The greatest common divisor and other triangular norms on the set of natural numbers

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Abstract

This paper deals with triangular norms and conorms defined on the extended set $\mathbb{N}$ of natural numbers ordered by divisibility. From the fundamental theorem of arithmetic, $\mathbb{N}$ can be identified with a lattice of functions from the set of primes to the complete chain $\{0, 1, 2, ..., +\infty\}$, thus our knowledge about (divisible) t-norms on this chain can be applied to the study of t-norms on $\mathbb{N}$. A characterization of those t-norms on $\mathbb{N}$ which are a direct product of t-norms on $\{0, 1, 2, ..., +\infty\}$ is given and, after introducing the concept of $T$-prime (prime with respect to a t-norm $T$), a theorem about the existence of a $T$-prime decomposition is obtained. This result generalizes the fundamental theorem of arithmetic.

Keywords: natural number, greatest common divisor, least common multiple, primes, factorization, bounded partially ordered set, triangular norms and conorms, divisible t-norms.

1 Introduction

Number theory starts with the natural numbers $1, 2, 3, 4, 5, ...$ generating from 1 by successively adding 1. On the set of natural numbers we have the operations $+$ and $\times$, which are simple in themselves but lead to more sophisticated concepts. For example, we say that $d$ divides $n$ if $n = dq$ for some natural number $q$. A natural number $p$ is called prime if the only natural numbers dividing $p$ are 1 and $p$ itself.

Divisibility and primes are behind many of interesting questions in mathematics, and also behind recent applications of number theory in cryptography, internet security, electronic money transfers, etc.

The sequence of prime numbers begins with $2, 3, 5, 7, 11, 13, 17, 19, ...$ and continues in a seemingly random manner. There is so little pattern in the sequence that one cannot even see clearly whether it continues forever. However, Euclid (around 300 BCE) proved that there are infinitely many primes. It is important to be able to factorize natural numbers into products of primes. For small numbers we can usually perform the factorization with a modest supply of pen and paper, for large numbers such as $45615161$ a copious supply of pen and paper is needed ($45615161 = 5879 \times 7759$) but for very large numbers such as $2^{64} + 1$ even a calculator may be unable to perform the factorization ($2^{64} + 1 = 274177 \times 67280421310721$). Although possibly unable to perform an explicit factorization into prime factors we know nevertheless that one always exists and this fact is made precise in a theorem, which on account of its importance, has been designated ”The Fundamental Theorem of Arithmetic”.

Triangular norms (t-norms for short) were introduced by Schweizer and Sklar (1961) in the framework of probabilistic metric spaces when generalizing a concept used by Menger (1942) in order to extend the triangle inequality in...
the definition of distances. Consequently, the first field where t-norms played a major role was the theory of probabilistic metric spaces and many results concerning t-norms were obtained in the course of the development of this field. Thus, a crucial result by Aczél - Ling (1965) is the characterization of continuous t-norms in terms of ordinal sums of continuous Archimedean t-norms. In his first paper on fuzzy sets, Zadeh (1965) suggested to use the minimum, the product and, in a restricted sense, the Lukasiewics t-conorm for defining the intersection and the union of fuzzy sets. The use of general t-norms and t-conorms for this purpose goes back to the late seventies (Trillas, Höhle), since then they have been used extensively for modelling the intersection and the union of fuzzy sets and the logical ”and” and ”or” in fuzzy logic. On the other hand, t-norms play also a significant role in other fields such as those dealing with distribution functions (going back to the origins of t-norms), aggregation operators, many-valued logics, a generalized theory of measures and integrals, etc.

When we deal with logics where the class of truth values is modelled by a discrete set, their connectives can also be interpreted from the so called discrete triangular norms, discrete triangular conorms and discrete strong negations. Our interest in this topic (see, e.g. [4, 5]) is in part motivated by the fact that in most practical situations it is necessary to reduce the range [0, 1] to some finite or countably infinite set of truth values. On the other hand, due to the close relation between fuzzy set theory and order theory, several authors have worked with t-norms on bounded posets ([1]). In all cases, as it is expected, we use in the definition of t-norm the set of axioms provided by Schweizer and Sklar, once adapted to the corresponding setting. We recall that the notion of ”triangular norm on a poset” coincides with the notion of ”commutative partially ordered monoid”. There is an extensive literature from this area, see e.g. [2].

