Abstract

The notion of a linear space is generalized to the case where the underlying algebra is a commutative monoid and the set of scalars is a semiring reduct of a BL-algebra or an MV-algebra. The notions of linear dependence and independence are also introduced and necessary and sufficient conditions when vectors form a basis of a semi-linear space are given. A linear mapping between two semi-linear spaces is defined and considered in the case when it is represented by a fuzzy relation. An example of a fuzzy system which can be characterized by fuzzy IF-THEN rules is modeled using the respective linear mapping.

Keywords: semi-linear space, semiring, BL-algebra, MV-algebra

Introduction

The notion of a linear space is one of the central notions in mathematics and its applications. Therefore, generalization of the linear space to a weaker structure, such as a commutative monoid over a semiring or over a BL-algebra is of interest. From the application point of view, these spaces may be suitable for solving semi-linear equations and systems of semi-linear equations with fuzzy coefficients.

Preliminaries

In the sequel, we will use the BL, dual BL and MV algebras as basic structures for “arithmetic” operations over elements of lattice ordered sets. The definitions given below summarize definitions originally introduced in [5, 7].
Definition 1
A BL-algebra is an algebra
\[ \mathcal{L} = \langle L, \land, \lor, \ast, \rightarrow, 0, 1 \rangle \]
with four binary operations and two constants such that
(i) \( \langle L, \land, \lor, 0, 1 \rangle \) is a lattice with 0 and 1 as the least and greatest elements w.r.t. the lattice ordering,
(ii) \( \langle L, \ast, 1, \lor, \land \rangle \) is a commutative lattice ordered monoid,
(iii) \( \ast \) and \( \rightarrow \) form an adjoint pair, i.e.
\[ z \leq (x \rightarrow y) \iff x \ast z \leq y \]
for all \( x, y, z \in L \),
(iv) \( x \ast (x \rightarrow y) = x \land y, \)
\( (x \rightarrow y) \lor (y \rightarrow x) = 1 \)
for all \( x, y \in L \).
The following operations of negation and biresiduation can be additionally defined:
\[ \neg x = x \rightarrow 0 \]
\[ x \leftrightarrow y = (x \rightarrow y) \land (y \rightarrow x). \]

The well known examples of BL-algebra are Gödel, Lukasiewicz and product algebras on \([0, 1]\). Moreover, it is known that in the case \( L = [0, 1] \), the operation \( \ast \) is a continuous t-norm.

The notion of a dual BL-algebra has been introduced in [7] as a special case of a dually residuated lattice ordered monoids.

Definition 2
A dual BL-algebra is an algebra
\[ \mathcal{L}^d = \langle L, \lor, \land, \oplus, \ominus, 1, 0 \rangle \]
with four binary operations and two constants such that
(i) \( \langle L, \lor, \land, 1, 0 \rangle \) is a lattice with 1 and 0 as the greatest and least elements w.r.t. the lattice ordering,
(ii) \( \langle L, \oplus, 0, \land, \lor \rangle \) is a commutative lattice ordered monoid,
(iii) for all \( x, y, z \in L \)
\[ z \geq (y \ominus x) \iff x \oplus z \geq y, \]
\[ x \oplus (y \ominus x) = x \lor y, \]
\[ (x \ominus y) \land (y \ominus x) = 0. \]

Let us give an example of the dual to Gödel algebra on \([0, 1]\):

Example 1
\( \mathcal{L}^d_G = ([0, 1], \lor, \land, \oplus_G, \ominus_G, 1, 0) \) and
\[ x \oplus_G y = x \lor y, \]
\[ x \ominus_G y = \begin{cases} y, & x < y, \\ 0, & x \geq y. \end{cases} \]

Finally, we will introduce MV-algebras as special BL-algebras (see e.g. [5]).

Definition 3
A BL-algebra \( \mathcal{L} = \langle L, \lor, \land, \ast, \rightarrow, 0, 1 \rangle \) where the double negation law
\[ x = \neg(\neg x), \]
is valid for all \( x \in L \), is called an MV-algebra.

