A general theory of conditional decomposable information measures

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Abstract

We start from a general concept of conditional event in which the “third” value is not the same for all conditional events, but depends on \( E|H \). Following the same line adopted in previous papers to point out conditional uncertainty measures ([8], [9], [3], [4]), we obtain in a natural way the axioms defining a generalized \( (\odot, \oplus) \)-decomposable conditional information measure, which, for any fixed conditioning event \( H \) is a \( \odot \)-decomposable information measure in the sense of Kampé de Feriet and Forte ([15],[17]).

Keywords: Conditional events, Conditional information measure, independence.

1 Introduction

What is usually emphasized in the literature - when any conditional measure \( f(E|H) \) is taken into account is only the fact that \( f(\cdot|H) \) is, for every given \( H \), a measure with the same properties of \( f(\cdot) \). But this is a very restrictive view of conditional measure , corresponding trivially to just a modification of the world. On the contrary the real richness of conditional measure is based on the possibility to regard the conditioning event \( H \), which play the role of hypothesis, as a variable , with the same state of E. It must be regarded not just as a given fact, but as an (uncertain) event, for which the knowledge of its truth value is not required.

Usually conditional measures are obtained starting from an unconditional measure as a derived concept. This fact is particularly evident in information theory, in which usually the concept of independence precedes that of conditioning.

One starts from an (unconditional) information measure \( I: A \to [0, +\infty] \), (A Boolean algebra) and (pretend to) define, for every \( E \) and \( H \in A \) (\( H \) the ”conditional measure” \( I(E|H) \) trough the values of \( I(E \wedge H) \) and \( I(H) \) by using some operation \( \otimes \). More precisely \( I(E|H) \) is defined as the unique solution of the equation

\[
I(EH) = x \cdot I(H).
\]

Nevertheless, for every choice of \( \otimes \) there are some events \( H \) for which the equation admits a not unique solution.

Then usually one tries, trough some ad-hoceries, to extend the conditioning to all pairs of events without taking into account the inconsistencies that come out from this procedure.

Where is the problem? In fact one can not pretend to capture the complexity of the concept of conditional measure (obviously interesting when we consider different conditioning events) by means of the essentially poor concept of unconditional measure.

We propose to invert the process: defining conditional information measures \( I(\cdot|\cdot) \) as a
primitive concept, by means of some properties (axioms) and so studying the connections with the relevant unconditional information measures \( I(.) = I(.;\Omega) \).

The problem now become: which axioms to define a conditional information measure?

Our aim is to find “reasonable” axioms for generalized conditional information measures. The starting point is a “logical” one, in the sense that we consider a family \( \mathcal{T} \) of conditional events \( E|H \), each one being represented by a suitable three-valued random variable whose values are 1, 0, \( t(E|H) \). By using a monotone transformation \( \lambda \) of those random variables and introducing two commutative, associative, and increasing operations \( \odot \) and \( \oplus \) we get “automatically” (so to say) conditions on \( \lambda(t(E|H)) \) that can be taken as the “natural” axioms for a conditional information measures.

We show also that, as proved for conditional decomposable uncertainty measures, also conditional information measures can be characterized in terms of a class of unconditional information measures.

Finally we introduce the concept of coherence information measures only partial assessed and prove that the above characterization is an efficient tool to provide algorithms to check coherence.

2 Kampe de Feriet information measures

We recall the definition of (generalized) information measure given by Kampé de Feriet and Forte.

**Definition 1** A function \( I \) from an arbitrary set of events \( \mathcal{E} \), and with values on \( \mathbb{R}^+ = [0, +\infty] \), is an information measure if it is antimonotone, i.e. the following condition holds: for every \( A, B \in \mathcal{E} \)

\[
A \subseteq B \implies I(B) \leq I(A).
\]  

So, if both \( \emptyset \) and \( \Omega \) belong to \( \mathcal{E} \), it follows that

\[
0 \leq I(\Omega) = \inf_{A \in \mathcal{E}} I(A) \leq \sup_{A \in \mathcal{E}} I(A) = I(\emptyset).
\]

