Abstract

In Sugeno’s rules, conclusions are usually considered as being crisp numerical values. On the other hand, in Mamdani’s rules, conclusions are viewed as fuzzy numbers or more rarely as words. This paper makes an attempt at smoothing out this distinction by proposing to handle fuzzy rules with conclusions of the three different types using Mamdani’s reasoning as well as Sugeno’s interpolation mechanism. The extension of Sugeno’s fuzzy systems is carried out using fuzzy arithmetic operators for implementing the weighted mean of fuzzy numbers or words. Concerning Mamdani’s systems, the compositional rule of inference is applied indifferently at the numerical and linguistic levels.

After the presentation of the different possible implementations of a rule-based system, the developed ideas are illustrated using a fuzzy rulebase that approximates the arithmetic mean of two quantities.

Keywords: Fuzzy rules, Fuzzy reasoning, Mamdani’s systems, Sugeno’s systems, Fuzzy arithmetic, Computing with words

1 Introduction

Fuzzy rule-based systems are usually categorized into two families: Sugeno’s systems and Mamdani’s systems. Both types of systems are often distinguished according to the way the rule conclusions are expressed. On one hand the family of Sugeno’s systems is based on conclusions that can be numerically computed for given input variables. On the other hand, Mamdani’s rules use a linguistic specification of the conclusions with fuzzy subsets. Both families are associated with specific computation mechanisms. Concerning Sugeno’s systems, a numerical interpolation is carried out to fill the gap between the identified behaviors expressed by the rule conclusions when Mamdani’s systems are based on fuzzy reasoning. Hence, in most applications of Mamdani’s systems a defuzzification step is achieved for computing a crisp numerical output. By doing so, it becomes possible to compare the input-output mapping associated with both families of fuzzy systems. In this framework, different studies showed that there exists particular configurations for which both families of fuzzy systems are equivalent.

The major aim of this paper is to extend the usual interpolation mechanism implemented in Sugeno’s systems for dealing with Mamdani’s rules, that is rules whose conclusions are linguistically expressed. Actually, two different interpretations of such conclusions are considered. In the first one, the symbol $C_k$ involved in the conclusion « $z$ is $C_k$ » is associated with a specific fuzzy subset, namely a fuzzy number. In the second one, $C_k$ is viewed as a word, i.e. a crisp element belonging to some dictionary. These two points of view result in two different implementations of the interpolation mechanism inherent to Sugeno’s systems. The same distinction is proposed for Mamdani’s systems, leading also to two distinct inference mechanisms. To sum up the paper content at a glance, six computation techniques are presented according to table 1. Each cell contains the number of the section where the corresponding computation mechanism is detailed.
cells are associated with the usual implementation of both families of fuzzy systems.

Table 1: Paper content

<table>
<thead>
<tr>
<th>Computation</th>
<th>Interpolation (Sugeno)</th>
<th>Reasoning (Mamdani)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a crisp number</td>
<td>2.1</td>
<td>3.1</td>
</tr>
<tr>
<td>a fuzzy number</td>
<td>2.2</td>
<td>3.1</td>
</tr>
<tr>
<td>a word</td>
<td>2.3</td>
<td>3.2</td>
</tr>
</tbody>
</table>

The paper is organized according to table 1. Section 2 aims at presenting the usual Sugeno’s interpolation mechanism and its extension for handling fuzzy numbers and words. Section 3 is devoted to Mamdani’s fuzzy systems with an insight into its possible use for computing with words. Section 4 gives an illustration of the developed ideas using a simple rule base that implements a fuzzy arithmetic mean of two quantities. The final section contains a brief discussion about future works and a conclusion.

