \mathcal{L} \) which is the equivalent algebraic semantics of a logic \( \mathcal{L} \), we have:

- If \( \mathcal{L} \) is locally finite, then it has the FEP.
- If \( \mathcal{L} \) has the FEP, then it has the FMP.
- If \( \mathcal{L} \) has the FMP, then \( \mathcal{L} \) is decidable.

Table 4 shows which of these properties are true for the above mentioned axiomatic extensions of MTL.

<table>
<thead>
<tr>
<th>Logic</th>
<th>LF</th>
<th>FEP</th>
<th>FMP</th>
<th>Decidable</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTL</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>IMTL</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>WNM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>NM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>BL</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>L</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>II</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>G</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>CPC</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

This result is a generalization of the well-known theorem proved in [19] that states that all continuous t-norms (i.e. BL-chains over \([0,1]\)) are decomposable as ordinal sum of their Archimedean components, which are the three basic t-norms: Lukasiewicz, product and minimum t-norms. However, in Theorem 9 the decomposition is done by means of Wajsberg hoops, i.e. the \( \bar{0} \)-free subreducts of the MV-chains. This is not true in general in all MTL-chains; nevertheless in [18] some results in this direction are proved. Recall the definition of Archimedean component and totally decomposable chain:

**Definition 10.** Let \( A \) be an MTL-chain or a totally ordered semihoop. We define a binary relation \( \sim \) on \( A \) by letting for every \( a, b \in A \),

\[
\begin{align*}
& a \sim b \text{ if, and only if, there is } n \geq 1 \text{ such that } a^n \leq b \leq a \\
& \text{or } b \leq a \leq b^n.
\end{align*}
\]

It is easy to check that \( \sim \) is an equivalence relation. Its equivalence classes are called Archimedean classes. Given \( a \in A \), its Archimedean class is denoted as \( [a]_\sim \).

**Definition 11 ([18]).** A totally ordered semihoop is indecomposable if, and only if, it is not isomorphic to any ordinal sum of two non-trivial totally ordered semihoops.

**Theorem 12 ([18]).** For every MTL-chain \( A \), there is the maximum decomposition as ordinal sum of indecomposable totally ordered semihoops, with the first one bounded.

**Corollary 13 ([18]).** Let \( A \) be an MTL-chain. If the partition \( \{[a]_\sim : a \in A \setminus \{\top^A\} \} \) given by the Archimedean classes gives a decomposition as ordinal sum, then it is the
maximum one. In this case we say that $A$ is totally decomposable.

One can produce varieties where the chains are decomposable as ordinal sum of Archimedean components. Indeed, the $\Omega$ operator is introduced to produce new logics from the known ones by considering the ordinal sums of their corresponding semihoops. Namely, given any axiomatic extension $L$ of MTL, $\Omega(L)$ is defined as the variety of MTL-algebras generated by all the ordinal sums of $\Omega$-free subreducts of $L$-chains with the first bounded. $\Omega(L)$ denotes its corresponding logic.

3 The $n$-contraction

In [16] Kowalski and Ono studied some varieties of bounded integral commutative residuated lattices. In particular, they considered for every $n \geq 2$ the varieties defined by the following equations:

$$(E_n) \ x^n \approx x^{n-1}$$

$$(EM_n) \ x \lor \neg x^{n-1} \approx \top$$

$(E_2)$ corresponds, in fact, to the law of contraction, which defines the variety of Heyting algebras. Therefore, for every $n \geq 3$ the equation $(E_n)$ corresponds to a weak form of contraction that we will call $n$-contraction. Notice that $(EM_2)$ is the algebraic form of the law of the excluded middle, and for every $n \geq 3$ $(EM_n)$ corresponds to a weak form of this law.

In [4] Ciabattoni, Esteva and Godo brought the equations $(E_n)$ to the framework of fuzzy logics. Indeed, for each $n \geq 2$, they defined the $n$-contraction axiom as:

$$\varphi^{n-1} \rightarrow \varphi^n \ (C_n)$$

and they called $C_n$MTL (resp. $C_n$IMTL) the extension of MTL (resp. IMTL) obtained by adding this axiom.

Given $n \geq 2$, the equivalent algebraic semantics of $C_n$MTL (resp. $C_n$IMTL) is the class of $n$-contractive MTL-algebras (resp. IMTL-algebras), i.e. the subvariety of MTL (resp. IMTL) defined by the equation: $x^{n-1} \approx x^n$. Strong standard completeness for these logics was also proved in [4].

It is easy to see that $C_2$MTL is Gödel logic and $C_2$IMTL is the classical propositional calculus CPC. Moreover, for every $n \geq 3$, WNM is a strict extension of $C_n$MTL, NM is a strict extension of $C_n$IMTL, $C_n$MTL is a strict extension of $C_{n+1}$MTL and $C_n$IMTL is a strict extension of $C_{n+1}$IMTL.

We say that an axiomatic extension of MTL $L$ is $n$-contractive if $\vdash_l (C_n)$. Of course, given any $L$ we can make it $n$-contractive by adding the schema $(C_n)$. We call the resulting logic $C_nL$.