A typical example of an MV-algebra is Lukasiewicz algebra on \([0, 1]\).

Main Definitions

In this section we will give definitions of semi-rings and semi-linear spaces and consider numerous examples of both structures.

Definition 4
A semiring \( \mathcal{R} = \langle R, +, \cdot, 0, 1 \rangle \) is an algebra where
(SR1) \( \langle R, +, 0 \rangle \) is a commutative monoid,
(SR2) \( \langle R, \cdot, 1 \rangle \) is a monoid,
(SR3) for all \( a, b, c \in R \)
\[ a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a. \]
A semiring is commutative if \((R, \cdot, 1)\) is a commutative monoid.
A semiring is a semiring with annihilator if the neutral element of \((R, +, 0)\) is an annihilator, i.e.
\[ 0 \cdot a = a \cdot 0 = 0 \]
for all \(a \in R\).

A typical example of a commutative semiring is a set \(\mathbb{N}\) of non-negative integers with addition and multiplication. Below, we will give examples of other semirings which can be taken as reducts of BL-algebras, dual BL-algebras and MV-algebras.

**Example 2**
1. Let \(L = \langle L, \lor, \land, *, \rightarrow, 0, 1 \rangle\) be a BL-algebra. Then its \(\lor\)-reduct
\[ L_\lor = \langle L, \lor, *, 0, 1 \rangle \]
is a commutative semiring.

2. Let again \(L\) be a BL-algebra. The following algebra (\(\land\)-reduct of \(L\)) is a commutative semiring:
\[ L_\land = \langle L, \land, *, 1, 1 \rangle. \]

3. Let \(L^d = \langle L, \lor, \land, \oplus, \ominus, 1, 0 \rangle\) be a dual BL-algebra. Then its \(\land\)-reduct
\[ L^d_\land = \langle L, \land, \oplus, 1, 0 \rangle \]
is a commutative semiring.

4. Let again \(L^d\) be a dual BL-algebra. The following algebra (\(\lor\)-reduct of \(L^d\)) is a commutative semiring:
\[ L^d_\lor = \langle L, \lor, *, 1, 1 \rangle. \]

5. Let \(L = \langle L, \lor, \land, \oplus, \ominus, \neg, 0, 1 \rangle\) be an MV-algebra. Then its \(\lor\)-reduct
\[ L_\lor = \langle L, \lor, \ominus, 0, 1 \rangle \]
and its \(\land\)-reduct
\[ L_\land = \langle L, \land, \oplus, 1, 0 \rangle. \]
are commutative semirings.

Let us remark that the semirings in Example 2 are the idempotent semirings ([6]), because their operation of “addition” is idempotent.

**Definition 5**
Let \(A \neq \emptyset\) be a set of elements and \(\mathcal{R} = \langle R, +, \cdot, 0, 1 \rangle\) a semiring. We say that \(A\) is a (left) semimodule over \(\mathcal{R}\) if two operations are defined:

(a) addition \(+\) such that for each two elements \(a, b \in A\) there is a uniquely determined element \(a + b \in A\) called their sum,
(b) multiplication \(\cdot\) by an element from \(R\) such that for any \(a \in A\) and \(p \in R\) there is a uniquely determined element \(p \cdot a\) called their product.

These operations fulfil the following properties for all \(a, b, c \in A\) and \(p, q \in R\):

(SL1) \(a + b = b + a\),
(SL2) \(a + (b + c) = (a + b) + c\),
(SL3) there exists the (neutral) element \(0 \in A\) such that \(a + 0 = a\),
(SL4) \(p \cdot (a + b) = p \cdot a + p \cdot b\),
(SL5) \((p + q) \cdot a = p \cdot a + q \cdot a\),
(SL6) \(p \cdot (q \cdot a) = (p \cdot q) \cdot a\),
(SL7) \(1 \cdot a = a\).

A right semimodule over \(\mathcal{R}\) may be defined analogously.

