Abstract

Ranking functions are qualitative degrees of uncertainty ascribed to events charged by uncertainty and taking as their values non-negative integers in the sense of ordinal numbers. Introduced are ranking functions induced by real-valued possibilistic measures and it is shown that different possibilistic measures with identical ranking functions yield the same results when applied in decision procedures based on qualitative comparison of the magnitudes of the possibilistic measures in question ascribed to the uncertain events.

Keywords: Possibilistic distribution, rank degree, ranking distribution, rank equivalent possibilistic measures, qualitative decision making under uncertainty.

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1 Introduction and Motivation

Consider the case when we have to bet on just one $H_{i_0}$ from a finite list $H_1, H_2, \ldots, H_n$ of mutually disjoint and exhaustive hypotheses on the ground of some empirical data charged by uncertainty in the sense of randomness, let $P(H_1), \ldots, P(H_n)$ be the resulting posteriori probabilities ascribed to particular hypotheses. Applying the most simple $0−1$ loss function and aiming to minimize the expected loss, we arrive (quite naturally) to the maximum likelihood decision procedure: to bet on $H_{i_0}$ such that $P(H_{i_0}) \geq P(H_i)$ holds for each $i = 1, 2, \ldots, n$ (choosing $i_0$ somehow, if there are two or more $i$’s with this property).

What is lost under this decision making is the difference or ratio between $P(H_{i_0})$ and other values $P(H_i)$, e.g., if $n = 2$, then $H_1$ is preferred to $H_2$ when $P(H_1) = 0.51$ and $P(H_2) = 0.49$ as well as in the case when $P(H_1) = 0.99$ and $P(H_2) = 0.01$. Consequently, what is also lost is a great portion of the effort perhaps expended in order to specify the values $P(H_1), \ldots, P(H_n)$, or at least to obtain their good and reliable enough approximations and estimations. On the other side, some more conditions perhaps imposed on the structures under consideration (e.g., statistical independence, Laplace principle, \ldots) may be omitted if only the ordering of the values $P(H_i)$ according to their magnitudes matters.

Ranking functions as introduced by Spohn [6], [7] can be taken just as the characteristic enabling to distinguish from each other two probability distributions (over the same finite set) resulting in different orderings of the elements when ordering them according to the magnitude of probability values ascribed to them by the two probability distributions in
question. In what follows, our aim will be to apply the idea of ranking functions to the case when the uncertainty charging the data under consideration is quantified and processed not by probability measures, but by possibilistic measures. The basic idea and notations are those from [3] where also a list of interesting references dealing with probabilistically motivated ranking functions can be found.

2 Rank Reducible Real-Valued Possibilistic Distributions

**Definition 2.1** (Real-valued) possibilistic distribution over a non-empty space \( \Omega \) is a mapping \( \pi : \Omega \rightarrow [0,1] \) such that \( \bigvee_{\omega \in \Omega} \pi(\omega) = 1 \); here and below \( \bigvee, \bigwedge, \vee \) (infimum, resp.) denotes the supremum (infinum, resp.) operation in \( [0,1] \). Possibilistic distribution \( \pi \) over \( \Omega \) is called rank reducible, if there exists a finite or infinite sequence \( \alpha(\pi) = (\alpha_0 > \alpha_1 > \ldots) \) of positive real numbers such that, for each \( \omega \in \Omega \), either \( \pi(\omega) = 0 \) or \( \pi(\omega) = \alpha_k \) for some \( k = 0, 1, 2, \ldots \), and for each \( \alpha_k \), there exists \( \omega \in \Omega \) such that \( \pi(\omega) = \alpha_k \). The possibilistic distribution \( \pi \) on \( \Omega \) induces the (real-valued) possibilistic measure \( \Pi \) on \( P(\Omega) \), setting \( \Pi(A) = \bigvee_{\omega \in A} \pi(\omega) \) for each \( \emptyset \neq A \subset \Omega \) and \( \Pi(\emptyset) = 0 \). If \( \pi \) is rank reducible, also \( \Pi \) is called rank reducible.

Obviously, for each rank reducible possibilistic distribution \( \pi \) the sequence \( \alpha(\pi) \) is uniquely defined and \( \alpha_0 = 1 \), consequently, for each \( \omega \in \Omega \) such that \( \pi(\omega) > 0 \) holds the index \( k \in \mathbb{N} = \{0, 1, 2, \ldots\} \), for which \( \pi(\omega) = \alpha_k \) is uniquely defined. Far not every possibilistic distribution on \( \Omega \) is rank reducible, as a counter-example take \( \pi \) such that, for some \( \alpha_1 > 0 \) and for infinitely many values \( \alpha_1 \geq \alpha > \alpha_0 \), the sets \( \{\omega \in \Omega : \pi(\omega) = \alpha\} \) and \( \{\omega \in \Omega : \pi(\omega) = \alpha_0\} \) are nonempty.