**Theorem 14.** If $L$ is an $n$-contractive axiomatic extension of MTL, then for every $\Gamma \cup \{ \varphi, \psi \} \subseteq Fm_L$ we have:

$$\Gamma, \varphi \vdash_L \psi \text{ if, and only if, } \Gamma \vdash_L \varphi^{n-1} \rightarrow \psi.$$ 

In [16] Kowalski and Ono prove also the following result:

**Proposition 15 (Prop 1.11, [16]).** Let $K$ be a variety of residuated lattices. Then, $K$ has the property EDPC (equationally definable principal congruences) if, and only if, $K \models (E_n)$, for some $n \geq 2$.

According to a bridge theorem of Abstract Algebraic Logic, an algebraizable logic has the global deduction-detachment theorem if, and only if, its equivalent algebraic semantics has the EDPC. Therefore, in our framework of fuzzy logics as axiomatic extensions of MTL, the contractive logics are a good choice in the sense that they are the only finitary extensions of MTL enjoying the global deduction-detachment theorem.

We will consider also the axioms corresponding to $(EM_n)$:

$$\varphi \lor \neg \varphi^{n-1} \ (S_n)$$

Given any axiomatic extension $L$ of MTL, $S_nL$ will be its extension with $(S_n)$.

For extensions of Łukasiewicz logic the axioms $(C_n)$ and $(S_n)$ are equivalent as the following proposition states, but this not generalizable to all extensions of MTL as we will see.

**Proposition 16.** Let $A$ be an MV-chain. The following are equivalent:
(i) \( A \models (E_n) \).
(ii) \( A \in \mathbb{I}_n \).
(iii) \( A \models (EM_n) \).

4 New results

4.1 \( n \)-contractive chains

In this section we will study some basic properties of the \( n \)-contractive chains. Observe that this is an important and big class of chains since it contains all the finite MTL-chains.

**Proposition 17.** All finite MTL-chains are \( n \)-contractive for some \( n \).

Given an MTL-algebra \( A \), \( a \in A \) is idempotent if \( a^2 = a \). \( \text{Id}(A) \) will be the set of all idempotent elements of \( A \). Obviously, \( 0^A, 1^A \in \text{Id}(A) \). The idempotent elements determine the Archimedean classes in standard BL-chains. We will obtain something similar for \( n \)-contractive MTL-chains. First we realize that they are easily described in \( n \)-contractive chains and, in addition, their number can be expressed equationally, as the following propositions show.

**Proposition 18.** Let \( A \) be an \( n \)-contractive MTL-algebra. Then, \( \text{Id}(A) = \{a^{n-1} : a \in A\} \).

**Definition 19.** For every \( n \geq 3 \) and \( k \geq 2 \), we define the next formula:
\[
I^n_k(x_0, \ldots, x_k) := \bigvee_{i<k} (a_i^{n-1} \rightarrow x_{i+1}^{n-1}).
\]

**Proposition 20.** For every \( n \geq 3 \), every \( k \geq 2 \) and every \( n \)-contractive MTL-chain \( A \) the following are equivalent:

1. \( A \models I^n_k(x_0, \ldots, x_k) \approx T \).
2. \( | \text{Id}(A) | \leq k \).

Now we obtain the following nice description of Archimedean classes in terms of the idempotent elements:

**Proposition 21.** Let \( A \) be an \( n \)-contractive MTL-chain. Then, for every \( a, b \in A \):

1. \( a \sim b \) if, and only if, \( a^{n-1} = b^{n-1} \), and
2. \( a^{n-1} = \text{min}[a] \).

**Corollary 22.** Let \( A \) be an \( n \)-contractive MTL-chain and let \( a \in A \). If \( [a] \) has supremum, then it is the maximum.

Therefore, Archimedean classes with supremum in \( n \)-contractive chains are always intervals of the form \([b^{n-1}, b]\). Moreover, this implies that given a standard \( n \)-contractive chain \( A \), in the set \( \text{Id}(A) \) none of the elements has neither predecessor nor successor. In particular, \( 1 \) is an accumulation point of idempotent elements.

Next proposition characterizes the subdirectly irreducible \( n \)-contractive algebras.

**Proposition 23.** An \( n \)-contractive MTL-chain is subdirectly irreducible if, and only if, its set of idempotent elements has a coatom.

**Corollary 24.** There are no subdirectly irreducible standard MTL-chains.

An important subclass of subdirectly irreducible algebras is the class of simple algebras, those without non-trivial congruences. The generalized excluded middle equations \((EM_n)\) describe exactly the simple \( n \)-contractive chains.

**Proposition 25.** Let \( A \) be an MTL-chain. The following are equivalent:

1. \( A \models (EM_n) \).
2. \( A \) is \( n \)-contractive and simple.

Recall that an algebra is semisimple if, and only if, it is representable as a subdirect product of simple algebras.

**Corollary 26.** For each \( n \geq 2 \), the class of semisimple \( n \)-contractive MTL-algebras is the variety \( S_n \).