A nonempty subset \(B\) of a left (right) semimodule \(A\) over \(\mathcal{R}\) is called a subsemimodule if \(B\) is closed under addition and multiplication.

**Definition 6**
Let semiring \(\mathcal{R}\) be a reduct of a BL-algebra (dual BL-algebra, MV-algebra) \(L = \langle L, \lor, \land, *, \rightarrow, 0, 1 \rangle\). Then a semimodule over \(L\) is called a semilinear space.

The elements of a semilinear space are called vectors and elements of \(L\) scalars.
Example 3
1. Let \( \mathcal{L} = (L, \lor, \land, *, \to, 0, 1) \) be a BL-algebra (MV-algebra) on \( L \), \( \mathcal{L}_\lor = (L, \lor, *, 0, 1) \) its semiring reduct. Let us consider the set of all \( n \)-dimensional vectors \( A = L^n, n \geq 1 \), and define
\[
(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 \lor b_1, \ldots, a_n \lor b_n),
\]
\[
p \cdot (a_1, \ldots, a_n) = (p \ast a_1, \ldots, p \ast a_n)
\]
where \( p \in L \). The neutral element in \( A \) is the vector \((0, \ldots, 0)\). It is easy to see that \( L^n \) is a semi-linear space over \( \mathcal{L}_\lor \).

2. Let \( X \neq \emptyset \), \( \mathcal{L} \) be a BL-algebra (MV-algebra) on \( L \) and \( \mathcal{L}_\lor = (L, \lor, *, 0, 1) \) its semiring reduct. Let us consider the set of all \( L \)-valued functions \( A = L^X \) and put
\[
\begin{align*}
A(x) + B(x) &= A(x) \lor B(x), \\
p \cdot A(x) &= p \ast A(x)
\end{align*}
\]
where \( p \in L \). The neutral element in \( A \) is the function which is identically equal to 0. It is easy to see that \( L^X \) is a semi-linear space over \( \mathcal{L}_\lor \).

3. Let \( X \neq \emptyset \), \( \mathcal{L} \) be a BL-algebra (MV-algebra) on \( L \) and \( \mathcal{L}_\lor = (L, \lor, *, 0, 1) \) its semiring reduct. Let \( S \) be a similarity on \( X \) given by its membership function \( S : X \times X \to L \). We consider a set \( E_S = \{ f | X \to L \} \) of those \( L \)-valued functions \( f \) which are \( k \)-extensional for some \( k = 1, 2, \ldots \) with respect to \( S \). Recall that \( k \)-extensionality of a function \( f \) with respect to \( S \) means that for all \( x, y \in X \)
\[
S^k(x, y) \leq f(x) \leftrightarrow f(y)
\]
holds true. It is easy to see that \( E \) is a semi-linear subspace of the semi-linear space \( A = L^X \) over \( \mathcal{L}_\lor \).

Linear Dependence and Independence

Let \( A \) be some left semi-linear space over a semiring \( \mathcal{R} \). We will denote vectors from \( A \) by bold characters to distinguish them from scalars.

By a linear combination of vectors \( a_1, \ldots, a_n \in A \) we mean the following expression
\[
\alpha_1 \cdot a_1 + \cdots + \alpha_n \cdot a_n
\]
where \( \alpha_1, \ldots, \alpha_n \in \mathcal{R} \) are scalars called also coefficients. This linear combination uniquely determines a certain vector from \( A \).

Definition 7

By the definition, a single vector \( a \) is linearly independent. Vectors \( a_1, \ldots, a_n, n \geq 2 \), are linearly independent if none of them can be represented by a linear combination of the others.

Otherwise, we say that vectors \( a_1, \ldots, a_n \) are linearly dependent.

An infinite set of vectors is linear independent if any finite subset of it is linear independent.