2 Sugeno-like fuzzy systems

2.1 Computation mechanism

Sugeno’s fuzzy systems are based on a collection of “If ... then ...” rules whose premise is linguistically expressed but whose conclusion is numeric (real value, linear or polynomial equation...). Without loss of generality we restrict ourselves to rules with two inputs of the form:

\[ R^S_{(i,j,k)}: \text{If } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z = c_k \]  

(1)

where \( A_i \) and \( B_j \) are linguistic values with fuzzy triangular membership functions and \( c_k \) a real value. For precise inputs \( x_0 \) and \( y_0 \), the output \( z \) is computed according to the following mechanism:

\[ z = SUG(x_0, y_0) = \frac{\sum_{(i,j,k) \in I} w_{i,j} \cdot c_k}{\sum_{(i,j,k) \in I} w_{i,j}} \]  

(2)

with \( w_{i,j} = \mu_{A_i}(x_0) \cdot \mu_{B_j}(y_0) \). For the sake of simplicity, a strict triangular partitioning of the input universes of discourse is assumed, which leads to:

\[ z = SUG(x_0, y_0) = \sum_{(i,j,k) \in I} w_{i,j} \cdot c_k \]  

(3)

The idea developed in this paper consists in extending the Sugeno’s computational mechanism for dealing with rules of the form:

\[ R^M_{(i,j,k)}: \text{If } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z = c_k \]  

(4)

where the conclusion \( c_k \) is now a linguistic value with a fuzzy triangular membership function. The direct extension of equation (3) leads to:

\[ z_L = LSUG(x_0, y_0) = \sum_{(i,j,k) \in I} w_{i,j} \otimes c_k \]  

(5)

where \( z_L = LSUG(x_0, y_0) \) denotes the linguistic output of the fuzzy system. It is worth noting that equation (5) involves specific operators for handling addition and product by a scalar in a linguistic framework. In other words, the operators \( \otimes \) and \( \oplus \) represent the linguistic counterpart of the real-valued operators + and \( \cdot \).

2.2 Computing with fuzzy numbers

One possible interpretation of equation (5) consists in making no distinction between symbol \( c_k \) and its fuzzy meaning, that is its membership function. In this case, fuzzy numbers have to be combined using a weighted fuzzy mean operator which can be implemented according to fuzzy arithmetic. The application of the extension principle leads to the following definition of the operators \( \otimes \) and \( \oplus \):

\[ \mu_w \otimes_C (z) = \mu_C(z/w), w \neq 0 \]  

(6)

\[ \mu_{C_1 \oplus_C C_2}(z) = \sup_{v \in S}(\min(\mu_{C_1}(z-v), \mu_{C_2}(z))) \]  

(7)

Using triangular fuzzy subsets with a parametric representation, where \( C = (l, m, r) \) denotes the fuzzy subset \( C \) with modal value \( m \) and support \( [l, r] \), the development of equations (6) and (7) leads to the following exact results:

\[ w \otimes C = w \otimes (l, m, r) = (w.l, w.m, w.r) \]  

(8)

\[ C_1 \oplus C_2 = (l_1, m_1, r_1) \oplus (l_2, m_2, r_2) = (l_1+l_2, m_1+m_2, r_1+r_2) \]  

(9)

Finally, according to equations (8) and (9), the output of the implemented Sugeno-like fuzzy system (equation (5)) is a triangular fuzzy subset whose parameters are computed using a weighted mean operator. For example, the modal value of \( z_L \) is given by:

\[ m = \sum_{(i,j,k) \in I} w_{i,j} \cdot m_k \]  

(10)

where \( m_k \) is the modal value of the symbols \( c_k \).
volved in the conclusion of the rules (4). The same equation holds for dealing with the left and right parameters, i.e. \( l \) and \( r \). Comparing with equation (3), it appears clear that the modal value of the computed result is equal to the result obtained using a usual Sugeno’s system with constant conclusions. The difference between both implementations resides in the imprecision attached to the computed result determined from the one associated with the combined conclusions.

**2.3 Computing with words**

Another possible implementation of equation (5) consists in defining linguistic operators \( \oplus \) and \( \otimes \) for directly dealing with symbols \( C_k \). In this framework, the «addition» of two words should provide another word. In this case, the \( \oplus \) operator is defined by a function from \( \text{LZ} \times \text{LZ} \) to \( \text{LZ} \) where \( \text{LZ} = \{ C_1, C_2, \ldots, C_K \} \) is the set of all fuzzy linguistic subsets defined on \( \text{LZ} \). According to this approach, essentially used for dealing with qualitative equations, it is not possible to model any gradual behavior without defining a large number of words, which is prejudicial to the system interpretability. An alternative proposal consists in expressing the result of the «addition» of two words as a fuzzy linguistic set, that is a fuzzy set defined on a set of words, i.e \( \text{LZ} \) in the present case. Such an approach is developed in the remainder of this section.

