Strong implications from continuous uninorms

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Abstract

This work deals with strong implications (S-implications in short) derived from uninorms continuous at $]0,1[^2$. The general expression of such implications is found and several properties are studied. In particular, the distributivity of the S-implications over conjunctive and disjunctive uninorms is investigated.

Keywords: Uninorm, t-norm, t-conorm, S-implication, R-implication, strong negation, distributivity.

1 Introduction

The most usual kinds of implication functions used in fuzzy logic are strong implications or S-implications given by

$$I(x,y) = S(N(x),y) \quad \text{for all} \quad x,y \in [0,1],$$

where N is a strong negation, and residual implications or R-implications given by

$$I(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$$

for all $x,y \in [0,1]$. Commonly, these implications are performed by t-norms and t-conorms (see for instance [6]) and they are successfully used in several aggregation problems, like aggregation of fuzzy relations, mathematical morphology and others.

On the other hand, uninorms are a special kind of aggregation operators that have proved to be useful in many fields like expert systems, neural networks, aggregation, fuzzy system modelling, measure theory, mathematical morphology, etc. They are interesting because of their structure as a special combination of a t-norm and a t-conorm (see [2] or [6]), and because they must be conjunctive ($U(1,0)=0$) or disjunctive ($U(1,0)=1$). This allows to define fuzzy implications functions from uninorms and, in fact, several studies have been made in this direction in [1] and [8]. In these references only three classes of uninorms are used, namely, uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, representable uninorms and idempotent ones.

However, another important class, that includes representable uninorms, is the class of uninorms continuous in $]0,1[^2$ introduced and characterized in [5]. This work wants to deal with strong implications derived from this kind of uninorms (that we will call here continuous uninorms although they are continuous only in the open square $]0,1[^2$). The study of R-implications is not included for lack of space. The uninorm used to obtain S-implications must be disjunctive and thus two possible kinds of continuous uninorms work. We give the expression of S-implications for both kinds of continuous uninorms. Moreover, several properties are studied and specially the distributivity of such implications over conjunctive and disjunctive uninorms.

2 Preliminaries

We suppose the reader to be familiar with basic results concerning t-norms and t-conorms that can be found, for instance, in [6]. Anyway, given any $a, b \in]0,1[$, we will use the following notations: $\varphi_a : [0,a] \to [0,1]$ will denote the increasing bijection given by $\varphi_a(x) = x/a$, and $\psi_{a,b} : [a,b] \to [0,1]$ the one given by $\psi_{a,b}(x) = \frac{x-a}{b-a}$. Moreover, given any increasing bijection $\psi : [a,b] \to [0,1]$ and any binary operator $F : [0,1]^2 \to [0,1]$, $F_\psi : [a,b]^2 \to [a,b]$ called the ψ-transform of F, will denote the operator given by

$$F_\psi(x,y) = \psi^{-1}(F(\psi(x), \psi(y))).$$

Definition 1 *A* uninorm *is a two-place function $U : [0,1] \times [0,1] \longrightarrow [0,1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0,1]$, called* neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.*

It is clear that the function U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0,1\}$ and a uninorm U is called *conjunctive* when $U(1,0) = 0$ and *disjunctive* when $U(1,0) = 1$. The structure of any uninorm U with neutral element $e \in]0,1[$ is always as follows. It works like a t-norm in the interval $[0,e]$, like a t-conorm in the interval $[e,1]$ and it takes values between the minimum and the maximum in all other cases.

The known classes of uninorms most commonly used are:

- Uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, those with the 0-section and 1-section continuous except perhaps at the point $x = e$.
- Idempotent uninorms, those satisfying $U(x,x) = x$ for all $x \in [0,1]$.
- The class $\mathcal{U}(e)$, introduced in [4], given by those uninorms with neutral element e satisfying $U(x,y) \in \{x,y\}$ for all x, y such that $\min(x,y) \le e \le \max(x,y)$. Note that this class includes both previous classes.
- Representable uninorms, see definition 2 below.
- Continuous in $]0,1[^2$, characterized in [5].

Because we will mainly use the last two classes of uninorms in the paper, we recall them.