Kampé de Feriet [15] claims that the above inequality (1) is necessary and sufficient to build up an information theory; nevertheless, only to attribute an universal value to \( I(\Omega) \) and \( I(\emptyset) \) the further conditions \( I(\emptyset) = +\infty \) and \( I(\Omega) = 0 \) are given. The choice of these two values is obviously aimed at reconciling with the Wiener-Shannon theory. In general, axiom (1) implies only that

\[
I(A \lor B) \leq \min\{I(A), I(B)\};
\]

we can specify the rule of composition by introducing a binary operation \( \odot \) to compute \( I(A \lor B) \) by means of \( I(A) \) and \( I(B) \).

**Definition 2** An information measure defined on an additive set of events \( \mathcal{A} \) is \( \odot \)-decomposable if there exists a binary operation \( \odot \) on \( [0, +\infty] \times [0, +\infty] \) such that, for every \( A, B \in \mathcal{A} \) with \( A \land B = \emptyset \), we have

\[
I(A \lor B) = I(A) \odot I(B).
\]

So “min” and the rule of Weiner-Shannon theory \( x \odot y = -\log(e^{-x/c} + e^{-y/c}) \) are only two of the possible choices of \( \odot \).

3 Conditional events

An event can be singled-out by a (nonambiguous) proposition \( E \), that is a statement that can be either true or false (corresponding to the two “values” 1 or 0). Since in general it is not known whether \( E \) is true or not, we are uncertain on \( E \), in the sense that we don’t have complete information on \( E \).

In general, it is not enough directing attention just toward an event \( E \) in order to assess “convincingly” its measure of uncertainty or of information \( f \), but it is also essential taking into account other events which may possibly contribute in determining the “information” on \( E \). Then the fundamental tool must be conditional measures, since the true problem is not that
of assessing \( f(E) \), but rather that of assessing \( f(E|H) \), taking into account all the relevant “information” carried by some other event \( H \).

In order to deal adequately with conditional measures, we need to introduce the concept of conditional event, denoted by \( E|H \), with \( H \neq \emptyset \) (where \( \emptyset \) is the impossible event): it is a generalization of the concept of event, and can be defined through its truth–value \( T(E|H) \).

When we assume \( H \) true, we take \( T(E|H) \) equal to 1 or 0 according to whether \( E \) or its contrary \( E^c \) is true, and when we assume that \( H \) is false we take \( T(E|H) \) equal to a suitable function \( t(E|H) \) with values in \([0, 1]\).

This truth–value \( T(E|H) \) extends the concept of indicator \( I_E = T(E|\Omega) \) concerning an (unconditional) event \( E \). Moreover, notice that \( H \neq \emptyset \) does not mean that \( I_H \) cannot take the value 0: it means only that \( I_H \) is not “constantly” equal to 0, i.e. (in other words) that it is not known that \( H \) is false (otherwise \( H = \emptyset \)).

This “definition” of conditional event (introduced in [8]), differs from many seemingly “similar” 3–valued ones adopted in the relevant literature since 1935, starting with de Finetti [13]. In fact in this definition one does not assign the same third value \( u \) (“undetermined”) to all conditional events, but make it suitably depend on \( E|H \).

There is a “natural” interpretation of \( T(E|H) \) in terms of a betting scheme, that may help in clarifying its meaning and in assigning its truth values: if an amount \( t(E|H) \) – which suitably depends on \( E|H \) – is paid to make a conditional bet on \( E|H \), we get, when \( H \) is true, an amount 1 if also \( E \) is true (the bet is won) and an amount 0 if \( E \) is false (the bet is lost), and we get back the amount \( t(E|H) \) if \( H \) turns out to be false (the bet is called off).

Since the conditional event \( E|H \), or better its boolean support that is the (ordered) pair \((E, H)\), induces a (unique) partition of the certain event \( \Omega \), that is \((E \land H, E^c \land H, H^c)\): one puts in fact

\[
t(E|H) = t(E \land H, E^c \land H, H^c).
\]

It follows \( t(E|H) = t(E \land H|H) \), and so \( T(E|H) = T((E \land H)|H) \).