**2.3.1 Notations**

Let us first introduce the notations used for expressing fuzzy linguistic subsets defined on \( \text{LZ} \). Let \( E \) be such a fuzzy subset, with extensional definition:

\[
E = \alpha_1/C_1 + \alpha_2/C_2 + \ldots + \alpha_K/C_K
\]

(11)

\[
= \sum_{k=1}^{K} \alpha_k/C_k
\]

In others words, any symbol \( C_k \) belongs to \( E \) with the degree \( \alpha_k \), i.e. \( \mu_E(C_k) = \alpha_k \). For the sake of simplicity, any symbol associated with a zero membership degree is usually removed from equation (11).

Finally, let \( F(\text{LZ}) \) be the set of all fuzzy linguistic subsets defined on \( \text{LZ} \). Any word \( C_k \) can be viewed as the singleton \( \{C_k\} \) whose extensional definition is simply \( 1/C_k \). It means that any word can be assimilated with a degenerated fuzzy linguistic subset, that is an element in \( F(\text{LZ}) \).

In this framework, both operators \( \oplus \) and \( \otimes \) involved in equation (5) will be defined for handling fuzzy linguistic subsets. It means that the addition will induce a relation from \( F(\text{LZ}) \times F(\text{LZ}) \) to \( F(\text{LZ}) \) while the product by a degree will generate a relation from \( [0, 1] \times F(\text{LZ}) \) to \( F(\text{LZ}) \).

The methodology further proposed for defining these two relations makes use of the assumption that all processed symbols have a numerical meaning. In this context, it is possible to define linguistic operations so as to guarantee coherence with corresponding numerical operations. With this aim in view, it is important to clarify the links existing between linguistic and numerical worlds and to define a kind of mapping between objects of both universes. That is the purpose of the next section.

**2.3.2 Mapping between words and numbers**

Interfaces are here proposed for transforming numbers into words and conversely. More precisely, links between numbers and fuzzy linguistic subsets are considered and specified by means of two functions:

- The first one is in charge of converting a number into a fuzzy linguistic subset. An element \( z \) in \( \text{Z} \) is thus associated with a fuzzy subset \( D(z) \) of terms in \( \text{LZ} \), that is an element in \( F(\text{LZ}) \). \( D(z) \) is called a descriptor by Zadeh, and can be deduced from the fuzzy meaning of the used terms, namely:

\[
D : \text{Z} \rightarrow F(\text{LZ})
\]

(12)

\[
z \rightarrow D(z) = \sum_{k=1}^{K} \alpha_k/C_k
\]

Actually, the description mapping \( D \) implements a special kind of *fuzzification*, called *linguistic* or *symbolic fuzzification* [3].

- The second function \( H \) produces a crisp numerical value, i.e. an element in \( \text{Z} \), from a fuzzy linguistic subset, i.e. an element in \( F(\text{LZ}) \), that is \( H : f(\text{LZ}) \rightarrow \text{Z} \). In the same way that the description \( D \) implements a symbolic fuzzification, the converse mapping \( H \) carries out a *linguistic* or *symbolic defuzzification* [3]. Several methods of symbolic defuzzification can be chosen. Here, we restrict ourselves to the *height* method defined as:

\[
\forall E \in F(\text{LZ}), \quad H(E) = \frac{\sum_{k=1}^{K} \mu_E(C_k) \cdot m_k}{\sum_{k=1}^{K} \mu_E(C_k)}
\]

(13)
Given by:

In the assumed case of a strict triangular partitioning of the numeric universe of discourse Z, the pair of interfaces (D, H) satisfies the following equation:

\[ \forall z \in Z, H(D(z)) = z \quad (14) \]

which characterizes optimal interfaces. It is worth noting that the image of Z under D is not F(LZ) but only a subset of F(LZ). Otherwise expressed, the description function D is not surjective. It means that:

\[ \exists F \in F(LZ), D(H(F)) \neq F \quad (15) \]

For example, an element F in F(LZ) that contains more than two words with a non null membership degree can not be mapped to using the description D. Actually, by restricting the definition domain of H to those fuzzy linguistic subsets that can be obtained by the description function D, it can be verified that:

\[ \forall E \in D(Z), D(H(E)) = E \quad (16) \]

where D(Z) ⊂ F(LZ) is the image of Z under D. Properties (14), (15) and (16) of the defined optimal interfaces are illustrated in figure 1.

In the assumed case of a strict triangular partitioning of Z, the restricted definition domain of H is given by:

\[ D(Z) = \{ \alpha/C_{k-1} + (1-\alpha)/C_k; k = 2, ..., K, \alpha \in [0,1] \} \quad (17) \]

\[ \mu_{1}(x) = \sup_{(x,y) \in X \times Y} T_1(T_2(\mu_{E_1}(x), \mu_{E_2}(y)), \mu_{F}(x,y,z)) \quad (22) \]

where \( T_1 \) and \( T_2 \) represent fuzzy t-norms and \( \Gamma \) the graph of the fuzzy relation built from the rules. Choosing a conjunctive interpretation of the rules, it follows:

\[ \mu_{T}(x,y,z) = \bigwedge_{(i,j,k) \in I} T_3(\mu_{A_i}(x), \mu_{B_j}(y), \mu_{C_k}(z)) \quad (23) \]

\[ \forall E_1, E_2 \in F(LZ), E_1 \otimes E_2 = D(H(E_1) + H(E_2)) \quad (18) \]

\[ \forall E \in F(LZ), \forall w \in [0,1], w \otimes E = D(w \cdot H(E)) \quad (19) \]

Using the so-defined operators, equation (5) can be rewritten in the form:

\[ z_L = \sum_{(i,j,k) \in I} w_{i,j} \otimes C_k \]

\[ = D(\sum_{(i,j,k) \in I} H(D(w_{i,j} \cdot H(1/C_k)))) \]

and reformulated as:

\[ z_L = D(\sum_{(i,j,k) \in I} w_{i,j} \cdot m_k) \quad (20) \]

Comparing with equation (3), it appears that the proposed linguistic computation mechanism leads to the linguistic description of the result that would be obtained with a usual Sugeno system with constant numeric conclusions, provided that the modal values of the involved symbols \( C_k \), i.e. \( m_k \), correspond with the constant conclusions \( c_k \) in (1).

3 Mamdani-like fuzzy systems

3.1 Reasoning with fuzzy numbers

The usual way for dealing with Mamdani’s fuzzy rules (4) consists in using the compositional rule of inference. Considering fuzzy observed data \( E_x \) and \( E_y \), the assigned conclusion concerning \( z \) is the element F in F(Z) given by:

\[ \forall z \in Z \]

\[ \mu_{E}(z) = \sup_{(x,y) \in X \times Y} T_1(T_2(\mu_{E_1}(x), \mu_{E_2}(y)), \mu_{F}(x,y,z)) \quad (22) \]

2.3.3 Linguistic operations

Having laid down functions for transforming numbers into fuzzy linguistic subsets and conversely, it becomes possible to proceed with the definition of the linguistic operators \( \Theta \) and \( \otimes \) according to the three following steps:

- Fuzzy linguistic arguments are transformed into numbers using the function H.
- The arithmetic operation is carried out with the so-obtained numeric operands.
- The result is retransformed into a fuzzy linguistic subset by means of the description function H.
where $T_3$ and $T_4$ represent fuzzy $t$-norms and $\perp$ a fuzzy $t$-conorm. In the particular case considered here, observed data $x_0$ and $y_0$ are crisp, which induces that equation (24) is reduced to:

$$\forall z \in Z \mu_F(z) = \mu_T(x_0, y_0, z)$$ (24)