**Definition 2.2** Let \( \pi \) be a rank reducible possibilistic distribution over \( \Omega \), let \( \kappa(\omega) = k \), if \( \pi(\omega) = \alpha_k \), let \( \kappa(\omega) = \infty \), if \( \pi(\omega) = 0 \). The value \( \kappa(\omega) \) is called the rank of the element \( \omega \in \Omega \) w.r.t. \( \pi \) and the mapping \( \kappa : \Omega \rightarrow N^* = \{0, 1, 2, \ldots\} \cup \{\infty\} \) is called the ranking distribution defined by \( \pi \) on \( \Omega \). The induced ranking function \( \kappa \) on \( P(\Omega) \) is defined by \( K(A) = k \), if \( \Pi(A) = \alpha_k > 0 \), and \( K(A) = \infty \), if \( \Pi(A) = 0 \).

As \( \Pi(A) = \bigvee_{\omega \in A} \pi(\omega) \), obviously \( \Pi(A) \) must be a value from \( \alpha(\pi) \) or 0, so that \( K(A) \) is defined for every \( A \subset \Omega \). The relations \( K(\emptyset) = \infty, \Pi(\{\omega\}) = \pi(\omega), \) and \( K(\omega) = \kappa(\omega) \) obviously hold for each \( \omega \in \Omega \).

**Definition 2.3** Rank reducible possibilistic distributions \( \pi_1, \pi_2 \) over \( \Omega \) are called rank equivalent (\( \pi_1 \approx \pi_2 \), in symbols), if their ranking distributions are identical, i.e., if \( \kappa_1(\omega) = \kappa_2(\omega) \) for each \( \omega \in \Omega \).

As can be easily seen, \( \approx \) defines an equivalence relation on the space of rank reducible possibilistic distributions over \( \Omega \). If \( \pi_1 \approx \pi_2 \) holds, also \( K_{\pi_1} \) and \( K_{\pi_2} \) are identical, i.e., \( K_{\pi_1}(A) = K_{\pi_2}(A) \) for each \( A \subset \Omega \).

**Theorem 2.1** Rank reducible possibilistic distributions \( \pi_1, \pi_2 \) on \( \Omega \) are rank equivalent if and only if, for each \( A, B \subset \Omega \), the inequalities \( \Pi_1(A) \leq \Pi_1(B) \) and \( \Pi_2(A) \leq \Pi_2(B) \) hold simultaneously, i.e., \( \Pi_1(A) \leq \Pi_1(B) \) is valid iff \( \Pi_2(A) \leq \Pi_2(B) \) holds.

**Proof:** Let us prove that given a rank reducible possibilistic distribution \( \pi \) on \( \Omega \), \( K(A) = \bigwedge \{\kappa(\omega) : \omega \in A\} \) holds for every \( A \subset \Omega \). Setting \( \alpha_\infty = 0 \) to cover also the case when \( \pi(\omega) = 0 \) we obtain that \( K(A) = k \iff \Pi(A) = \alpha_k \), however, \( \Pi(A) = \bigvee_{\omega \in A} \pi(\omega) \), so that the inequality \( \pi(\omega) \leq \alpha_k \) for each \( \omega \in A \) follows. Hence, for each \( \omega \in A, \pi(\omega) = \alpha_i \) for some \( i \geq k \) and \( \pi(\omega) = \alpha_k \) for some \( \omega \in A \), consequently, \( \kappa(\omega) = i, i \geq k \), for each \( \omega \in A \) and there exists \( \omega \in A \) such that \( \kappa(\omega) = k \), so that \( K(A) = k = \bigwedge \{\kappa(\omega) : \omega \in A\} \). Let \( A, B \subset \Omega \) be such that \( \Pi_1(A) \leq \Pi_1(B) \) holds. As \( \Pi_1(A) = \alpha_{k_1(A)} \) and \( \Pi_1(B) = \alpha_{k_1(B)} \) holds for some \( k_1(A), k_1(B) \), the inequality \( \kappa_1(A) \geq k_1(B) \) follows. But, \( k_1(A) = K_1(A) \) and \( k_1(B) = K_1(B) \) by definition and the identities \( K_1(A) = K_2(A), K_1(B) = K_2(B) \) hold as \( \pi_1 \) and \( \pi_2 \) are supposed to be rank equivalent. Hence, \( K_2(A) \geq K_2(B) \) follows, so that \( \Pi_2(A) = \beta_{K_2(A)} \leq \Pi_2(B) = \beta_{k_2(B)} = \Pi_2(B) \) follows, here \( \{\beta_0, \beta_1, \ldots\} \) is the scale.
of values uniquely defined by \( \pi_2 \). As the role of \( \pi_1 \) and \( \pi_2 \) is completely exchangeable, we obtain that, for \( \pi_1 \) and \( \pi_2 \) rank equivalent, \( \Pi_1(A) \leq \Pi_1(B) \) holds iff \( \Pi_2(A) \leq \Pi_2(B) \) holds.