In this paper we deal with t-norms (and t-conorms) defined on the extended set N of natural numbers ordered by divisibility. One of our interests is to give a characterization of those t-norms on the lattice N which are a direct product of t-norms on the discrete chain \{0, 1, 2, ..., +∞\}, and also, after introducing the concept of T-prime (prime with respect to a t-norm T), to present a theorem about the existence of a T-prime decomposition with respect to a divisible t-conorm.

The paper is organized as follows. In Section 2, we briefly recall definitions, examples and basic properties concerning t-norms and t-conorms defined on a bounded partially ordered set. In Section 3, characterizations of divisible t-norms and t-conorms on \{0, 1, 2, ..., +∞\} are presented and, from them, we solve a discrete version of the Frank’s equation. In Section 4, we work with t-norms on the lattice of natural numbers which are a direct product of t-norms on \{0, 1, 2, ..., +∞\}. The fundamental theorem of arithmetic is, of course, a crucial tool in order to obtain a characterization of those t-norms. Finally, in Section 5 we introduce the notion of T-prime, and a T-prime decomposition with respect to a divisible t-conorm S is given. In case of T = gcd and S = Prod. we recover the classical decomposition of every natural number as a product of a finite number of primes.

For an exhaustive overview of classes, properties and applications of t-norms we recommend a recent monograph [3].

## 2 Triangulars norms and conorms on a partially ordered set

Let \((P; \leq)\) be a non-trivial bounded partially ordered set (poset) with \(e\) and \(u\) as minimum and maximum elements respectively.

**Definition 1** A triangular norm (briefly t-norm) on P is a binary operation \(T : P \times P \rightarrow P\) such that for all \(x, y, z \in P\) the following axioms are satisfied:

1) \(T(x, y) = T(y, x)\)
2) \(T(T(x, y), z) = T(x, T(y, z))\)
3) \(T(x, y) \leq T(x', y')\) whenever \(x \leq x'\) and \(y \leq y'\)
4) \(T(x, u) = x\)
A triangular conorm (t-conorm for short) is a binary operation \( S : P \times P \to P \) which, for all \( x, y, z \in P \), satisfies 1)-3) and \( S(x, e) = x \).

**Definition 2** A t-norm \( T \) (a t-conorm \( S \)) on \( P \) is divisible if the following condition holds:

For all \( x, y \in P \) with \( x \leq y \) there is \( z \in P \) such that \( x = T(y, z) \) (\( y = S(x, z) \)).

**Example 1** A basic example of (non-divisible) t-norm on any poset \( P \) is the drastic \( T_D \) defined by:

\[
T_D(x, y) = \begin{cases} 
  x & \text{if } y = u \\
  y & \text{if } x = u \\
  e & \text{otherwise}
\end{cases}
\]

Analogously,

\[
S_D(x, y) = \begin{cases} 
  x & \text{if } y = e \\
  y & \text{if } x = e \\
  u & \text{otherwise}
\end{cases}
\]

is the drastic t-conorm.

If \( P \) is bounded lattice, i.e., \( P \) is a bounded poset in which every finite subset has a greatest lower bound (infimum) and a lowest upper bound (supremum), then \( T_\wedge(x, y) = x \wedge y \) (\( x \wedge y = \inf \{x, y\} \)) is a t-norm and \( S_\vee(x, y) = x \vee y \) (\( x \vee y = \sup \{x, y\} \)) is a t-conorm. Trivially, the infimum \( T_\wedge \) and supremum \( S_\vee \) are divisible: \( x \leq y \) is equivalent to \( x \wedge y = x \) or \( x \vee y = y \).

**Proposition 1** Let \( T \) be a t-norm on a poset \( P \). Then:

a) \( T_D \leq T \) (pointwise order). Thus, \( T_D \) is the smallest t-norm.

b) \( T(x, y) \leq x, T(x, y) \leq y \ \forall x, y \in P \).

c) \( T(x, e) = e \ \forall x \in P \) (\( e \) is an annihilator).

d) If \( P \) is a lattice, then \( T \leq T_\wedge \). So, in this case, \( T_\wedge \) is the largest t-norm.

e) If \( P \) is a lattice, then \( T(x, x) = x \ \forall x \in P \) if and only if \( T = T_\wedge \).