Example 4
1. Let \( \mathcal{L} \) be a BL-algebra (MV-algebra). In the semi-linear space of \( n \)-dimensional vectors \( A = L^n \) over the semiring \( \mathcal{L}_\lor \), the following vectors are linearly independent:
\[
\begin{align*}
e_1 &= (1, 0, 0, \ldots, 0) \\
e_2 &= (0, 1, 0, \ldots, 0) \\
&\ldots \\
e_n &= (0, 0, 0, \ldots, 1)
\end{align*}
\]

2. Let \( \mathcal{L} \) be an MV-algebra. In the semi-linear space of \( n \)-dimensional vectors \( A = L^n \)
over the semiring $\mathcal{L}$, the following vectors are linearly independent:

\[
\begin{align*}
  f_1 &= (0, 1, 1, \ldots, 1) \\
  f_2 &= (1, 0, 1, \ldots, 1) \\
  \vdots \\
  f_n &= (1, 1, 1, \ldots, 0)
\end{align*}
\]

In the rest of this section we will confine ourselves to the semi-linear space $L^n$ over $\mathcal{L}$ where $\mathcal{L}$ is a BL-algebra.

**Theorem 1**

Let $A = L^n$ be the semi-linear space of $n$-dimensional vectors over $\mathcal{L}$ where $\mathcal{L}$ is a BL-algebra. Let vector $b = (b_1, \ldots, b_n) \in L^n$ be represented by a linear combination of vectors $a_1, \ldots, a_m \in L^n$ where $a_i = (a_{i1}, \ldots, a_{il})$, $i = 1, \ldots, m$. Then $b$ can be represented by the linear combination of $a_1, \ldots, a_m$ with coefficients

\[
\hat{x}_j = \bigwedge_{i=1}^n (a_{ij} \rightarrow b_i), \quad j = 1, \ldots, m. \tag{2}
\]

**Corollary 1**

Let $A = L^n$ be the semi-linear space of $n$-dimensional vectors over $\mathcal{L}$ where $\mathcal{L}$ is a BL-algebra. Then a vector $b \in L^n$ can be represented by a linear combination of vectors $a_1, \ldots, a_m \in L^n$ if and only if the system (2) is solvable and the greatest solution is given by (2).

It is worth noticing that if a vector $b \in L^n$ can be represented by a linear combination of vectors $a_1, \ldots, a_m \in L^n$ then the representation is not necessarily unique.

**Corollary 2**

Let the conditions of Theorem 1 be fulfilled and $b = (b_1, \ldots, b_n) \in L^n$ be a vector represented by a linear combination of vectors $a_1, \ldots, a_m$. Then for each $j = 1, \ldots, n$

\[
b_j \leq a_{1j} \lor \cdots \lor a_{mj}. \tag{3}
\]

**Corollary 3**

Let $A = L^n$ be the semi-linear space of $n$-dimensional vectors over $\mathcal{L}$ where $\mathcal{L}$ is a BL-algebra. Then the zero vector $0 = (0, \ldots, 0) \in L^n$ is representable by the linear combination of arbitrary vectors $a_1, \ldots, a_m \in L^n$ with the respective coefficients

\[
\hat{x}_i = \bigwedge_{j=1}^n -a_{ij}, \quad i = 1, \ldots, m.
\]

By the criterion, suggested below, it is possible to investigate whether the given system of vectors is linear independent.

**Theorem 2**

Let $A = L^n$ be the $\mathcal{L}$-semilinear space of $n$-dimensional vectors where $\mathcal{L}$ is a BL-algebra. Vectors $a_1, \ldots, a_m \in L^n$, $m \geq 2$, are linearly independent if and only if

\[
(\forall l \in \{1, \ldots, m\})(\exists i \in \{1, \ldots, n\}) \left( a_{li} \not\leq \bigvee_{j=1, j \neq l}^n a_{ji} * \left( \bigwedge_{k=1}^n a_{jk} \rightarrow a_{lk} \right) \right). \tag{4}
\]