Let for each \( A, B \subset \Omega \) the equivalence \( \Pi_1(A) \leq \Pi_1(B) \) iff \( \Pi_2(A) \leq \Pi_2(B) \) be valid and suppose, in order to arrive at a contradiction, that there exists \( \omega_0 \in \Omega \) such that \( \kappa_1(\omega_0) > 0, \kappa_2(\omega_0) = 0 \) holds. Then \( K_1(\Omega - \{\omega_0\}) = 0 \) (as \( K_1(\Omega) = 0 = K_1(\Omega - \{\omega_0\}) \land K_1(\{\omega_0\}) \)), so that \( \Pi_1(\Omega - \{\omega_0\}) = 1 > \Pi_1(\{\omega_0\}) \), as \( \Pi_1(\{\omega_0\}) \leq 1 \) holds. However, \( \Pi_2(\Omega - \{\omega_0\}) \leq 1 = \Pi_2(\{\omega_0\}) \), so that the pair \( \{\omega_0\}, \Omega - \{\omega_0\} \) of subsets of \( \Omega \) violates the conditions imposed on \( \Pi_1 \) and \( \Pi_2 \). Hence, \( \{ \omega \in \Omega : \kappa_1(\omega) = 0 \} = \{ \omega \in \Omega : \kappa_2(\omega) = 0 \} \).

Applying the principle of induction, let the identity

\[
\Omega_i^1 = \{ \omega \in \Omega : \kappa_1(\omega) = i \} = \{ \omega \in \Omega : \kappa_2(\omega) = i \} \equiv \Omega_i^2 \quad (2.1)
\]

hold for each \( i = 1, 2, \ldots, k \). Let \( \omega_0 \in \Omega_0 = \Omega - \bigcup_{i=0}^k \Omega_i^1 = \Omega - \bigcup_{i=0}^k \Omega_i^2 \) be such that \( \kappa_1(\omega_0) > k + 1, \kappa_2(\omega) = k + 1 \). Then \( K_1(\Omega - \{\omega_0\}) = k + 1 \), as \( K_1(\Omega_0) = K_1(\Omega_0 - \{\omega_0\}) \land K_1(\{\omega_0\}) = k + 1 \), so that \( \Pi_1(\Omega - \{\omega_0\}) \leq \Pi_1(\{\omega_0\}) \) holds. However, \( K_2(\Omega - \{\omega_0\}) \geq k + 1 = K_2(\{\omega_0\}) \) holds, so that \( \Pi_2(\Omega_0 - \{\omega_0\}) \leq \Pi_2(\{\omega_0\}) \) follows, hence, the pair \( \{\omega_0\}, \Omega - \{\omega_0\} \) of subsets of \( \Omega \) violates, again, the conditions imposed on \( \Pi_1 \) and \( \Pi_2 \). So, the identity \( \kappa_1(\omega) = i \) iff \( \kappa_2(\omega) = i \) holds for each \( i = 1, 2, \ldots, \), hence, also for \( \kappa_1(\omega) = \infty, i = 1, 2, \ldots, \), as \( \{ \omega \in \Omega : k_j(\omega) = \infty \} = \Omega - \bigcup_{i=0}^\infty \{ \omega \in \Omega : k_j(\omega) = i \} \) for both \( j = 1, 2 \). Consequently, \( \pi_1 \) and \( \pi_2 \) are rank equivalent and the theorem is proved. \( \square \)

Hence, rank reducible possibilistic measures with identical ranking distributions yield identical results when applying them to decision-making procedures based on qualitative (i.e., “greater than”, “smaller than”, ...) comparisons of the values ascribed by such measures to the sets (random events) in question.

### 3 Cartesian Products of Rank Reducible Possibilistic Distributions

Let \( T_n = \langle [0,1]^n, \leq_s \rangle \) be the space of all \( n \)-tuples of reals from \([0,1]\) together with the pointwisely defined partial ordering \( \leq_s \), according to which \( \langle x_1, \ldots, x_n \rangle \leq_s \langle y_1, \ldots, y_n \rangle \) holds iff \( x_i \leq y_i \) holds for each \( i = 1, \ldots, n \). As can be easily seen, \( T_n \) is a complete lattice \([1] \) with pointwisely defined supremum \( \bigvee_s \) and infimum \( \bigwedge_s \), i.e., for each \( \emptyset \neq A \subset [0,1]^n \),

\[
\bigvee_s A = \bigvee_{\langle x_1, \ldots, x_n \rangle \in A} \langle x_1, \ldots, x_n \rangle = \left( \bigvee_{\langle x_1, \ldots, x_n \rangle \in A} x_1, \ldots, \bigvee_{\langle x_1, \ldots, x_n \rangle \in A} x_n \right) \quad (3.1)
\]

and dually for \( \bigwedge_s A \).