Analogously, if \( S \) is a t-conorm on \( P \) then we have:

a) \( S_D \geq S \). Thus, \( S_D \) is the largest t-conorm.

b) \( S(x, y) \geq x, S(x, y) \geq y \ \forall x, y \in P \).

c) \( S(x, u) = u \ \forall x \in P \) (\( u \) is an annihilator).

d) If \( P \) is a lattice, then \( S \geq S_\vee \). Thus, \( S_\vee \) is the smallest t-conorm.

e) If \( P \) is a lattice, then \( S(x, x) = x \ \forall x \in P \).

**Remark 1** Let \( T \) be a t-norm on a poset \( P \). We can define the relation

\[
x \leq_T y \iff \text{there is } z \in P \text{ such that } x = T(y, z).
\]

Thus, we clearly obtain a partial ordering relation on \( P \) in such a way that \( x \leq_T y \) implies \( x \leq y \). Thus, a t-norm \( T \) on \( P \) is divisible if and only if \( \leq \) and \( \leq_T \) are equal.

Similar ordering \( \leq_S \) can be defined from a t-conorm:

\[
x \leq_S y \iff \text{there is } z \in P \text{ such that } y = S(x, z).
\]

Trivially, a t-conorm \( S \) on \( P \) is divisible if and only if \( \leq \) and \( \leq_S \) coincide.

**Remark 2** Given a strong negation \( n \) on \( P \), i.e., an involutive and non-increasing function \( n : P \to P \), then for each t-norm \( T \) on \( P \) one obtains a t-conorm \( S_n \) on \( P \) which is the \( n \)-dual to \( T \) in the following sense: \( S(x, y) = n(T(n(x), n(y))) \). Observe that applying this construction to the t-conorm \( S \), we get back the t-norm \( T \) we started with.

### 3 Triangular norms and conorms on the chain \( L = \{0, 1, 2, ..., +\infty\} \)

In this section we recall the main results about divisible t-norms defined on the bounded infinite chain \( L = \{0, 1, 2, ..., +\infty\} \) equipped with the usual order, with \( x < +\infty \) for all non-negative integer \( x \) and \( x + (+\infty) = (+\infty) + x = +\infty \) for all \( x \) in \( L \). In a similar way as for finite t-norms, a representation theorem is obtained for the divisible case. However, it is worth to be noticed that now essential differences arise with respect to the finite setting. For instance, there is no any Archimedean divisible t-norm on \( L \), and, \( S(x, y) = x + y \) is the only Archimedean divisible t-conorm. On the other hand, there is clearly no strong negation on \( L \) and consequently we have no duality between t-norms and t-conorms. That is, we need to study t-norms and t-conorms separately.
We can classify all divisible t-norms and t-conorms on \( L \) as follows.

**Proposition 2** A t-norm \( t \) on \( L = \{0, 1, 2, \ldots, +\infty\} \) is divisible if and only if there exists an infinite set \( I = \{a_0, a_1, \ldots, +\infty \mid 0 = a_0 < a_1 < \ldots < +\infty\} \) of elements of \( L \) such that:

\[
t(x, y) = \begin{cases} 
\max(a_i, x + y - a_i) & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \\
\min(x, y) & \text{otherwise}
\end{cases}
\]

**Proposition 3** A t-conorm \( s \) on \( L = \{0, 1, 2, \ldots, +\infty\} \) is divisible if and only if one of the following conditions holds:

(i) There exists an infinite set \( I = \{a_0, a_1, \ldots, +\infty \mid 0 = a_0 < a_1 < \ldots < +\infty\} \) of elements of \( L \) such that:

\[
s(x, y) = \begin{cases} 
\min(a_{i+1}, x + y - a_i) & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \\
\max(x, y) & \text{otherwise}
\end{cases}
\]

(ii) There exists a finite set \( I = \{a_0, a_1, \ldots, a_{m+1} \mid 0 = a_0 < a_1 < \ldots < a_{m+1} = +\infty\} \) of elements of \( L \) such that:

\[
s(x, y) = \begin{cases} 
\min(a_{i+1}, x + y - a_i) & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \\
\max(x, y) & \text{otherwise}
\end{cases}
\]

We denote by \( T_L \) and \( S_L \) the set of t-norms and t-conorms on \( L = \{0, 1, 2, \ldots, +\infty\} \) respectively.