Definition 2 ([2]) *A uninorm U with neutral element $e \in]0,1[$ is said to be* representable *if there is an increasing continuous mapping $h : [0,1] \to [-\infty, +\infty]$ (called an* additive generator *of U), with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by*

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$.

Remark 1 *Representable uninorms were initially introduced under another name in [3]. A representable uninorm is clearly continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$, and strictly increasing in $]0,1[^2$. Moreover, there exists a strong negation N with fixed point e such that for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$*

$$U(x,y) = N(U(N(x), N(y))).$$

This negation N is given by $N(x) = h^{-1}(-h(x))$, where h is an additive generator of U.

Uninorms continuous in $]0,1[^2$ were characterized in [5] as follows.

Theorem 1 *([5]) Suppose U is a uninorm continuous in $]0,1[^2$ with neutral element $e \in]0,1[$. Then either one of the following cases is satisfied:*

(a) There exist $\lambda \in [0,e[$, $u \in [0,\lambda]$, two continuous t-norms T and T' and a representable uninorm U^R such that U can be represented as $U(x,y) =$

$$\begin{cases} T_{\varphi_u}(x,y) & \text{if } x,y \in [0,u] \\ T'_{\psi_{u,\lambda}}(x,y) & \text{if } x,y \in [u,\lambda] \\ {U^R}_{\psi_{\lambda,1}}(x,y) & \text{if } x,y \in]\lambda,1[\\ 1 & \text{if } \min(x,y) \in]u,1] \\ & \text{and } \max(x,y) = 1 \\ \min(x,y) \text{ or } 1 & \text{if } (x,y) = (u,1) \\ & \text{or } (x,y) = (1,u) \\ \min(x,y) & \text{elsewhere.} \end{cases} \quad (1)$$

(b) There exist $v \in]e,1]$, $\omega \in [v,1]$, two continuous t-conorms S and S' and a repre-

sentable uninorm U^R such that U can be represented as $U(x,y) =$

$$\begin{cases} S_{\psi_{v,\omega}}(x,y) & \text{if } x,y \in [v,\omega] \\ S'_{\psi_{\omega,1}}(x,y) & \text{if } x,y \in [\omega,1] \\ {U^R}_{\varphi_v}(x,y) & \text{if } x,y \in]0,v[\\ 0 & \text{if } \max(x,y) \in [0,\omega[\\ & \text{and } \min(x,y) = 0 \\ \max(x,y) \text{ or } 0 & \text{if } (x,y) = (0,\omega) \\ & \text{or } (x,y) = (\omega,0) \\ \max(x,y) & \text{elsewhere.} \end{cases} \quad (2)$$

Denote by $\mathcal{CU}$ the class of these uninorms and, particularly, by $\mathcal{CU}^{\min}$ the class of uninorms with form (1) and by $\mathcal{CU}^{\max}$ the class of uninorms with form (2). A uninorm U in $\mathcal{CU}^{\min}$ (or in $\mathcal{CU}^{\max}$) will be denoted as $U = (e,u,\lambda,T,T',U^R)$ (or $U = (e,v,\omega,U^R,S,S')$) to represent its parameters.

Remark 2 *Any uninorm U in $\mathcal{CU}^{\min}$ with $u = 0$ or U in $\mathcal{CU}^{\max}$ with $v = 1$ is a representable uninorm.*

Definition 3 *A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication operator, or an implication, if it satisfies:*

I1) I is nonincreasing in the first variable and nondecreasing in the second one.

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, any implication satisfies $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

The following proposition can be found in [1].

Proposition 1 *Let U be a representable uninorm with neutral element $e \in]0,1[$ and additive generator h. Let U^* be the disjunctive representable uninorm with the same additive generator h, and N be the strong negation given by $N(x) = h^{-1}(-h(x))$. Then,*

i) *The residual implicator I_U is given by $I_U(x,y) =$*

$$\begin{cases} h^{-1}(h(y) - h(x)) & \text{if } (x,y) \notin \{(0,0),(1,1)\} \\ 1 & \text{otherwise.} \end{cases}$$

ii) *$I_U(x,y) = I_{U^*}(x,y) = U^*(N(x),y)$ for all $x,y \in [0,1]$.*

3 S-implications from continuous uninorms

Definition 4 *Given a disjunctive uninorm U and a strong negation N, the operator defined by*

$$I(x,y) = U(N(x),y) \quad \text{for all } x,y \in [0,1]$$

is an implication operator, called the strong implication of U and N.