In the original Mamdani’s work, the dual fuzzy operators min/max were chosen as $t$-norm and $t$-conorm. In that case, the inferred fuzzy result $F$ is given by:

$$\forall z \in Z \mu_F(z) = \max_{(i, j, k) \in I} (\mu_{A_i}(x_0), \mu_{B_j}(y_0), \mu_{C_k}(z))$$ (25)

Another choice that guarantees some linearity properties is based on the use of the $t$-norm product and the $t$-conorm bounded sum, i.e. $T_3(a, b) = T_4(a, b) = a \cdot b$ and $\perp(a, b) = \min(1, a+b)$. Equations (23) and (24) are then rewritten as:

$$\forall z \in Z \mu_F(z) = \min(1, \Sigma_{(i, j, k) \in I} \mu_{A_i}(x_0) \cdot \mu_{B_j}(y_0), \mu_{C_k}(z))$$ (26)

In the specific case where the rule conclusions are real values $c_k$ (a type-II system for Sugeno [8]), symbols $C_k$ are defined as singletons, which leads to:

$$\mu_F(z) = \left\{ \begin{array}{ll} \perp_{(i, j, k) \in I} T_4(\mu_{A_i}(x_0), \mu_{B_j}(y_0)) & \text{if } z = c_k \ (27) \\ 0 & \text{otherwise} \end{array} \right.$$ (27)

In this context, the choice of bounded sum/product inference and center of gravity defuzzification allows an exact equivalence with Sugeno’s formalism.

### 3.2 Reasoning with words

Considering now that symbols $C_k$ are words, the compositional rule of inference can be directly applied at the linguistic level. It means that the crisp observations $x_0$ and $y_0$ have also to be expressed at the linguistic level. One way of handling the required numeric-to-linguistic transformation consists in using the fuzzy description $D$ over the defined sets of terms $LX$ and $LY$. It follows:

$$\forall C_k \in LZ \quad \mu_F(C_k) = \sup_{(A_i, B_j) \in LX \times LY} T_1(T_2(\mu_{D_{x_0}}(A_i), \mu_{D_{y_0}}(B_j)), \mu_T(A_i, B_j, C_k))$$ (28)

where $\Gamma$ is now the graph of a linguistic relation built from the rules. Actually, each rule (4) defines an element in $\Gamma$. In other words, the 3-tuple $(A_i, B_j, C_k)$ belongs to $\Gamma$, i.e. $\mu_T(A_i, B_j, C_k) = 1$, when there exists a rule linking input symbols $A_i$ and $B_j$ with output symbol $C_k$. As the input linguistic domain $LX \times LY$ is a countable set, the supremum coincide with the maximal element. In this case, equation (28) can be generalized by using any $t$-conorm instead of max, which leads to:

$$\forall C_k \in LZ \quad \mu_F(C_k) = \sup_{(A_i, B_j) \in LX \times LY} T_1(T_2(\mu_{D_{x_0}}(A_i), \mu_{D_{y_0}}(B_j)), \mu_T(A_i, B_j, C_k))$$ (29)

This linguistic formalism can also be used to handle weighted linguistic rules of the form:

If $x$ is $A_i$ and $y$ is $B_j$ then $z = C_k$ with weight $\alpha_{ijk}$ (30)

In this case, the weight $\alpha_{ijk}$ represents the strength of the link between the symbols involved in the rule, that is:

$$\mu_T(A_i, B_j, C_k) = \alpha_{ijk}$$ (31)

### 4 Illustration

The purpose of this section is to compare the different possible implementations of a rule-based system. With this aim in view, a unique target function is chosen for testing both families of fuzzy systems and their variants, namely:

- **Sugeno’s systems**
  - a. with constant conclusions
  - b. with symbolic conclusions viewed as fuzzy numbers
  - c. with symbolic conclusions viewed as words

- **Mamdani’s systems**
  - d. with symbolic conclusions viewed as singletons
  - e. with symbolic conclusions viewed as fuzzy numbers
  - f. with symbolic conclusions viewed as words