**Corollary 4**

Let $A = [0,1]^n$ be the $\mathcal{L}$-semilinear space of $n$-dimensional vectors and $\mathcal{L}$ be a BL-algebra on $[0,1]$. Vectors $a_1, \ldots, a_m \in L^n$ are linearly independent if and only if

\[
(\forall l \in \{1, \ldots, m\})(\exists i \in \{1, \ldots, n\}) \left( a_{li} \not> \bigvee_{j=1, j \neq l}^n a_{ji} * \left( \bigwedge_{k=1}^n a_{jk} \rightarrow a_{lk} \right) \right).
\]

**Basis in a Semi-linear Space**

**Definition 8**

A linear independent set of generators of a semi-linear space $A$ is called a basis of $A$.

In what follows we fix a BL-algebra $\mathcal{L}$ and consider the semi-linear space of $n$-dimensional vectors $A = L^n$ over $\mathcal{L}$.

An example of a basis in $L^n$ is given by vectors $e_1, \ldots, e_n$ in Example 3 (cf. (1)). It immediately follows from the last definition that an arbitrary element from a semi-linear space $L^n$ can be represented by a linear combination of elements of its basis.

We investigate conditions assuming or guaranteeing that vectors $a_1, \ldots, a_m \in L^n$ form a basis.
Suppose that a vector \( b = (b_1, \ldots, b_n) \in L^n \) is represented by a linear combination of \( a_1, \ldots, a_m \). Then there exist coefficients \( x_1, \ldots, x_m \in L \) such that
\[
b = x_1 \ast a_1 \lor \cdots \lor x_m \ast a_m
\]

If we take \( b = (1, \ldots, 1) \) then the necessary condition on basic vectors can be obtained:
\[
a_{ij} \lor \cdots \lor a_{mj} = 1, \quad j = 1, \ldots, n.
\]

In the below given theorem, we will prove a necessary and sufficient condition that vectors \( a_1, \ldots, a_m \) constitute a basis of \( L^n \).

**Theorem 3**

Let \( A = L^n \) be the semi-linear space of \( n \)-dimensional vectors over \( L \lor \) where \( L \) is a BL-algebra. Vectors \( a_1, \ldots, a_m \in L^n \) form a basis in \( A \) if and only if they are linear independent and
\[
\bigwedge_{j=1}^{n} \left( \bigvee_{i=1}^{m} a_{ij} \ast \left( \bigwedge_{l=1,l\neq j}^{n} -a_{il} \right) \right) = 1. \tag{5}
\]

We will deduce the necessary condition on a basis of \( L^n \).

**Corollary 5**

Let \( A = L^n \) be the semi-linear space of \( n \)-dimensional vectors over \( L \lor \) where \( L \) is a BL-algebra. If vectors \( a_1, \ldots, a_m \in L^n \) form a basis in \( A \) then they are linear independent and
\[
(\forall j \in \{1, \ldots, n\})(\exists i \in \{1, \ldots, m\})
\left( a_{ij} \notin \bigvee_{l=1,l\neq j}^{n} a_{il} \right).
\]

It turns out that when \( L \) is linearly ordered, a criterion that vectors \( a_1, \ldots, a_m \) constitute a basis of \( L^n \) has a very simple form.

**Theorem 4**

Let a support set \( L \) of basic algebras (BL or MV) be linearly ordered. Then in the semi-linear space of \( n \)-dimensional vectors \( L^n \) over the semiring \( L \lor \), the unique basis is given by (1).

Let us investigate the problem whether a system of linearly independent vectors can be extended to a basis. The simple example shows that this is not always the case.

**Lemma 1**

Let \( A = L^n \) be the semi-linear space of \( n \)-dimensional vectors over \( L \lor \) where \( L \) is a BL-algebra. Then the system of linearly independent vectors
\[
a_1 = (a, 0, 0, \ldots, 0)
\]
\[
a_2 = (0, a, 0, \ldots, 0) \tag{6}
\]
\[
\ldots\ldots\ldots
\]
\[
a_n = (0, 0, 0, \ldots, a).
\]
cannot be extended to a basis of \( L^n \).