In conclusion we require for \( t(\cdot|\cdot) \) only the following conditions:

i) the function \( t(E|H) \) depends only on the partition \( E \land H, E^c \land H, H^c \).

ii) the function \( t(\cdot|H) \) must be not identically equal to zero).

A useful representation of \( T(E|H) \) (that will be denoted from now on by \( I_{E|H} \)) can be given by means of a discrete real random quantity

\[
X = \sum_{h=1}^{\nu} x_h I_{E_h}
\]

written in its “canonical” form (i.e., the \( E_h \)’s are a partition of \( \Omega \)): just take, in the above formula, \( \nu = 3, E_1 = E \land H, E_2 = E^c \land H, E_3 = H^c \), and \( x_1 = 1, x_2 = 0, x_3 = t(E|H) \), so that

\[
I_{E|H} = 1 \cdot I_{E \land H} + 0 \cdot I_{E^c \land H} + t(E|H) \cdot I_{H^c}.
\]

We recall that it is possible to give an interpretation in terms of betting scheme of \( I_{E|H} \): if an amount \( p \) – which should suitably depend on \( E|H \) – is paid to bet on \( E|H \), we get, when \( H \) is true, either an amount 1 if also \( E \) is true (the bet is won) or an amount 0 if \( E \) is false (the bet is lost), and we get back the amount \( p \) if \( H \) turns out to be false (the bet is called off).

4 From conditional events to conditional information measures

Since \( t(E|H) \) can be thought as a measure of how much you believe in the truth of \( E|H \), it turns out, introducing partial operations between conditional events, that it is the “natural” candidate to be a conditional uncertainty measure. Through particular different choices of these operations we get, for example, conditional probability conditional possibility and conditional belief functions.

In the present context we necessarily follows a different line: considered a family \( T = \)
Consider now \( \lambda(T) = \{ \lambda(I_E|H) \} \) that is the family of the random quantities representing the conditional events transformed by \( \lambda \), whose range is \([0, +\infty]\).

We can give an interpretation in terms of betting scheme of

\[
\lambda(I_E|H) = 0 \cdot I_{E \land H} + (\infty) \cdot I_{E^c \land H} + \lambda(t(E|H)) \cdot I_{H^c}.
\]

(2)

Also for \( \lambda(I_E|H) \) we can give a convincing interpretation: if we have paid \( \lambda(t(E|H)) \) to obtain the following information: "in the hypothesis that \( H \) is true also \( E \) is true", the random quantity \( \lambda(I_E|H) \) represents the amount which we get back when we know the truth values of \( E \) and \( H \). In fact we get an amount \( 0 \), when \( H \) and \( E \) are true, an amount \( +\infty \) if \( H \) is true and \( E \) is false (we received the maximum danger and we obtain the maximum damages); finally we get back the amount \( \lambda(t(E|H)) \) if \( H \) turns out to be false (the information has no interest).

Given now two commutative, associative, and increasing operations \( \circ \) and \( \oplus \) from \([0, +\infty] \times [0, +\infty]\) to \([0, +\infty]\), having respectively \(+\infty\) and 0 as neutral elements and with \( \oplus \) distributive with respect to \( \circ \) (for instance \( \min \) and \( + \)), we define corresponding operations among the elements of \( \lambda(T) \): the “result” is a random quantity, but it does not, in general, belong to \( \lambda(T) \).