Let \( \Omega \) be a nonempty set, let \( \pi_1, \ldots, \pi_n \) be possibilistic distributions on \( \Omega \). Set \( \pi(\omega) = \langle \pi_1(\omega), \ldots, \pi_n(\omega) \rangle \in [0,1]^n \) for each \( \omega \in \Omega \). As

\[
\bigvee_{\omega \in \Omega} \pi(\omega) = \left( \bigvee_{\omega \in \Omega} \pi_1(\omega), \ldots, \bigvee_{\omega \in \Omega} \pi_n(\omega) \right) = (1, \ldots, 1) = 1 \quad (3.2)
\]

(the unit element of the complete lattice \( T_n \)), \( \pi \) defines a \( T_n \)-valued possibilistic distribution on \( \Omega \) (cf. \([2]\) for lattice-valued possibilistic measures in general). Setting \( \Pi(A) = \bigvee_{\omega \in A} \pi(\omega) \), we obtain easily that \( \Pi(A) = \langle \Pi_1(A), \ldots, \Pi_n(A) \rangle \in [0,1]^n \) for each \( A \subset \Omega \), hence, \( \Pi : \mathcal{P}(\Omega) \rightarrow [0,1]^n \) defines the \( T_n \)-valued possibilistic measure on \( \mathcal{P}(\Omega) \) induced by the \( T_n \)-valued possibilistic distribution \( \pi \) on \( \Omega \).

Let every of the possibilistic distributions \( \pi_1, \ldots, \pi_n \) be rank reducible, so that, for each \( \omega \in \Omega \) and \( 1 \leq i \leq n \), the value \( \kappa^i(\omega) \in \mathcal{N}^* \), i.e., the value of the rank distribution induced by \( \pi_i \) and ascribed to \( \omega \), is defined as above. The value of the rank distribution induced on \( \Omega \) by \( \pi \) is defined by \( \kappa(\omega) = \)
be rank reducible $A, B \subset \Omega$ for each $i = 1, 2, \ldots, n$ (taking $j < \infty$ for each $j \in \mathcal{N}$). Setting $K(A) = (K^1(A), \ldots, K^n(A))$ for each $A \subset \Omega$ and applying the same reasoning as in the case of Theorem 2.1, we obtain that

\[
\bigwedge_0 \{\kappa(\omega) : \omega \in A\} = \\
= \bigwedge_0 \{\kappa(\omega) : \omega \in A\} = \\
= \bigg(\bigwedge_{\omega \in A} \kappa(\omega), \ldots, \bigwedge_{\omega \in A} \kappa(\omega)\bigg) = \\
= (K^1(A), \ldots, K^n(A)) = K(A) \quad (3.3)
\]

holds for each $A \subset \Omega$.

Rank reducible $T_n$-valued possibilistic distributions $\pi^i = (\pi^i_1, \ldots, \pi^i_n)$, $i = 1, 2$, on a nonempty space $\Omega$ are called rank equivalent ($\pi^1 \approx \pi^2$, in symbols), if $\pi^1_\omega(\omega) = \pi^2_\omega(\omega)$ for each $\omega \in \Omega$, i.e., if $\kappa^1_\omega(\omega) = \kappa^2_\omega(\omega)$ for every $\omega \in \Omega$ and every $i = 1, 2, \ldots, n$.

**Theorem 3.1.** Let $\pi^1 = (\pi^1_1, \ldots, \pi^1_n), i = 1, 2$, be rank reducible $T_n$-valued possibilistic distributions on a nonempty space $\Omega$. These distributions are rank equivalent if and only if, for each $A, B \subset \Omega$, the inequality $\Pi^1(A) \leq \Pi^1(B)$ is valid iff $\Pi^2(A) \leq \Pi^2(B)$ is the case. Written in symbols, the equivalence

\[
\pi^1 \approx \pi^2 \iff (\forall A, B \subset \Omega) \\
[(\Pi^1(A) \leq \Pi^2(A)) \iff (\Pi^2(A) \leq \Pi^2(B))] \quad (3.4)
\]

holds.