Note that, for a uninorm $U \in \mathcal{CU}$, U is disjunctive if and only if one of the following items holds:

a) $U = (e,u,\lambda,T,T',U^R)$ is in $\mathcal{CU}^{\min}$, $\lambda = 0$ (consequently only a t-norm T is needed in its expression), and $U(1,0) = 1$.

b) $U = (e,v,\omega,U^R,S,S')$ is in $\mathcal{CU}^{\max}$.

In both cases, the general structure of the strong implication of such uninorm U can be easily derived and it is given in the following two propositions, respectively.

Proposition 2 *If $U = (e,u,0,T,U^R)$ is a disjunctive uninorm in $\mathcal{CU}^{\min}$, then $I_{U,N}$ is given by $I_{U,N}(x,y) =$*

$$\begin{cases} T_{\varphi_u}(N(x),y) & \text{if } x \in [N(u),1] \text{ and } y \in [0,u] \\ {U^R}_{\psi_{u,1}}(N(x),y) & \text{if } x \in]0,N(u)[\text{ and } y \in]u,1[\\ 1 & \text{if } x = 0 \text{ or } y = 1 \\ \min(N(x),y) & \text{otherwise.} \end{cases}$$

Proposition 3 *If $U = (e,v,\omega,U^R,S,S')$ is a disjunctive uninorm in $\mathcal{CU}^{\max}$, then $I_{U,N}$ is given by $I_{U,N}(x,y) =$*

$$\begin{cases} S_{\psi_{v,w}}(N(x),y) & \text{if } x \in [N(w),N(v)] \\ & \text{and } y \in [v,w] \\ S'_{\psi_{w,1}}(N(x),y) & \text{if } x \in [0,N(w)] \\ & \text{and } y \in [w,1] \\ {U^R}_{\varphi_v}(N(x),y) & \text{if } x \in]N(v),1[\\ & \text{and } y \in]0,v[\\ 0 & \text{if } x = 1, y \in [0,w[, \text{ or} \\ & y = 0, x \in]N(w),1] \\ 0 \text{ or } \max(N(x),y) & \text{if } (x,y) = (1,w) \text{ or} \\ & (x,y) = (N(w),0) \\ \max(N(x),y) & \text{otherwise.} \end{cases}$$

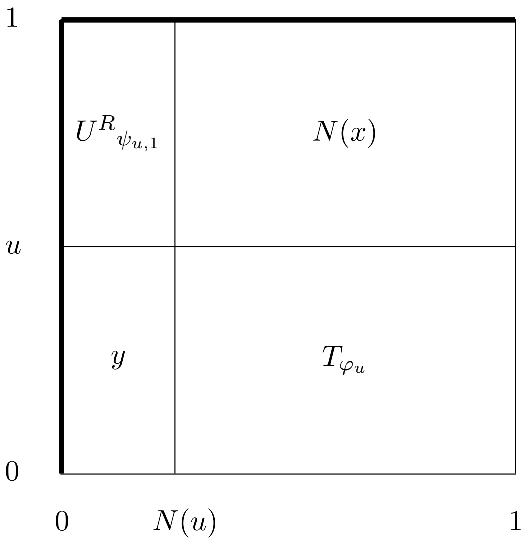

Figure 1: $I_{U,N}$ with $U \in \mathcal{CU}^{\min}$, where operators in the figure are applied to the pairs $(N(x), y)$.