The comparative study aims at focusing on the manner in which the imprecision inherently present in the rule based system is transmitted to the computed result. In this context, a bounded sum/product inference is considered in all Mamdani’s system implementations with the purpose of avoiding undesirable nonlinearities. Furthermore, a simple target function is chosen so as to suppress approximation problems. A linear system that implements the arithmetic mean of two numbers $x$ and $y$ is thus considered [5]. Moreover,
the use of the implemented mean operator is restricted to a specific case for which it makes sense to describe numbers with symbols. The numeric variables $x$ and $y$ are thus viewed as been student marks. In the french education system, the corresponding universe of discourse $X$ and $Y$ are usually $[0, 20]$. A possible alternative consists in marking with letters, i.e. $LX = LY = \{F, E, D, C, B, A\}$. The meaning generally attributed to each symbol is given in figure 2.

![Figure 2: Partition of the input universes of discourse](image)

Case a.

Applying the modal equivalence principle $[4]$ allows an automatic synthesis of a Sugeno’s system that exactly implements any linear function. Considering the target function $f(x, y) = (x + y) / 2$, the rule base given in table 2 is obtained.

Table 2: Exact implementation of the arithmetic mean

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>F</th>
<th>E</th>
<th>D</th>
<th>C</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>F</td>
<td></td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

All along this section, the crisp numerical observations $x_0 = 15$ and $y_0 = 8.8$ are considered. According to the defined partitions, the resulting fuzzy descriptions are given by:

$D(x_0) = 0.75/B + 0.25/C$
$D(y_0) = 0.8/D + 0.2/C$

The Sugeno’s system output $z$ is computed according to equation (2), where the single fired rules are the four rules framed in table 2. It follows:

$$z = (0.75*0.8*12) + (0.75*0.2*14) + (0.25*0.8*10) + (0.25*0.2*12) = 11.9$$

which, as expected, is the exact arithmetic mean of $x_0 = 15$ and $y_0 = 8.8$.

Case b.

Viewing rule conclusions as fuzzy numbers previously requires the definition of appropriate fuzzy subsets. With this aim in view, it is reasonable to transform each real value $c_k$ involved in a rule conclusion into a fuzzy subset called « around $c_k »$. According to table 2, eleven fuzzy numbers are defined. However, instead of using generic terms such as « around 10 », a naming related to student marks is preferred. Thus, the term set $LZ$ associated with the output variable $z$ is chosen as $LZ = \{F, F_E, E, E_D, D, D_C, C_B, B, B_A, A\}$. For example, the symbol «E_D» corresponds to a mark between E and D.

The corresponding fuzzy partition of the domain $Z$ is illustrated in figure 3. It is important to be aware that a linguistic mark, for example C, has different meaning depending on the underlying variable, $x$, $y$ on the first hand, $z$ on the other one. The linguistic rulebase derived from table 2 is given in table 3.

![Figure 3: Partition of the output domain $Z$](image)

Table 3: Linguistic rulebase

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>F</th>
<th>E</th>
<th>D</th>
<th>C</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>D_C</td>
<td>C</td>
<td>C_B</td>
<td>B</td>
<td>B_A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>D</td>
<td>D_C</td>
<td>C</td>
<td>C_B</td>
<td>B</td>
<td>B_A</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>E_D</td>
<td>D</td>
<td>D_C</td>
<td>C</td>
<td>C_B</td>
<td>B</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>E</td>
<td>E_D</td>
<td>D</td>
<td>D_C</td>
<td>C</td>
<td>C_B</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td>F_E</td>
<td>E</td>
<td>E_D</td>
<td>D</td>
<td>D_C</td>
<td>C</td>
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<tr>
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<td></td>
<td>F</td>
<td>F_E</td>
<td>E</td>
<td>E_D</td>
<td>D</td>
<td>D_C</td>
</tr>
</tbody>
</table>

The fuzzy arithmetic mean of $x_0 = 15$ and $y_0 = 8.8$ is computed using equation (5), that is:

$$z_L = (0.6\oplus C) \oplus (0.15\oplus C_B) \oplus (0.2\oplus D_C) \oplus (0.05\oplus C) = (l, m, r).$$