We have in fact, by (2), for any pair of conditional events \( E|H, A|K \) (to improve readability of the following two formulas, for any event \( E \) we put \( I_E = E \)):

\[
\lambda(I_E|H) \circ \lambda(I_{A|K}) = [0 \circ 0]EHAK + [0 \circ +\infty]EH\Lambda H + [+\infty \circ +\infty]E^c HAK + [+\infty \circ +\infty]E^c H\Lambda H + [0 \circ \lambda(t(A|K))]EHK^c + [0 \circ \lambda(t(E|H))]AKH^c + +[0 \circ \lambda(t(E|H))]A^c KH^c + +[\lambda(t(E|H)) \circ (t(A|K))]H^c K^c,
\]

and

\[
\lambda(I_E) \circ \lambda(I_{A|H}) = [0 \oplus 0]EHAK + [0 \oplus +\infty]EH\Lambda H + [0 \oplus +\infty]E^c HAK + [+\infty \oplus +\infty]E^c H\Lambda H + [0 \oplus \lambda(t(E|H))]AKH^c + [0 \oplus \lambda(t(E|H))]A^c KH^c + +[0 \oplus \lambda(t(E|H))]A^c H\Lambda K^c + +[0 \oplus \lambda(t(E|H))]AKH^c + +[\lambda(t(E|H)) \oplus \lambda(t(A|K))]H^c K^c.
\]

As it is easily seen, both right-hand sides of the latter two expressions are not of the kind (1), i.e. there does not exist a conditional event \( A|B \) such that they can be written (using the simplified notation) as

\[
0 \cdot AB + (\infty) \cdot A^c B + \lambda(t(A|B)) \cdot B^c.
\]

Nevertheless, if we operate by \( \circ \) only on events \( E|H \) and \( A|K \) such that \( H = K \) and \( E \land A \land H = 0 \) then we have:

\[
\lambda(I_E|H) \circ \lambda(I_{A|H}) =
\]

\[
0 \cdot EAH + (\infty) \cdot E^c A^c H + +[\lambda(t(E|H)) \circ \lambda(t(A|H))] \cdot H^c.
\]

So, if \( T \) contains also \( (E \land A)|H \), necessarily we must have:

\[
\lambda(t([E \lor A]|H)) = \lambda(t(E|H)) \circ \lambda(t(A|H)).
\]

Similarly, if for \( \oplus \) we consider only events \( E|H \) and \( A|K \) such that \( K = E \land H \), and if the family \( T \) containing \( E|H \) and \( A|E \land H \) contains also \( (E \land A)|H \), we necessarily have:

\[
\lambda(t([E \land A]|H))) = \lambda(t(E|H) \oplus \lambda(t(A|E \land H)).
\]

So, if we operate only with those elements of \( \lambda(T) \times \lambda(T) \) such that the range of each operation is \( \lambda(T) \), we “automatically” get conditions on \( \lambda(t(E|H)) \) that can be regarded as the “natural” axioms for a conditional information measure \( I \), defined on \( \mathcal{C} = \mathcal{E} \times \mathcal{H} \), with \( \mathcal{E} \) Boolean algebra, \( \mathcal{H} \subseteq \mathcal{E} \) additive set not containing \( \{0\} \).
Denoting $\lambda(t(.)) = I(.)$, we have:

(I1) $I(E|H) = I(E \land H|H)$, for every $E \in \mathcal{E}$ and $H \in \mathcal{H}^o$.

(I2) for any given $H \in \mathcal{H}^o$ $I(\overline{H})$ is a $\oplus$ "generalized decomposable measure of information", that is:

$I(\Omega|H) = 0$, $I(\emptyset|H) = +\infty$,

and, for any $E, A \in \mathcal{E}$, with $A \land E \land H = \emptyset$, we have

$I((E \lor A)|H) = I(E|H) \lor I(A|H)$

(I3) for every $A \in \mathcal{E}$ and $E, H, E \land H \in \mathcal{H}^o$,

$I((E \land A)|H) = I(E|H) \land I(A|(E \land H))$.

Definition 3 We call $(\ominus, \oplus)$-conditional information measure a function $I : \mathcal{C} \rightarrow [0, +\infty]$, with $\mathcal{C} = \mathcal{E} \times \mathcal{H}, \mathcal{E}$ a Boolean algebra, $\mathcal{H} \subseteq \mathcal{E}$ an additive set not containing $\{\emptyset\}$, satisfying conditions (I1), (I2), (I3).