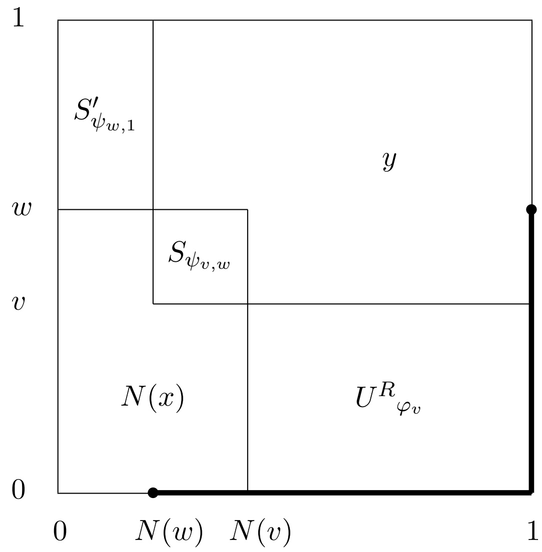

Figure 2: $I_{U,N}$ with $U \in \mathcal{CU}^{\max}$, where operators in the figure are applied to the pairs $(N(x), y)$

We can see the general structure of $I_{U,N}$ being $U \in \mathcal{CU}^{\min}$ and being $U \in \mathcal{CU}^{\max}$ in figures 1 and 2, respectively.

From its definition, it is clear that strong implications $I_{U,N}$ always satisfy contrapositive symmetry with respect to N,

$$I(N(y), N(x)) = I(x, y) \quad \text{for all } x, y \in [0,1]$$

because U is commutative and N involutive, and the exchange principle

$$I(x, I(y,z)) = I(y, I(x,z))$$

for all $x, y, z \in [0,1]$, because U is commutative and associative.

Other interesting property is the distributivity of such implications over conjunctive and disjunctive uninorms. That is,

$$I_{U,N}(U_c(x,y), z) = U_d(I_{U,N}(x,z), I_{U,N}(y,z)) \tag{3}$$

for all $x, y, z \in [0,1]$, where U_c and U_d are uninorms in one of the classes considered, such that U_c is conjunctive, U_d is disjunctive and their underlying t-norms and t-conorms are continuous.

This property was already studied for strong implications derived from other types of uninorms in [9]. Thus, we complete here that study for continuous uninorms. First of all, we have the following result.

Theorem 2 *With the previous notations, $I_{U,N}$, U_c and U_d satisfy equation (3) if and only if U_c and U_d are N-dual and U is distributive over U_d.*

Then, we have to solve the distributivity equation of a uninorm $U \in \mathcal{CU}$ over a uninorm U_d:

$$U(x, U_d(y,z)) = U_d(U(x,y), U(x,z)) \tag{4}$$

for all $x, y, z \in [0,1]$. We will distinguish two cases depending on which class is in the uninorm U.

3.1 Distributivity when $U \in \mathcal{CU}^{\min}$

We start with the case that $U \in \mathcal{CU}^{\min}$, that is, $\lambda = 0$.

Lemma 1 *Let $U = (e, u, 0, T, U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then $U(e_d, e_d) = e_d, e_d \leq u$ and $U_d(e, e) < 1$.*

Lemma 2 *Let $U = (e, u, 0, T, U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a* *disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then $U_d(x,x) = x$ for all $x \in [0,u]$.*

Corollary 1 *Let $U = (e,u,0,T,U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then U_d is nor representable, nor in $\mathcal{CU}$.*

Lemma 3 *Let $U = (e,u,0,T,U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. For all $y < e_d < z < 1$ we have that $U_d(y,z) \neq z$.*

Corollary 2 *Let $U = (e,u,0,T,U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then U_d is not in $\mathcal{U}_{\max}$ and, if it is in $\mathcal{U}(e_d)$, it must be given by $U_d(x,y) =$*

$$\begin{cases} \min(x,y) & \text{if } x < e_d \text{ and } y < 1, \text{ or,} \\ & y < e_d \text{ and } x < 1 \\ S_{\psi_{u,1}}(x,y) & \text{if } (x,y) \in [u,1]^2 \\ \max(x,y) & \text{otherwise.} \end{cases} \quad (5)$$

Theorem 3 *Let $U = (e,u,0,T,U^R)$ be a disjunctive uninorm in $\mathcal{CU}^{\min}$, and let U_d be a disjunctive uninorm in one of the classes considered in the Preliminaries. Then, U is distributive over U_d if and only if $e_d \leq u$ and*

(i) $U(e_d,e_d) = e_d$, that is, there exist T' and T'' t-norms such that $T = (\langle 0, \frac{e_d}{u}, T' \rangle, \langle \frac{e_d}{u}, 1, T'' \rangle)$.