**Proof:** Let $\pi^1 \approx \pi^2$ be the case, so that $\kappa^1_\omega(\omega) = \kappa^2_\omega(\omega)$ for every $\omega \in \Omega$ and every $i = 1, 2, \ldots, n$. Hence, $\pi^1_\omega$ and $\pi^2_\omega$ are rank equivalent for each $i = 1, 2, \ldots, n$ so that, for each $A, B \subset \Omega$, the inequalities $\Pi^1(A) \leq \Pi^1(B)$ and $\Pi^2(A) \leq \Pi^2(B)$ hold simultaneously. So, if $\pi^1 \approx \pi^2$ and $\Pi^1(A) = (\Pi^1_1(A), \ldots, \Pi^1_n(A)) \leq \Pi^1(B) = (\Pi^1_1(B), \ldots, \Pi^1_n(B))$ is the case, then $\Pi^1_i(A) \leq \Pi^1_i(B)$ holds for each $i = 1, 2, \ldots, n$, consequently, also $\Pi^2_i(A) \leq \Pi^2_i(B)$ holds for each $i = 1, 2, \ldots, n$ and $\Pi^2(A) \leq \Pi^2(B)$ follows. Exchanging mutually the role of $\Pi^1$ and $\Pi^2$ we obtain immediately that $\pi^1 \approx \pi^2$ and $\Pi^2(A) \leq \Pi^2(B)$ imply that $\Pi^1(A) \leq \Pi^1(B)$ holds, hence, the implication $\pi^1 \approx \pi^2 \Rightarrow \ldots$ in (3.4) is proved.

The inverse implication will be proved by contradiction. Let $n = 2$, let there exist $A, B \subset \Omega$ such that $\Pi^1(A) \leq \Pi^1(B)$, but not $\Pi^2(A) \leq \Pi^2(B)$ (the case with $\Pi^2(A) \leq \Pi^2(B)$ but not $\Pi^1(A) \leq \Pi^1(B)$ is the same just with the roles of $\Pi^1$ and $\Pi^2$ mutually interchanged). Hence, as $\Pi^1(A) = (\Pi^1_1(A), \Pi^1_2(A))$ for both $i = 1, 2$, both the inequalities $\Pi^1_i(A) \leq \Pi^1_i(B)$ and $\Pi^2_i(A) \leq \Pi^2_i(B)$ hold, but either $\Pi^2_1(A) \leq \Pi^2_1(B)$ or $\Pi^2_2(A) \leq \Pi^2_2(B)$ does not hold. Let us suppose, without any loss of generality, that $\Pi^2_1(A) \leq \Pi^2_1(B)$ does not hold, hence, that $\Pi^2_1(A) > \Pi^2_1(B)$ is the case.

As both the real-valued possibilistic distributions $\pi^1_1$ and $\pi^2_1$ are rank reducible, for each $\emptyset \neq A \subset \Omega$, there exists $\omega_A \subset A$ such that $\Pi^1_1(A) = \pi^1_1(\omega_A)$ and $\Pi^2_1(A) = \pi^2_1(\omega_A^*).$ Supposing that $\Pi^2_1(A) > \Pi^2_1(B)$ holds, we obtain that $\Pi^2_1(A) > 0$ and $A \neq \emptyset$ follows, let us suppose that $B \neq \emptyset$ as well (the case when $B = \emptyset$ will be treated separately below).

So, we suppose that $\Pi^1_1(A) \leq \Pi^1_1(B)$ and $\Pi^2_1(A) > \Pi^2_1(B)$ hold simultaneously. Hence, there exist $\omega_A \subset A$ and $\omega_B \subset B$ such that $\pi^1_1(\omega_1) \leq \pi^1_1(\omega_B)$ holds for each $\omega_1 \subset A$ and $\pi^2_1(\omega_A) > \pi^2_1(\omega_B)$ holds for each $\omega_B \subset B$. As can be easily seen, for each rank reducible real-valued possibilistic distribution $\pi$ on $\Omega$ the relations $\kappa(\omega_1) \geq \kappa(\omega_2)$ iff $\pi(\omega_1) \leq \pi(\omega_2)$ and $\kappa(\omega_1) > \kappa(\omega_2)$ iff $\pi(\omega_1) < \pi(\omega_2)$ are valid for each $\omega_1, \omega_2 \subset \Omega$. Hence, our assumption that $\Pi^1_1(A) \leq \Pi^1_1(B)$ and $\Pi^2_1(A) > \Pi^2_1(B)$ hold simultaneously implies that there exist $\omega_A \subset A$ and $\omega_B \subset B$ such that $\kappa^1_1(\omega_A) \geq \kappa^1_1(\omega_B)$ for each $\omega_A \subset A$ and $\kappa^2_1(\omega_A) < \kappa^2_1(\omega_B)$ for each $\omega_B \subset B$. Applying the first inequality to $\omega_1 = \omega_A \subset A$ and the other one to $\omega_2 = \omega_B \subset B$, we obtain that the inequalities $\kappa^1_1(\omega_A) \geq \kappa^1_1(\omega_B)$ and $\kappa^2_1(\omega_A) < \kappa^2_1(\omega_B)$ should be valid simultaneously, but this con-
tradicts the assumption $\pi^1 \approx \pi^2$ according to which $\kappa^1$ and $\kappa^2$ should be identical on the whole space $\Omega$.