From Proposition 2 and 3 we can now prove

**Proposition 4** A divisible pair \((t, s) \in T_L \times S_L\) is a solution of the Frank’s equation

\[
t(x, y) + s(x, y) = x + y, \quad x, y \in L \tag{1}
\]

if and only if there is an infinite set \( I = \{a_0, a_1, \ldots, +\infty \mid 0 = a_0 < a_1 < \ldots < +\infty\} \subset L \) such that \( t \) and \( s \) have the form:

\[
t(x, y) = \begin{cases} 
\max(a_i, x + y - a_i) & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \\
\min(x, y) & \text{otherwise}
\end{cases}
\]

\[
s(x, y) = \begin{cases} 
\min(a_{i+1}, x + y - a_i) & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \\
\max(x, y) & \text{otherwise}
\end{cases}
\]

Proof. It is obvious that \((t, s)\) given by (2) is a divisible solution of (1): for all \((x, y) \in [a_i, a_{i+1}]^2\) we have \(t(x, y) + s(x, y) = \max(a_i, x + y - a_i) + \min(a_{i+1}, x + y - a_i) = x + y\) and \(t(x, y) + s(x, y) = \min(x, y) + \max(x, y) = x + y\) otherwise. Reciprocally, if \((t, s)\) is a divisible pair which is a solution of (1) then \( t \) and \( s \) have the same set of idempotent elements, thus from Proposition 2 and 3 they must be of the form described in (2).

\(\Diamond\)

Proofs and more details on t-norms defined on bounded chains can be found in [5].

### 4 Triangular norms and conorms on the extended set of natural numbers

Let \( N = \{1, 2, \ldots, \infty\} \) be the extended set of natural numbers, i.e., the set of natural numbers with the symbol \( \infty \) adjoined, with \( n < \infty \) for all natural number \( n \). From now on, to say that \( n \) is a natural number is equivalent to say that \( n \) is an element (finite or infinite) of \( N \). On the other hand, we extend the usual operations as follows: \( n + \infty = \infty + n = \infty, n \cdot \infty = \infty \cdot n = \infty \), for all \( n \in N \).

Let \( n \) and \( d \) natural numbers. We shall say that \( d \) divides \( n \) if there exists a natural number \( q \) such that \( n = dq \). In this case we write \( d|n \). This relation turns \( N \) into a bounded lattice, with \( 1 \) and \( \infty \) as minimum and maximum elements respectively.

If \( m \) and \( n \) are natural numbers, by the greatest common divisor of \( m \) and \( n \) we mean the natural number \( d \) which is a common divisor of \( m \) and \( n \), and such that, if \( r \) is a divisor of \( m, n \), then \( r \) divides \( d \). We then write \( \gcd(m, n) = d \). We remark four basic properties of this binary operation on \( N \):

1. \( \gcd(m, n) = \gcd(n, m) \)
2. \( \gcd(\gcd(m, n), s) = \gcd(\gcd(m, n), s) \)
3. \( \gcd(m, n), n|n' \) then \( \gcd(m, n) | \gcd(m', n') \)
4. \( \gcd(m, \infty) = m \).

In other words, the \( \gcd \) is a triangular norm on the lattice \( N (\gcd = T_{\wedge}) \). Note that \( \gcd \) is not a t-norm on the bounded chain \( (N, \leq) \) where \( \leq \) is the usual order.
Let us consider \( p \in \mathbb{P} \). Suppose that \( p_1, p_2, \ldots, p_r \) are all the primes that divide \( m \) or \( n \) and that \( m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, n = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \), \( a_i, b_i \geq 0 \), are their prime factorizations.

Then, as it is well known, we have:
\[
\text{gcd}(m, n) = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}
\]
where \( c_i = \min(a_i, b_i) \ i = 1 \ldots r \).

In order to extend the formula above for computation of gcd in all cases, we assume the following conventions:
\( \infty = p_1^{\infty} \cdots p_r^{\infty} \), \( 1 = p_1^0 \cdots p_r^0 \), for all finite set of primes \( p_1, \ldots, p_r \).