(ii) There exists S a t-conorm such that U_d is given by equation (5).

(iii) U^R is distributive over S, that is, $S = \max$ or S is strict and the additive generator s of S satisfying $s\left(\frac{e-u}{1-u}\right) = 1$, is also a multiplicative generator of U^R.

The general structure of $U \in \mathcal{CU}^{\min}$ and U_d such that U is distributive over U_d is depicted in figure 3.

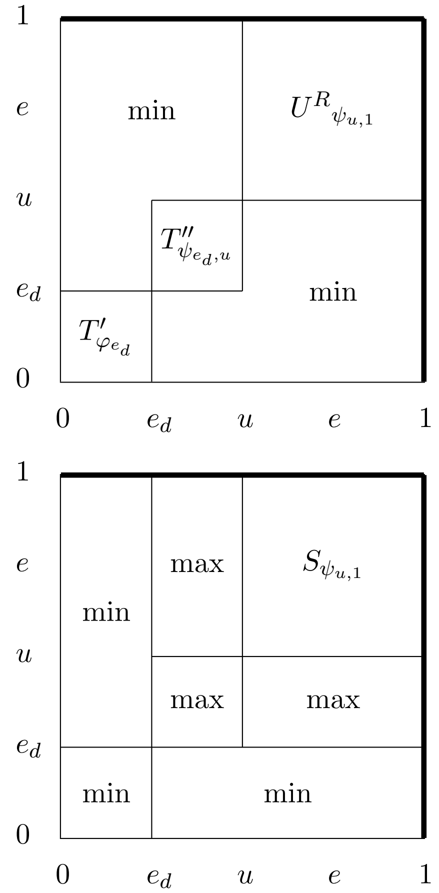

Figure 3: $U \in \mathcal{CU}^{\min}$ (up) distributive over $U_d \in \mathcal{U}(e)$ (down).

3.2 Distributivity when $U \in \mathcal{CU}^{\max}$

Now we consider the case when $U \in \mathcal{CU}^{\max}$. Similarly as in the previous case we obtain the following results.

Lemma 4 *Let $U = (e,v,\omega,U^R,S,S')$ be a disjunctive uninorm in $\mathcal{CU}^{\max}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then $e < v \leq e_d$, $U(e_d,e_d) = e_d$ and $U_d(e,e) > 0$.*

Lemma 5 *Let $U = (e,v,\omega,U^R,S,S')$ be a disjunctive uninorm in $\mathcal{CU}^{\max}$, and let U_d be a disjunctive uninorm with neutral element e_d, such that U is distributive over U_d. Then $U_d(x,x) = x$ for all $x \in [v,1]$. Consequently, U_d can not be continuous in $]0,1[^2$.*

Lemma 6 *Let $U = (e,v,\omega,U^R,S,S')$ be a disjunctive uninorm in $\mathcal{CU}^{\max}$, and let U_d be a* *disjunctive uninorm with neutral element* e_d*, such that* U *is distributive over* U_d*. For all* $0 < y < e_d < z$ *we have that* $U_d(y, z) \neq y$*.*

Lemma 7 *Let* $U = (e, v, \omega, U^R, S, S')$ *be a disjunctive uninorm in* $\mathcal{CU}^{\max}$*, and let* U_d *be a disjunctive uninorm with neutral element* e_d*, such that* U *is distributive over* U_d*, and* $e_d < \omega$*. Then* $U_d(x, 0) = U(x, 0)$ *for all* $x \in [0, 1]$*.*

Corollary 3 *Let* $U = (e, v, \omega, U^R, S, S')$ *be a disjunctive uninorm in* $\mathcal{CU}^{\max}$*, and let* U_d *be a disjunctive uninorm with neutral element* e_d*, such that* U *is distributive over* U_d*, and* $e_d < \omega$*. Then* U_d *is not in* $\mathcal{U}_{\min}$ *and, if it is in* $\mathcal{U}(e_d)$*, it must be given by* $U_d(x, y) =$

$$\begin{cases} T_{\varphi_v}(x, y) & \text{if } (x, y) \in [0, v]^2 \\ \max(x, y) & \text{if } x > e_d \text{ and } y > 0, \text{ or,} \\ & x > e_d \text{ and } x > 0 \\ \max(x, y) & \text{if } 0 \in \{x, y\} \text{ and } \max(x, y) > \omega \\ U(0, \omega) & \text{if } (x, y) \in \{(0, \omega), (\omega, 0)\} \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{6}$$