**Linear Mappings (Homomorphisms)**

**Definition 9**

Let \( R = \langle R, +, \cdot, 0, 1 \rangle \) be a semiring and \( A, B \) semi-linear spaces over \( R \). A mapping \( H : A \rightarrow B \) is linear (a homomorphism) if the followings holds:

(i) \( H(m + m') = H(m) + H(m') \), for all \( m, m' \in A \).

(ii) \( H(r \cdot m) = r \cdot H(m) \), for all \( m \in A \) and \( r \in R \).

**Example 5**

Let \( L = \langle L, \lor, \land, *, \rightarrow, 0, 1 \rangle \) be a BL-algebra on \( L \) and \( L \lor = \langle L, \lor, *, 0, 1 \rangle \) its semiring reduct. Let \( A = L^n \), \( B = L^m \) be the semi-linear spaces of \( n \) (respectively, \( m \))-dimensional vectors and \( H \in L^{m \times n} \), \( H = (h_{ij}) \) be a matrix. For each \( a \in A \), \( a = (a_1, \ldots, a_n) \) we define
\[
H(a)_i = \bigvee_{j=1}^{n} h_{ij} \ast a_j, \quad i = 1, \ldots, m.
\]

It is easy to see that \( H : A \rightarrow B \) is a linear mapping.

The following theorem can be proved in the same way as in the case of linear spaces.
Theorem 5
Let $\mathcal{R} = \langle \mathcal{R}, +, \cdot, 0, 1 \rangle$ be a semiring and $A, B$ semi-linear spaces over $\mathcal{R}$ with bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ respectively. Then each linear mapping $H : A \rightarrow B$ can be represented by an $m \times n$-matrix $H \in \mathbb{R}^{mn}$ such that for any $a \in A$, $b \in B$:

$$H(a) = \bigvee_{j=1}^{m} (H \cdot \alpha)_j \cdot b_j$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $(H \cdot \alpha)_j = \bigvee_{i=1}^{n} (H_{ji} \cdot \alpha_i)$.

Oppositely, each matrix $H \in \mathbb{R}^{nm}$ represents a linear mapping between the semi-linear spaces $A, B$ such that (7) holds.

Theorem 5 characterizes linear mappings between semi-linear spaces with finite bases. Below we will consider the case where semi-linear spaces are spaces of functions without finite bases and characterize linear mappings between them by fuzzy relations.

For the rest of this section, let $X, Y \neq \emptyset$, $\mathcal{L}$ be a complete BL-algebra on $L$ and $\mathcal{L}_\vee$ its semiring reduct. Let $A = L^X$, $B = L^Y$ be semi-linear spaces of $L$-valued functions on $X$ and $Y$ respectively (see Example 3).

Definition 10 \([2]\)
Let $H : A \rightarrow B$ be a mapping between semi-linear spaces of $L$-valued functions on $X$ and $Y$ respectively. If

$$H(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} H(f_i)$$

holds for every family $\{f_i | i \in I\}$ of elements in $L^X$ then $H$ is called sup-continuous.

Theorem 6
Let $A = L^X$, $B = L^Y$ be semi-linear spaces of $L$-valued functions over $\mathcal{L}_\vee = \langle L, \vee, \cdot, 0, 1 \rangle$ and $R \in L^{X \times Y}$ be a fuzzy relation. The mapping $H_R : A \rightarrow B$ given by

$$H_R(f)(y) = \bigvee_{x \in X} (f(x) \cdot R(x, y))$$

is a homomorphism. Moreover, $H_R$ is sup-continuous.

PROOF: Let $f, g \in L^X$ and $r \in L$. To prove that $H_R$ is a homomorphism it is sufficient to verify (i) and (ii) in Definition 9.