Remark 1 We note that, if we fix $\ominus$, then the choice of operation $\oplus$ is not free (for instance) in the case of Wiener-Shannon information measure, we can prove that the only possible choice for $\oplus$ is $\lor$. The constrains are given by the requirement of distributivity and, obviously, by the axioms. In fact we pretend that the following relation holds for every $A \lor E \subseteq H \subseteq K$:

$$[I(A|H) \lor I(E|H)] \lor I(H|K) =$$

$$= [I(A|H) \lor I(H|K)] \lor [I(E|H) \lor I(H|K)].$$

Nevertheless, if for instance we choose $\ominus = \min$, then we have many different possible choices for $\lor$. In the literature there are some other axiomatizations for conditional information measures, we refer to that given by P. Benvenuti in [1]. Our axioms (I1), (I2) coincide respectively with AIV and AI, introduced in [1]. In the quoted paper the author introduces the further following axiom:

AIV there exists a function $\psi$ from $[0, +\infty] \times [0, +\infty] \times [0, +\infty]$ to $[0, +\infty]$, such that

$I(E|H \lor K) =$

$$\psi[I(H|\Omega), I(K|\Omega), I(E|H), I(E|K)]$$

This axiom in fact involves only conditional events whose conditioning event have strictly positive information, so it is not able to completely manage the conditional information, in the sense that we have no rules to assess $I(E|H)$ when $I(H|\Omega) = 0$. Moreover, if we pretend to compute $I(E|H \lor K)$, involving only the values $I(H|\Omega)$, $I(K|\Omega)$ and not also $I(H|H \lor K)$ and $I(K|H \lor K)$ we obtain the same difficulties when we would compute $P(H|H \lor K)$ and $P(K|H \lor K)$, in the case that $P(H|\Omega) = P(K|\Omega) = 0$.

Nevertheless, if $\lor$ is strictly increasing in $[0, +\infty]$ and we restrict our attention to the events $E|H$ with $I(H|\Omega) \neq 0$, and $I(H|\Omega) \neq +\infty$, then it is easy to prove the following result:

Proposition 1 Let $I$ be a $(\ominus, \oplus)$-conditional information measure, with $\oplus$ strictly increasing in $[0, +\infty]$, then, for the events $E|H \lor K$, with $I(H|\Omega)$, $I(K|\Omega)$ in $[0, +\infty]$, condition AIV holds. On the other hand, if AIV holds, then I3 is satisfied.

Condition I3 implies that a conditional information measure $I(\cdot|H)$ is not singled-out (only) by the information measure of its conditioning event $H$, but its value is bound to the values of other conditional information measures $I(\cdot|E \land H)$, for suitable events $E$.

Definition 4 Let $\mathcal{B}$ be a finite algebra and $\mathcal{C}_a$ the set of atoms in $\mathcal{B}$. The class $\mathcal{I} = \{I_0, \ldots, I_k\}$ of information measures, defined on $\mathcal{B}$, is said nested if (for $j = 1, \ldots, k$) the following conditions hold:

for every $A \in \mathcal{B}$

- $I_0(A) = I(A|H_0^0)$ with $H_0^0 = \bigvee_{j=1}^n H_j$;
- $I_j(A) = I(A|H_0^j)$ with $H_0^j = \bigvee_{r \in \mathcal{C}} \{H_r : I_{j-1}(H_r) = +\infty\}$ $(j > 0)$;
- for any $C \in \mathcal{C}_a$ there exists a (unique) $j = j_C$ such that $I_{j_C}(C) < +\infty$.

Theorem 1 Let $\mathcal{I} = \{I_0, \ldots, I_k\}$ be a nested class of information measures on $\mathcal{B}$. Let $f$ be
a function defined on \( \mathcal{B} \times \mathcal{H} \) (with \( \mathcal{H} \subseteq \mathcal{B} \) additive set not containing \( \emptyset \)) into \([0, +\infty]\), such that \( f(A, B) \) is solution of all the equations

\[
I_\alpha(A \land B) = x \oplus I_\alpha(B) \tag{3}
\]

\( \alpha = 0, \ldots, k \) and \( f(A, B) \) is the unique solution of equation (3) related to \( j_B \).