Thus, if \( m \notin \{1, \infty\} \) and \( m = p_1^{a_1} \cdots p_r^{a_r}, a_i > 0 \), then \( m = \text{gcd}(m, n) = p_1^{c_1} \cdots p_r^{c_r} \) where \( c_i = \min(a_i, \infty) = a_i \) (\( \infty = p_1^{\infty} \cdots p_r^{\infty} \)). In the same way, \( 1 = \text{gcd}(m, 1) = p_1^{c_1} \cdots p_r^{c_r} \) where \( c_i = \min(a_i, 0) = 0 \) (\( 1 = p_1^0 \cdots p_r^0 \)).

In the case \( m, n \in \{1, \infty\} \), we choose a finite set of primes \( p_1, \ldots, p_r \) and from \( 1 = p_1^0 \cdots p_r^0 \), \( \infty = p_1^{\infty} \cdots p_r^{\infty} \) we have \( \text{gcd}(m, n) = p_1^{c_1} \cdots p_r^{c_r} \) with \( c_i = \min(a_i, b_i) \) where \( a_i, b_i \in \{0, +\infty\} \).

Thus, we have for all \( m, n \in \mathbb{N} \): \( T_\varphi(m, n) = \text{gcd}(m, n) = p_1^{c_1} \cdots p_r^{c_r}, \ n = p_1^{b_1} \cdots p_r^{b_r}, \ a_i, b_i \in \{0, 1, 2, \ldots, +\infty\} \).

Let \( \varphi = (t_p)_{p \in \Pi} \) be a family of t-norms on \( L = \{0, 1, 2, \ldots, +\infty\} \) with index set the set of primes \( \Pi = \{2, 3, 5, 7, 11, \ldots\} \).

**Definition 3** Given a family \( \varphi = (t_p)_{p \in \Pi} \) of t-norms on \( L \), we can define on \( \mathbb{N} \) a binary operation as follows:
\[
T_\varphi(m, n) = t_{\prod_{p|\text{gcd}(m,n)}}(a_{p|\text{gcd}(m,n)}, b_{p|\text{gcd}(m,n)})
\]
where \( m = p_1^{a_1} \cdots p_r^{a_r}, \ n = p_1^{b_1} \cdots p_r^{b_r} \).

**Proposition 5** Given a family \( \varphi = (t_p)_{p \in \Pi} \) of t-norms on \( L \), let us consider \( T_\varphi \) defined in (3). Then, \( T_\varphi \) is a t-norm on \( \mathbb{N} \) that is divisible if and only if \( t_p \) is divisible for all \( p \in \Pi \).

**Example 2**

1. \( T_\varphi(m, n) = \text{gcd}(m, n) \) is the t-norm on \( \mathbb{N} \) derived from \( t_p = t_M \) for all prime \( p \) \((t_M(a, b) = \min(a, b)) \)

2. \( T_D \) comes from \( t_p = t_D \) for all prime \( p \).

**Remark 3** If \( T \) is a t-norm on the poset \( \mathbb{N} \) then, according to Proposition 1, we have \( T(m,n) = \text{gcd}(m,n) \) for all \( m, n \in \mathbb{N} \). Thus, \( T(m,n) = p_1^{c_1} \cdots p_r^{c_r} \) with \( c_i = \min(a_i, b_i) \) where \( m = p_1^{a_1} \cdots p_r^{a_r}, \ n = p_1^{b_1} \cdots p_r^{b_r} \). What we have done in Definition 3 is to take as \( c_i \) the value of a t-norm (assigned to the prime \( p_i \)) at \( (a_i, b_i) \).

**Proposition 6** Let \( T \) be a t-norm on \( L \). Then, there exists a family \( \varphi = (t_p)_{p \in \Pi} \) of t-norms on \( L \) such that \( T = T_\varphi \) if and only if the following condition is satisfied for all primes \( p_1, \ldots, p_r \) and all \( a_1, \ldots, a_r, b_1, \ldots, b_r \in L \):
\[
T(p_1^{a_1} \cdots p_r^{a_r}, p_1^{b_1} \cdots p_r^{b_r}) = T(p_1^{a_1}, p_1^{b_1}) \cdots T(p_r^{a_r}, p_r^{b_r})
\]

(4)