If $B = \emptyset$, then $\Pi^1(B) = 0$, hence, also $\Pi^1(A) = 0$, so that $\pi^1(\omega) = 0$ and $\kappa^2(\omega) = \infty$ for every $\omega \in \Omega$. However $\Pi^2(A) > \Pi^1(B)$ yields that $\Pi^2(A) > 0$ holds, so that there exists $\omega_0 \in A$ with the property that $\pi^2(\omega_0) > 0$ and, consequently, $\kappa^2(\omega_0) < \infty$ holds, but $\kappa^1(\omega_0) = \infty$ — again a contradiction. Hence, neither $\pi^1 \approx \pi^2$ nor $\pi^2 \approx \pi^2$ is the case, so that the $\pi^1 \approx \pi^2 \Leftarrow \ldots$ implication in (3.4) as well as Theorem 3.1 as a whole are proved.

$\square$

4 Axiomatic and Empirical Approaches to Ranking Distributions and Ranking Functions

Ranking distributions and ranking functions were introduced as a secondary notion and useful tool when investigating decision making under uncertainty in the sense of randomness. Also ranking distributions and ranking functions can be introduced axiomatically as primary notions. Referring to [3], a definition may read as follows.

**Definition 4.1** Let $Y$ be a nonempty set, let $\mathcal{N} = \{0, 1, 2, \ldots\}$, let $\mathcal{N}^* = \mathcal{N} \cup \{\infty\}$. A mapping $\kappa : Y \to \mathcal{N}^*$ is called a ranking distribution on $Y$, if there exists $y \in Y$ such that $\kappa(y) = 0$ and if for each $k = 1, 2, \ldots$ the implication

$$\{y \in Y : \kappa(y) = k\} \neq \emptyset \Rightarrow \{y \in Y : \kappa(y) = k - 1\} \neq \emptyset$$

(4.1)

is valid. Each ranking distribution $\kappa$ on $Y$ induces ranking function $K$ on $\mathcal{P}(Y)$, setting

$$K(B) = \bigwedge_{y \in B} \kappa(y)$$

for each $\emptyset \neq B \subset Y$ and setting $K(\emptyset) = \infty$, here $\bigwedge$ denotes the standard infimum in $\mathcal{N}^*$ supposing that $k < \infty$ holds for each $k \in \mathcal{N}$. A ranking distribution $\kappa$ on $Y$ is simple, if for each $k \in \mathcal{N}$ there exists at most one $y \in Y$ such that $\kappa(y) = k$.

As can be easily seen, for each $\emptyset \neq B \subset \mathcal{P}(Y)$ and for $\bigcup B = \bigcup_{B \in B} B$,

$$K\left(\bigcup B\right) = \bigwedge_{y \in \bigcup B} \kappa(y) = \bigwedge_{B \in B} \left(\bigwedge_{y \in B} \kappa(y)\right) = \bigwedge_{B \in B} K(B).$$

(4.2)

For each ranking distribution $\kappa$ on $Y$ there exists a possibilistic distribution $\pi(\kappa)$ on $Y$ such that $\kappa$ is the ranking distribution induced by $\pi(\kappa)$, take simply $\pi(\kappa)(y) = (\kappa(y) + 1)^{-1}$, if $\kappa(y) < \infty$, and $\pi(\kappa)(y) = 0$, if $\kappa(y) = \infty$.

Let $y = (y_1, y_2, \ldots)$ be a finite or infinite sequence of elements of $Y$. Given $y \in Y$, define $\kappa(y)(y)$ as follows: (i) if $y$ does not occur in $y$, set $\kappa(y)(y) = \infty$, (ii) if $i_0$ is the index of the first occurrence of $y_i$ in $y$, i.e., if $y_i \neq y$ for every $j < i_0$ and $y_{i_0} = y$, set $\kappa(y)(y) = \text{card}(y_1, y_2, \ldots, y_{i_0-1})$, so that $\kappa(y)(y)$ is the number of different elements of $Y$ preceding $y$ in $y$. Equivalently, let $y^0 = (y^0_1, y^0_2, \ldots)$ be the sequence obtained from $y$ when erasing repeated occurrences of elements. For each $y \in Y$ occurring in $y$ there exists just one $i_0$ such that $y = y^0_{i_0}$, then set $\kappa(y)(y) = i_0 - 1$.

The following assertion is almost obvious (cf. Lemma 5.1 in [4]).