Proposition 16 implies that for every \( n \geq 2 \), \( S_n \cap \text{MC} = S_n \cap \text{MC} \). However, in MTL and in IMTL the situation is not so easy. In the first level the varieties corresponding to \((E_n)\) and \((EM_n)\) are still easy to compute. Indeed, \( S_2 = S_2 \text{ IMTL} = \middlearrow 2\text{IMTL} \).
Given any axiomatic extension $L$ of $\mathbf{MTL}$ and $n \geq 2$, the extensions obtained by $(S_n)$ and $(C_n)$ coincide. In particular, $S_n \mathbf{WCMTL} = C_n \mathbf{WCMTL}$.

It is straightforward to prove that the $\Omega$ operator and the schemata $(C_n)$ commute:

**Proposition 30.** Let $L$ be an axiomatic extension of $\mathbf{MTL}$. For every $n \geq 2$, $\Omega(C_n L) = C_n \Omega(L)$.

Therefore, we have $C_n \Omega(WCMTL) = \Omega(C_n WCMTL) = \Omega(S_n WCMTL)$.

Finally, we will consider for every $n \geq 2$ the logic $\Omega(S_n MTL)$ and we will show that it is also finitely axiomatizable.

**Proposition 31.** Let $A$ be an $n$-contractive $\mathbf{MTL}$-chain. The following are equivalent:

1. $A$ is totally decomposable (in the sense of Corollary 13).
2. $A$ is an ordinal sum of simple $n$-contractive chains.
3. $A \models (y^{n-1} \to x) \lor (x \to x \land y) \approx \top$.

**Corollary 32.** $\Omega(S_n MTL)$ is the variety generated by the totally decomposable $n$-contractive chains, and it is axiomatized by $(y^{n-1} \to x) \lor (x \to x \land y) \approx \top$.

Therefore $\Omega(S_n MTL)$-chains are the analog of the standard BL-chains in the class of $n$-contractive $\mathbf{MTL}$-algebras in the sense that they are also decomposable as ordinal sums of their Archimedean classes.

### 4.2 Combining weakly cancellative and $n$-contractive fuzzy logics

In [18] the variety $\mathbf{WCMTL}$ of weakly cancellative $\mathbf{MTL}$-algebras was defined to provide examples of indecomposable $\mathbf{MTL}$-chains. Besides, the $\Omega$ operator gave rise to the variety $\Omega(\mathbf{WCMTL})$ which was a kind of analog of $\mathbf{BL}$ in the sense that here all the chains were also decomposable as ordinal sums of weakly cancellative semihoops. Now it seems natural to consider the intersection of these varieties with the classes of $n$-contractive algebras (or equivalently the supremum of the corresponding logics) in order to obtain some new kinds of algebras with a nice and simpler structure. Therefore, we will consider for every $n \geq 2$ the logics $S_n \mathbf{WCMTL}$ and $C_n \mathbf{WCMTL}$.

**Proposition 28.** For every $n \geq 2$, $\{WC\}, (C_n) \vdash_{\mathbf{MTL}} (S_n)$.

**Corollary 29.** Given any axiomatic extension $L$ of $\mathbf{WCMTL}$ and $n \geq 2$, the extensions obtained by $(S_n)$ and $(C_n)$ coincide. In particular, $S_n \mathbf{WCMTL} = C_n \mathbf{WCMTL}$.

In this section we will present the current results of our study of some logical and algebraic properties of the considered logics. Standard completeness results are collected in Table 5, while results on finite embeddability property, finite model property and decidability are in Table 6. Both tables also show which problems remain open.

As mentioned above, the SSC was proved by Ciabattoni, Esteva and Godo in [4] for $C_n MTL$ and $C_n IMTL$ for every $n \geq 2$. For $\Omega(S_n MTL)$ we have used the real embedding method introduced in [14]. The logics satisfying some $(S_n)$ schema do not enjoy any standard completeness property because there are
Table 5: Standard completeness properties of n-contractive fuzzy logics.

<table>
<thead>
<tr>
<th>Logic</th>
<th>SC</th>
<th>FSSC</th>
<th>SSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>CnMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>CnIMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>SnMTL</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>SnIMTL</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Ω(SnMTL)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>SnWCMTL</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Ω(CnWCMTL)</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 6: FEP, FMP and decidability in n-contractive fuzzy logics.

<table>
<thead>
<tr>
<th>Logic</th>
<th>FEP</th>
<th>FMP</th>
<th>Decidable</th>
</tr>
</thead>
<tbody>
<tr>
<td>CnMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>CnIMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>SnMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>SnIMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Ω(SnMTL)</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>SnWCMTL</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Ω(CnWCMTL)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

no standard algebras in the corresponding varieties. The FEP can be proved directly using an easy semantical construction and the remaining properties follow from the FEP. As regards to local finiteness, we can prove that n-contractivity is a necessary condition for it; however the sufficiency remains an open problem, so local finiteness of varieties of n-contractive MTL-algebras is still unknown in general.

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References