Proof. Let us suppose \( T = T_\varphi \) where \( \varphi = (t_p)_{p \in \Pi} \). Then from (3) we have \( T(p_1^{a_1} \cdots p_r^{a_r}, p_1^{b_1} \cdots p_r^{b_r}) = T(p_1^{a_1}(a_1, b_1), p_1^{b_1}(a_1, b_1)) \cdots T(p_r^{a_r}(a_r, b_r), p_r^{b_r}(a_r, b_r)) \). Reciprocally, let us consider that the condition (4) holds. For each \( p \in \Pi \) we define a binary operation \( t_p \) on \( L \) as follows:
\( t_p(a, b) = c \) if and only if \( T(p^a, p^b) = p^c \). This is a well-defined operation on \( L \) which is a t-norm, and, trivially, the family \( \varphi = (t_p)_{p \in \Pi} \) is such that
\[
T(p_1^{a_1} \cdots p_r^{a_r}, p_1^{b_1} \cdots p_r^{b_r}) = T(p_1^{a_1}, p_1^{b_1}) \cdots T(p_r^{a_r}, p_r^{b_r}) = T(p_1^{a_1}(a_1, b_1), p_1^{b_1}(a_1, b_1)) \cdots T(p_r^{a_r}(a_r, b_r), p_r^{b_r}(a_r, b_r))
\]
in other words, \( T = T_\varphi \).

Analogously, the same can be done for t-conorms. Thus, the least common multiple \( S_\varphi(m, n) = \text{lcm}(m, n) \) is the t-conorm on \( \mathbb{N} \) derived from \( s_p = s_M \) for all prime \( p \) \((s_M(a, b) = \max(a, b)) \). The ordinary product \( m \cdot n \) on \( \mathbb{N} \) is the t-conorm derived from \( s_p = s_+ \forall p \) such that
\[
T_\varphi(m, n) \cdot S_\varphi(m, n) = m \cdot n \forall m, n \in \mathbb{N} \]

(5)

Obviously, a family \( \varphi = (t_p)_{p \in \Pi} \) is such that \( T_\varphi, S_\varphi \) satisfies (5) if and only if for all \( p \in \Pi \), \( (t_p, s_p) \) satisfies the Frank’s equation
\[
t_p(x, y) + s_p(x, y) = x + y, \ x, y \in L
\]
that for each \( p \) has the solution given in (2).

**Remark 4** Let us denote by \( t^I \) the divisible t-norm on \( L \) determined by \( I = \{a_0, a_1, ..., +\infty \mid 0 = a_0 < a_1 < ... < +\infty \} \). Given \( k \in L \colon 0 < k < +\infty \) then \( kI(x, y) = k^I(kx, ky) \) for all \( x, y \in L \) where \( kI = \{0 = ka_0 < ka_1 < ... < ka_m < ka_{m+1} < ... < +\infty \} \). Consider \( \varphi = (t^I(p))_{p \in \Pi} \) where \( t^I(p) \) means the divisible t-norm on \( L \) with \( I(p) \) the set of idempotent elements assigned to the prime \( p \) and \( k\varphi = (kI(p))_{p \in \Pi} \). In these conditions we can write according to (3):

\[
(T_\varphi(m, n))_k = p_1^{kI(m_1, b_{1i})}... p_r^{kI(m_r, b_{ri})} = p_1^{kI(m_1, b_{1i})}... p_r^{kI(m_r, b_{ri})} = T_{k\varphi}(m^k, n^k)
\]

where \( m = p_1^{a_1}... p_r^{a_r}, n = p_1^{b_1}... p_r^{b_r} \). That is, \( (T_\varphi(m, n))_k = T_{k\varphi}(m^k, n^k) \). In particular, if \( I(p) = L \), for all prime \( p \) then \( \gcd(m, n)^k = T_{kL}(m^k, n^k) \), where \( T_{kL} \) means the t-norm on \( N \) defined by \( (k^L)p \). Thus, the k-power of \( \gcd \) is the restriction of the t-norm \( T_{kL} \) to the domain \( \{0^k, 1^k, 2^k, 3^k, ..., +\infty \} \).

Consider \( L^\Pi = \{\Pi \to L \} \) the set of functions from \( \Pi \) to \( L \) (multisets on \( \Pi \)). It is a bounded lattice with the pointwise order \( \mu \leq \sigma \).