**Lemma 4.1** For each $y = (y_1, y_2, \ldots), \kappa(y) : Y \to \mathcal{N}^*$ defines a simple ranking distribution on $Y$.

Sequences $y^1 = (y^1_1, y^1_2, \ldots)$ and $y^2 = (y^2_1, y^2_2, \ldots)$ of elements of $Y$ are called rank equivalent, if $\kappa(y^1)(y) = \kappa(y^2)(y)$ for each $y \in Y$. A reasonable and intuitive interpretation behind may read that $(y_1, y_2, \ldots) = (X_1(\omega), X_2(\omega), \ldots)$ is a sequence of random samples, i.e., of empirical data charged by uncertainty (randomness). This case will be analyzed, in more detail, in the next chapter.

5 Randomized Ranking Distributions and Functions

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $(Y, \mathcal{Y})$ be a measurable space. A mapping $X : \Omega \to
is obvious. Let \( Y \) be a finite or countable space, and each \( \omega \in \Omega \) has \( \kappa(X(\omega))(y) = k \) for some \( k \). Then the relation

\[
P\left( \bigcap_{i=1}^{n} \{ \omega \in \Omega : X_i(\omega) \in B_i \} \right) = \prod_{i=1}^{n} P(\{ \omega \in \Omega : X_i(\omega) \in B_i \})
\]

holds. If the random variables \( X_1, X_2, \ldots \) are statistically independent and identically distributed, they are shortly denoted as \( i.i.d. \) random variables. Cf. [5] or other textbook on probability theory for more detail.

As can be easily proved, if \( Y \) is countable and \( Y = \mathcal{P}(Y) \), then for each \( k \in \mathbb{N}^* \) and each \( y \in Y \) the set \( \{ \omega \in \Omega : \kappa(X(\omega))(y) = k \} \) is in \( \mathcal{A} \), where \( X(\omega) = (X_1(\omega), X_2(\omega), \ldots) \). If \( k = \infty \), the relation

\[
\{ \omega \in \Omega : \kappa(X(\omega))(y) = \infty \} = \bigcap_{i=1}^{\infty} \{ \omega \in \Omega : X_i(\omega) \in Y - \{y\} \} \in \mathcal{A}
\]

is obvious. Let \( k < \infty \), let \( n \geq k \), then there exist at most countably many \( n \)-tuples \( \langle \alpha_1, \ldots, \alpha_n \rangle \) in \( Y^n \), consequently,

\[
\{ \omega \in \Omega : \kappa(X(\omega))(y) = k \} = \bigcup_{n=0}^{\infty} \left( \bigcup_{\langle \alpha_1, \ldots, \alpha_n \rangle \in \mathcal{Y}^n, \text{card}(\alpha_1, \ldots, \alpha_n) = k} \{ \omega \in \Omega : X_1(\omega) = \alpha_1, X_2(\omega) = \alpha_2, \ldots, X_n(\omega) = \alpha_n, X_{n+1}(\omega) = y \} \right).
\]

As each finite cylinder \( \{ \omega \in \Omega : X_1(\omega) = \alpha_1, \ldots, X_{n+1}(\omega) = y \} \) is in \( \mathcal{A} \), the set \( \{ \omega \in \Omega : \kappa(X(\omega))(y) = k \} \) is in \( \mathcal{A} \) as well.

Consequently, both the ranking distribution \( \kappa(X(\cdot))(\cdot) : \Omega \times Y \to \mathbb{N}^* \), defined in Chapter 4 when taking \( y = (X_1(\omega), X_2(\omega), \ldots) \) as well as the ranking function \( \kappa(X(\cdot))(\cdot) : \Omega \times \mathcal{P}(Y) \to \mathbb{N}^* \), induced by this ranking distribution, are random variables defined on the probability space \( (\Omega, \mathcal{A}, P) \) and taking their values in the measurable space \( (\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*)) \).

The sequence \( X = (X_1, X_2, \ldots) \) of random variables, each of them taking \( (\Omega, \mathcal{A}, P) \) into \( (Y, \mathcal{Y}) \), is called rank equivalent, if for each \( \omega_1, \omega_2 \in \Omega \) the sequences \( X_1(\omega_1), X_2(\omega_1), \ldots \) and \( X_1(\omega_2), X_2(\omega_2), \ldots \) are rank equivalent, i.e., if \( \kappa(X(\omega_1))(y) = \kappa(X(\omega_2))(y) \) for every \( \omega_1, \omega_2 \in \Omega \) and every \( y \in Y \). The sequence \( X \) is almost rank equivalent, if there exists \( \Omega_0 \subset \Omega, \Omega_0 \in \mathcal{A}, P(\Omega_0) = 1 \), such that the sequences \( X(\omega_1), X(\omega_2) \) are rank equivalent for every \( \omega_1, \omega_2 \in \Omega_0 \). Given a real number \( \varepsilon \geq 0 \), the sequence \( X \) is \( \varepsilon \)-rank equivalent, if there exists \( \Omega_\varepsilon \subset \Omega, \Omega_\varepsilon \in \mathcal{A}, P(\Omega_\varepsilon) = 1 - \varepsilon \), such that the sequences \( X(\omega_1), X(\omega_2) \) are rank equivalent for every \( \omega_1, \omega_2 \in \Omega_\varepsilon \) (hence, 0-rank equivalence coincides with almost rank equivalence).