Lemma 8 *Let* $U = (e, v, \omega, U^R, S, S')$ *be a disjunctive uninorm in* $\mathcal{CU}^{\max}$*, and let* U_d *be a disjunctive uninorm with neutral element* e_d*, such that* U *is distributive over* U_d*, and* $\omega \leq e_d$*. Then* $U_d(x, 0) = x$ *for all* $x > e_d$*, that is,* U_d *must be in* $\mathcal{U}_{\max}$*.*

Corollary 4 *Let* $U = (e, v, \omega, U^R, S, S')$ *be a disjunctive uninorm in* $\mathcal{CU}^{\max}$*, and let* U_d *be a disjunctive uninorm with neutral element* e_d*, such that* U *is distributive over* U_d*, and* $\omega \leq e_d$*. Then if it* U_d *is in* $\mathcal{U}(e_d)$*, then* $U \in \mathcal{U}_{\max}$ *and there exists a t-norm* T *such that* U_d *must be given by*

$$U_d(x, y) = \begin{cases} T_{\varphi_v}(x, y) & \text{if } (x, y) \in [0, v]^2 \\ \max(x, y) & \text{if } \max(x, y) \geq e_d \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 4 *Let* $U = (e, v, \omega, U^R, S, S')$ *be a disjunctive uninorm in* $\mathcal{CU}^{\max}$*, and let* U_d *be a disjunctive uninorm in one of the classes considered in the Preliminaries. Then,* U *is distributive over* U_d *if and only if* $e_d \geq v$*,* $U(e_d, e_d) = e_d$*, there exists* T *a t-norm such that* U^R *is distributive over* T *(that is,* $T = \min$ *or* T *is strict and the additive generator* t *of* T *satisfying* $t(\frac{e}{v}) = 1$ *is also a multiplicative generator of* U^R*), and one of the following two cases holds:*

i) $e_d < \omega$ *and then* U_d *is given by equation (6), or*

ii) $\omega \leq e_d$ *and then* U_d *is given by equation (7).*

The general structure of U and U_d such that U is distributive over U_d is depicted in figure 4 for case i), and figure 5 for case ii).

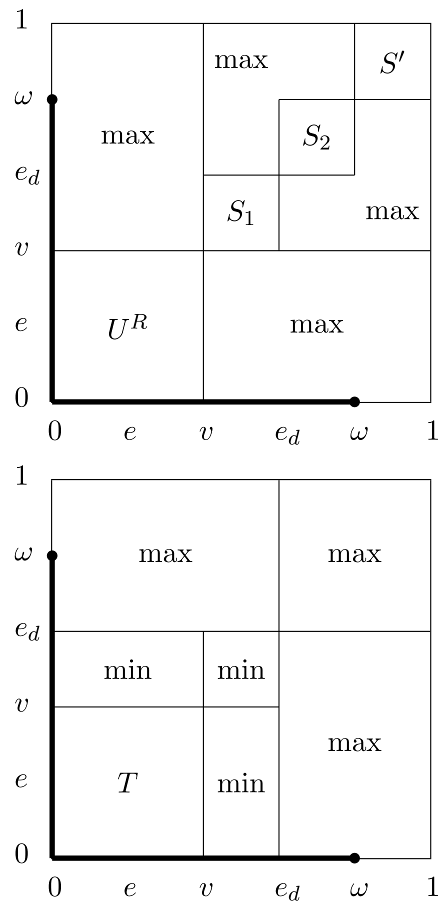

Figure 4: $U \in \mathcal{CU}^{\max}$ (top) distributive over $U_d \in \mathcal{U}(e)$ (bottom), with $v < e_d < \omega$.