To prove that $H_R$ is a homomorphism we will verify the property (8). Let $\{f_i | i \in I\}$ be a family of elements in $L^X$. For all $y \in Y$ we prove:

$$H_R(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \left( (\bigvee_{x \in X} f_i(x)) \cdot R(x, y) \right) = \bigvee_{i \in I} \left( \bigvee_{x \in X} (f_i(x) \cdot R(x, y)) \right) = \bigvee_{i \in I} H_R(f_i).$$

\[ \square \]

Remark 1
It is easy to see that the homomorphism (9) is expressed with help of the so called sup-t composition between a fuzzy set represented by its membership function $f$ and the fuzzy relation $R$.

If we compare Theorem 6 with Theorem 5 where the representation has been established in both directions, we may ask: which homomorphism between semi-linear spaces of $L$-valued functions can be represented by a sup-t composition with some fuzzy relation? That is: given a homomorphism $H : L^X \rightarrow L^Y$, which conditions on $H$ assure us that there exists a fuzzy relation $R(x, y)$ such that $H = H_R$ (cf. (9)). If such a relation exists, we say that $H$ is representable by a fuzzy relation, or, shortly, that $H$ is representable \([2]\).

Theorem 7 \([2]\)
If $H : L^X \rightarrow L^Y$ is a sup-continuous homomorphism then $H$ is representable, i.e. there exists a fuzzy relation $R \in L^{X \times Y}$ such that $H = H_R$.

Corollary 6
A sup-continuous homomorphism $H : L^X \rightarrow L^Y$ is representable if and only if there exists a fuzzy relation $R \in L^{X \times Y}$ such that $H = H_R$.

The following example \([2]\) shows that representable homomorphisms play an essential role in the analysis of fuzzy systems because using them, the sup-t composition between a fuzzy set and a fuzzy relation as well as the compositional rule of inference can be defined.
Example 6
Let us consider the inverted pendulum control problem. This system can be approximately modeled by a non-linear differential equation

\[ \ddot{y} - 10 \sin y = 0 \]

where \( y \) depends on time \( t \). The system can be controlled using a non-linear function \( u(t) = h(e(t), \dot{e}(t)) \) where \( e(t) = -y(t) \) is a pendulum deviation from the vertical position. We will replace it by means of a fuzzy PD-controller.

In our terms this means that we must construct a fuzzy system \( R \subseteq X \times Y \times Z \) where \( X \) is the range of deviation \( e(t) \), \( Y \) is the range of its velocity \( \dot{e}(t) \) and \( Z \) is the range of control action \( u(t) \) (this is a momentum sent to the inverted pendulum). The fuzzy control is given by a fuzzy function \( F \) defined by

\[ F(f_e(t), f_{\dot{e}}(t)) = H_R(f_e(t), f_{\dot{e}}(t)) \]

where \( f_e(t) \in X, f_{\dot{e}}(t) \in Y \) are singletons. \( F \) is actually a homomorphism representable by a certain fuzzy relation \( R \). For each \( t \), \( F(f_e(t), f_{\dot{e}}(t)) \subseteq Z \) is a fuzzy set that approximates value of the control function \( u(t) = h(e(t), \dot{e}(t)) \), i.e. each element \( z \in Z \) belonging to it in non-zero membership degree approximates \( u(t) \). Concretely, we take \( z = z_0 \) where \( z_0 \) is obtained by a defuzzification of \( F(f_e(t), f_{\dot{e}}(t)) \) (usually Center Of Gravity). Figure 1 demonstrates that this simple fuzzy system indeed controls the inverted pendulum.

Figure 1: Fuzzy control of inverted pendulum.

Conclusion

In this contribution we have introduced the notion of a semi-linear space and gave numerous examples of it. We have also defined the notions of linear dependence and independence and suggested conditions (necessary or sufficient) when vectors form a basis of a semi-linear space. A special attention has been paid to linear mappings and their representations characterized by fuzzy relations. An example of a fuzzy system which can be characterized by fuzzy IF-THEN rules is modeled by the respective linear mapping. Moreover, we have shown that this model is always used when a computation is based on the compositional rule of inference with the help of the sup-t composition between a fuzzy set and a fuzzy relation.

References