**Theorem 5.1** Let \( (\Omega, \mathcal{A}, P) \) be a probability space, let \( Y \) be a finite or countable space, let \( X = (X_1, X_2, \ldots) \) be a sequence of \( i.i.d. \) random variables, each of them taking \( (\Omega, \mathcal{A}, P) \) into \( (Y, \mathcal{Y}) \), let \( p(y) \) (\( p(B) \), resp.) denote the value \( P(\{ \omega \in \Omega : X_1(\omega) = y \}) \) (\( P(\{ \omega \in \Omega : X_1(\omega) \in B \} \), resp.) for each \( y \in Y \) (each \( B \subset Y \), resp.), the relation \( p(B) = \sum_{y \in B} p(y) \) is obvious), let \( y_1, y_2, \ldots, y_n \) be elements of \( Y \) such that \( p(y_i) > 0 \) is the case for every \( i = 1, 2, \ldots, n \). Then the relation

\[
P(\{ \omega \in \Omega : \kappa(X(\omega))(y_1) < \kappa(X(\omega))(y_2) < \cdots < \kappa(X(\omega))(y_n) \}) = 
\prod_{i=1}^{n-1} [p(y_i)(p(y_i) + p(y_{i+1}) + \cdots + p(y_n))^{-1}]
\]

holds.
Proof: Cf. Theorems 6.1 and 6.2 in [4]. □

Perhaps the most simple nontrivial case with \( n = 2 \) is worth being introduced explicitly because of an illustration and the intuition behind. Hence, in this case the relation

\[
P(\{\omega \in \Omega : \kappa(X(\omega))(y_1) < \kappa(X(\omega))(y_2)\}) = p(y_1)(p(y_1) + p(y_2))^{-1}
\]

holds, so that the ranking degrees \( \kappa(X(\omega))(y_1) \) and \( \kappa(X(\omega))(y_2) \) copy (in the natural reverse order), more and more strongly with the values \( p_1 - p_2 \) or \( p_1(p_1 + p_2)^{-1} \) increasing, the quantity ordering relation between \( p_1 \) and \( p_2 \). \((p_1 = p(y_i))\). Quite intuitively, if \( p_1 \approx 1 \) and \( p_2 \ll 1 \) is the case, the value \( p_1(p_1 + p_2)^{-1} \) is close to 1, so that \( y_1 \) “almost surely” occurs in \( X_1(\omega), X_2(\omega), \ldots \) before \( y_2 \).

6 Conclusions

Leaving aside the still open possibility of a qualitatively new interpretation of possibility degrees, let us consider the case when, like as in the standard probability theory, possibility degrees are taken as sizes of sets of favorable elementary random events, just with alternative demands imposed on the notion of size of sets (maxitivity instead of the standard demand of additivity). Because of the inequality \( P(A) \lor P(B) \leq P(A \cup B) \leq P(A) + P(B) \), obviously valid for the probabilities of any random events \( A, B \), the value \( P(A) \lor P(B) \) can be taken as the safe lower bound for \( P(A \cup B) \) in the sense that the inequality \( P(A \cup B) \geq \alpha \) for a given threshold value \( \alpha \) certainly holds if \( P(A) \lor P(B) \geq \alpha \) is the case no matter which the relation between \( A \) and \( B \) may be. Consequently, when considering a decision making under uncertainty, quantified by a probability measure but approximated from below as sketched above (e.g., because of the lack of knowledge concerning the statistical (in)dependence of the random events in question), possibilistic measures can be applied as a useful tool.

So, the problem which information concerning the possibility degrees is sufficient when considering qualitative decision problems, e.g., which is the most possible hypothesis from a list of given ones, mentioned in the introduction of this text, emerges again as being actual. We have given an answer saying that what decides is the corresponding ranking distribution ordering the random events in question, as all qualitative decisions based on possibilistic measures with identical ranking distributions will be the same. Moreover, we have sketched a way how to estimate such ranking distributions on the ground of sequences of random samples, obtaining also reasonable results concerning the reliability of qualitative decisions based on such estimations of ranking distributions. A more detailed analysis of the related problems seems to be useful and desirable and the author intends to focus his attention to them at his earliest occasion.

References