Let \( f : N \to L^\Pi \) be defined by:

\[
f(1) = \mu_1 = 0 \quad (0(p) = 0 \forall p \in \Pi) \quad f(\infty) = \mu_\infty = +\infty \quad (+\infty(p) = +\infty \forall p \in \Pi)
\]

\[
m \neq 1, \infty, m = p_1^{a_1}... p_r^{a_r}, f(m) = \mu_m := \begin{cases} a_i & \text{if } p = p_i, i = 1, ..., r \\ 0 & \text{otherwise} \end{cases}
\]

**Proposition 7** The function \( f \) is an isomorphism (of posets) from \( N \) to \( f(N) \).

**Remark 5** Observe that \( f(N) \) is the subset of \( L^\Pi \) that contains \( +\infty \) and such that \( +\infty \notin \text{Ran} \mu \) and \( \text{supp} \mu = \{p \in \Pi : \mu(p) \neq 0\} \) is a finite subset of \( \Pi \).

We also observe that, according to Proposition 7, most of the previous results can be formulated by means of sequences.

**Definition 4** Let \( \varphi = (t_p)_{p \in \Pi} \) a family of t-norms on \( L \) with index set \( \Pi \). From it we can define a t-norm \( \varphi_{\text{e}} \) on \( L^\Pi \) (an intersection \( \cap_{\varphi} \) of multisets on \( \Pi \)) as follows:

\[
(\mu \cap_{\varphi} \sigma)(p) = t_p(\mu(p), \sigma(p)) \forall p \in \Pi.
\]

Now, we can state the following

**Proposition 8** Let \( \varphi = (t_p)_{p \in \Pi} \) a family of t-norms on \( L \) and consider the induced intersection \( \cap_{\varphi} \) defined above, then \( f^{-1}(f(m) \cap_{\varphi} f(n)) = T_\varphi(m, n) \forall m, n \in N \), where \( T_\varphi \) is the t-norm on \( N \) defined in (3) with associated family \( \varphi = (t_p)_{p \in \Pi} \).

5 \quad T-primes and S-decomposition

**Definition 5** Let \( T \) be a t-norm on \( N \). We say that a number \( p \in N, p \geq 2 \), is \( T \)-prime if the following condition holds: \( T(m, p) = 1 \) for all \( m < p \).

**Remark 6**

i) \( \infty \) is not \( T \)-prime for all t-norm \( T \).

ii) In case \( T = \gcd \) the condition above is equivalent to the condition "the only natural numbers dividing \( p \) are 1 and \( p \) itself" (definition of prime).

iii) If the t-norm \( T \) satisfies (4) then the stated condition can be also formulated in the form: if \( p = p_1^{a_1}... p_s^{a_s} \) is the standard prime factorization of \( p \) then \( p \) is a \( T \)-prime if and only if \( T(m, p) = 1 \) for all \( m = p_1^{a_1} < p \leq p_i^{a_i+1}, \forall i = 1, ..., s \).

**Proposition 9** Denoting by \( \Pi(T) \) the set of \( T \)-primes, we have:

a) \( \Pi(\gcd) = \Pi, \Pi(T_D) = N \setminus \{1, \infty\} \).

b) If \( T_1, T_2 \) are t-norms such that \( T_1 \leq T_2 \) then \( T_1(m, n) \) divides \( T_2(m, n) \) for all \( m,n \in N \) then \( \Pi(T_2) \subset \Pi(T_1) \). In particular, \( \Pi \subset \Pi(T) \subset N \setminus \{1, \infty\} \) for all t-norm \( T \).

**Example 3** Consider the t-norm \( T \) defined by \( T(m, n) = p_1^{I(a_1, b_1)}... p_r^{I(a_r, b_r)} \) where \( m = p_1^{a_1}... p_r^{a_r}, n = p_1^{b_1}... p_r^{b_r} \) and \( t \) is the divisible t-norm on \( L = \{0, 1, 2, ..., +\infty\} \) with set of idempotent elements \( I = \{0 < k < k + 1 < k + 2 < ... < +\infty\} \). Then \( T(m, n) = 1 \) if and only if for \( i = 1, ..., r \) it is \( a_i + b_i \leq k \) or \( \min(a_i, b_i) = 0 \). In this case, and according to Remark 6, iii), we can state: \( p = p_1^{b_1}... p_s^{b_s} \) is \( T \)-prime if and only if for all \( i = 1, ..., s \) it
is either \( \alpha_i = 0 \) or \( a_i + b_i \leq k \) where \( \alpha_i = \max \{ \alpha : p_i^\alpha < p \} \). In particular,

1) \( p^a, p \) prime, \( a \geq 1 \), is \( T \)-prime if and only if \( a \leq k^{k+1} \).