3.3 Other related distributivities

Related to equation (3) there are several distributivity equations involving implications

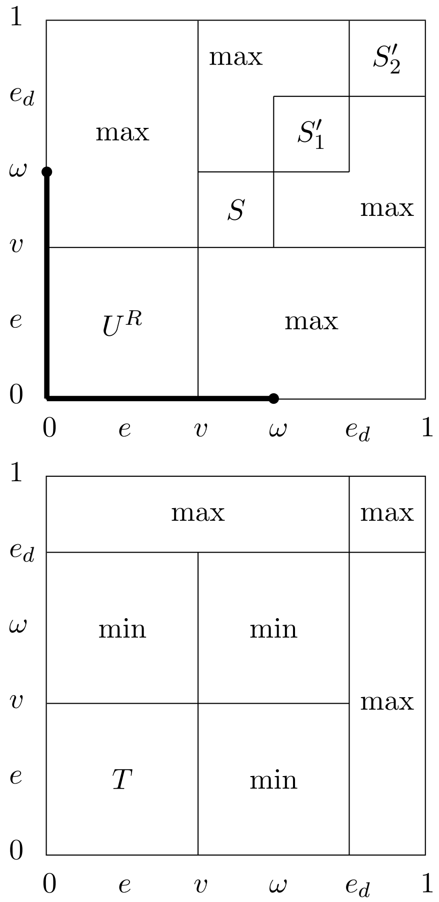

Figure 5: $U \in \mathcal{CU}^{\max}$ (top) distributive over $U_d \in \mathcal{U}_{\max}$ (bottom), with $\omega \leq e_d$.

and uninorms. Namely,

$$I_{U,N}(x, U_d(y,z)) = U'_d(I_{U,N}(x,y), I_{U,N}(x,z)) \quad (8)$$

$$I_{U,N}(U_d(x,y), z) = U_c(I_{U,N}(x,z), I_{U,N}(y,z)) \quad (9)$$

$$I_{U,N}(x, U_c(y,z)) = U'_c(I_{U,N}(x,y), I_{U,N}(x,z)) \quad (10)$$

where uninorms with subscript c are conjunctive and those with subscript d are disjunctive. All of these properties were studied for strong implications derived from other types of uninorms in [9]. As in such reference, it can be easily proved using contrapositive symmetry of strong implications that equation (8) is equivalent to equation (3) taking U_c the N-dual of U_d and consequently we have the following theorem.

Theorem 5 *Let U be a continuous disjunctive uninorm, and $U_d = (e_d, T_d, S_d)$ and $U'_d = (e'_d, T'_d, S'_d)$ two disjunctive uninorms. Then $I_{U,N}$, U_d and U'_d satisfy equation (8) if and only if $U_d = U'_d$ and U is distributive over U_d.*

Thus, Theorems 3 and 4 give all the solutions of equation (8). Analogously equation (9) is equivalent to equation (10) using this duality, and so, we only need to solve this last equation. Again, we easily have,

Lemma 9 *Let U be a continuous disjunctive uninorm and U_c, U'_c two conjunctive uninorms. Then, $I_{U,N}$, U_c and U'_c satisfy equation (10) if and only if $U_c = U'_c$ and U is distributive over U_c.*

Thus, we need to solve the distributivity equation of U over U_c. However, in this case there are no solutions among the four classes of uninorms considered, as it can be proved following the lemmas below.

Lemma 10 *Let U be a continuous disjunctive uninorm and U_c a conjunctive one. If U is distributive over U_c, necessarily $U \in \mathcal{CU}^{\max}$.*

Lemma 11 *Let $U = (e, v, \omega, U^R, S, S')$ be a disjunctive uninorm in $\mathcal{CU}^{\max}$, and let U_c be a conjunctive uninorm. If U is distributive over U_c, then*

- *$e < v \leq e_c$, $U(e_c, e_c) = e_c$ and $U_c(e,e) > 0$.*
- *$U_c(x,x) = x$ for all $x \in [v,1]$ and consequently U_c can not be in $\mathcal{CU}$.*
- *U_c can not be in $\mathcal{U}(e_c)$.*

Let us finally note that a similar study can be done for R-implications derived from uninorms continuous in $]0,1[^2$, that is not included here for lack of space. However, again two different kinds of implications are obtained and in both cases new solutions of the distributivity equation can be found.

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