According to equation (10), $z_L$ is the fuzzy triangular number plotted in figure 4 whose parameters are:

$$m = (0.6*12) + (0.15*14) + (0.2*10) + (0.05*12) = 11.9$$
\[
l = (0.6\times10) + (0.15\times12) + (0.2\times8) + (0.05\times10) = 9.9
\]
\[
r = (0.6\times14) + (0.15\times16) + (0.2\times12) + (0.05\times14) = 13.9
\]

Case c.
The linguistic arithmetic mean of \(x_0 = 15\) and \(y_0 = 8.8\) is still computed using equation (5), that is:
\[
z_L = (0.6\odot C) \oplus (0.15\odot C_B) \oplus (0.2\odot D_C) \oplus (0.05\odot C)
\]
However, the operators \(\odot\) and \(\oplus\) are now interpreted as linguistic operators as proposed in section 2.3.3. According to equation (21), it follows:
\[
z_L = D(11.9) = 0.05 / D_C + 0.95 / C
\]

Case d.
Symbolic conclusions involved in table 3 are now viewed as singletons. Considering inputs \(x_0 = 15\) and \(y_0 = 8.8\), the inferred fuzzy set \(F\) is computed using equation (27) with product as \(t\)-norm and bounded sum as \(t\)-conorm. The obtained result is plotted in figure 5.

Case e.
This case corresponds to usual Mamdani’s systems as described in section 3.1. The fuzzy arithmetic mean of \(x_0 = 15\) and \(y_0 = 8.8\), computed using equation , is given in figure 6.

Table 4: Fuzzy arithmetic mean of \(x_0 = 15\) and \(y_0 = 8.8\)

<table>
<thead>
<tr>
<th>Interpolation (Sugeno)</th>
<th>Reasoning (Mamdani)</th>
</tr>
</thead>
<tbody>
<tr>
<td>crisp number</td>
<td>Case a.</td>
</tr>
<tr>
<td>f. 11.9</td>
<td></td>
</tr>
<tr>
<td>fuzzy number</td>
<td>Case b.</td>
</tr>
<tr>
<td>z. 11.9 13.9</td>
<td></td>
</tr>
<tr>
<td>word</td>
<td>Case c.</td>
</tr>
<tr>
<td>d. C</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>C_B</td>
<td></td>
</tr>
</tbody>
</table>

It can be verified that using the height defuzzification method H for finally computing a crisp numerical value \(z\) leads to equal results, i.e. \(z = 11.9\), whatever the chosen implementation of the fuzzy system. Hence, there is no means to decide which implementation should be chosen considering only the defuzzified value obtained. However, it appears clear that the uncertainty inherent to the representation of the target function \(f(x,y) = \frac{(x+y)}{2}\) by means of fuzzy rules is propagated differently depending on whether Sugeno’s interpolation or Mamdani’s reasoning is used. In both cases, the «computing with words» approach provides a fuzzy linguistic result that allows the reconstruction of the corresponding fuzzy numeric subset (see [2] for Mamdani’s rules).

5 Conclusion
Six different computation mechanisms have been investigated for implementing rule-based systems and illustrated using a simple example. From a computational point of view, each solution is feasible when triangular membership functions are considered. Furthermore, all obtained
results are coherent with presently used rule-based systems, most of the time artificially preci-
siated (implicitly in usual Sugeno’s systems, ex-
plicitly in Mamdani’s systems where a defuzzification step is often included).
Of course, the practical use of suggested imple-
mentations required that further work be devel-
oped. For Sugeno’s interpolation, it is important
to identify to which systems the proposed ap-
proach can be applied. In this framework, recent
studies on fuzzy arithmetic (for example [6], [7])
will probably be useful for handling non-triangu-
lar fuzzy conclusions. For the «computing with
words» implementations, an important point con-
cerns the assessment of the information content.
Indeed, if reasoning at a linguistic level does not
induce any loss of information, the symbolic im-
plementation used in case f. is probably the more
efficient one for chaining fuzzy systems.

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