2) \( p_1 p_2 \) with \( p_1 < p_2 \) primes, is \( T \)-prime if and only if \( p_2 \leq p_1^{k-1} \).

**Example 4** Let \( T \) be the t-norm given in example 3 with \( k = 2 \), that is, \( t(a, b) = \min(a, b) \) for all \( (a, b) \neq (1, 1) \) and \( t(1, 1) = 0 \). Then \( \Pi(T) = \Pi \).

It is clear the role of the Fundamental Theorem of Arithmetic played throughout this paper. In this way, we will finish it with a Proposition that provides a generalization of the mentioned theorem in terms of t-norms and t-conorms.

**Proposition 10 (existence of \( T \)-prime \( S \)-decomposition)**

Let \( T \) be a t-norm and \( S \) a divisible t-conorm on \( N \). For each natural number \( n \notin \{ 1, \infty \} \) there exist determined \( T \)-primes \( p_1, p_2, \ldots, p_r \), \( r \geq 1 \), such that \( n \) can be written in one of the following forms:

a) \( n = S(p_1, p_2, \ldots, p_r) \)

b) \( n = S(p_1, p_2, \ldots, p_r, n_r) \) where \( n_r \) is a determined not \( T \)-prime number such that \( n_r = S(p_r, n_r) \).

Proof. If \( n \) itself is a \( T \)-prime there is nothing to do: \( n = S(n) \) and \( n \) can be written in the form a). If not, consider \( m_1 = \min \{ m < n : T(m, n) > 1 \} \). Let us show that \( T(m_1, n) \) is \( T \)-prime. Suppose that there exists \( m < T(m_1, n) \) such that \( T(m, T(m_1, n)) > 1 \), then \( 1 < T(m, T(m_1, n)) = T(T(m, m_1), n) \), but we have \( m < m_1 \) and \( m < n \) therefore \( T(m, m_1) < n \) and from definition of \( m_1 \) we have a contradiction. Now we can write \( n = S(p_1, n_1) \) where \( p_1 = T(m_1, n) \) is \( T \)-prime and \( n_1 = \min \{ q \in N : n = S(p_1, q) \} \). We can repeat the arguments for \( n_1 \): if \( n_1 \) is not \( T \)-prime we consider \( m_2 = \min \{ m < n_1 : T(m, n_1) > 1 \} \) then \( n = S(p_1, p_2, n_2) \) where \( p_2 = T(m_2, n_1) \) is \( T \)-prime, \( n_2 = \min \{ q \in N : n_1 = S(p_2, q) \} \) and so on. Since natural numbers cannot decrease forever, we eventually get a \( S \)-decomposition of \( n \) in one of the forms described in a) and b).

Observe that in case \( T = \gcd \) and \( S = \text{Prod} \) the decomposition above is the classical one: let us suppose \( n \neq 1 \), if \( n \) is prime nothing to do. If not, \( m_1 = \min \{ m < n : \gcd(m, n) > 1 \} \) is such that \( p_1 = \gcd(m_1, n) \) is the smallest non-trivial divisor of \( n \). It is prime and \( n = p_1 \cdot n_1 \). Repeating the process for \( n_1 \) we obtain \( n = p_1 \cdot p_2 \cdot n_2 \), and so on.

**Example 5** Let \( T \) be the t-norm given in example 3 with \( k = 5 \) and the corresponding t-conorm \( S \) given in (5): \( S(m, n) = \frac{m \times n}{T(m, n)} \). Let us find the \( T \)-prime \( S \)-decomposition of 216 according to the process given in Proposition 10. It is \( m_1 = \min \{ m < 216 : T(m, 216) > 1 \} = 8 \), thus \( 216 = S(T(8, 216), n_1) = S(2, 108) \). We repeat the decomposition for \( 108 \) obtaining \( 108 = S(2, 54) \) then \( 216 = S(2, 2, 54) \) and so on. The final decomposition is \( 216 = S(2, 2, 3, 18) \) where 2, 3, and 18 are \( T \)-primes.

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**References**