Then, \( f \) is a conditional information measure on \( \mathcal{B} \times \mathcal{H} \).

Proof: The proof of \( I1 \) and \( I2 \) is trivial. To prove \( I3 \), consider a triple \( I((E \land A)|H), I(E|H), I(A|(E \land H)) \) and let \( \alpha = j_H \).

We are interested in proving the equality

\[
f((E \lor A)|H) = f(E|H) \oplus f(A|(E \land H))
\]
or equivalently

\[
I_\alpha(H) \oplus f((E \lor A)|H) = I_\alpha(H) \oplus f(E|H) \oplus f(A|(E \land H))
\]

By using equation (3) we have

\[
f((E \lor A)|H) \oplus I_\alpha(H) = I_\alpha(E \land A \land H)
\]

and

\[
f(E|H) \oplus I_\alpha(H) \oplus f(A|(E \land H)) = I_\alpha(E \land H) \oplus f(A|(E \land H)).
\]

Now, if \( I_\alpha(E \land H) = +\infty \), then necessarily also \( I_\alpha(E \land A \land H) = +\infty \), and so the equality holds.

On the contrary, if \( I_\alpha(E \land H) < +\infty \), then by equation (3) we have:

\[
I_\alpha(E \land H) \oplus f(A|(E \land H)) = I_\alpha(E \land A \land H).
\]

5 Coherent conditional information measures and their characterization

The concept of coherence, developed on uncertainty measures (starting from de Finetti relatively to probability), is the tool to manage partial assessments, that is functions defined in arbitrary sets of (conditional) events (i.e. sets without particular Boolean structure).

Coherence requires that some numbers attributed to some events are in fact the restriction of a specific uncertainty measure on a Boolean algebra or (for conditional events) on a product of an algebra and an additive set.

The interest is obviously related to find necessary and sufficient conditions assuring coherence, possibly easily computable (see for instance [13, 14, 20, 19, 8, 7, 9, 11, 2, 5], precise and imprecise probabilities, [4, 12], for possibilities and necessities, [18] for belief function).

We extend the concept of coherence to (conditional) information measures.

**Definition 5** Let \( \mathcal{E} \) be an arbitrary set of conditional events and \( I : \mathcal{E} \to [0, +\infty] \). The function \( I \) is coherent \((\odot, \oplus)-\)conditional information measure iff it can be extended on \( \mathcal{B} \times \mathcal{H} \supset \mathcal{E} \) (with \( \mathcal{B} \) a Boolean algebra, \( \mathcal{H} \subseteq \mathcal{B} \) additive set not containing \( \emptyset \)) as a \((\odot, \oplus)\)-conditional information measure.

The next theorem gives a characterization of a coherent \((\odot, \oplus)\)-conditional information measure in terms of a class of unconditional \( \odot \)-decomposable information measures.

**Theorem 2** Let \( \mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \) be a finite set of conditional events, \( C_o \) and \( \mathcal{B} \) denote, respectively, the set of atoms and the algebra generated by \( \{E_1, H_1, \ldots, E_n, H_n\} \).

For a real function \( I : \mathcal{F} \to [0, +\infty] \) the following two statements are equivalent:

a) \( I \) is a coherent conditional information measure on \( \mathcal{F} \);

b) there exists (at least) a nested class \( \mathcal{I} = \{I_1, \ldots, I_k\} \) of information measures on \( \mathcal{B} \), such that for any \( E_i|H_i \in \mathcal{F} \), \( I(E_i|H_i) \) is solution of all the equations

\[
I_\alpha(E_i \land H_i) = x \oplus I_\alpha(H_i) \tag{4}
\]

\( \alpha = 0, \ldots, k \) and \( I(E_i|H_i) \) is the unique solution of equation (4) for \( \alpha = j_H \);

c) there exists a sequence of compatible systems \( S_\alpha \) (with \( \alpha = 0, \ldots, k \)), with unknowns \( x_\alpha^i = I_\alpha(C_r) \)

\[
S_\alpha = \begin{cases} 
\bigoplus C_r \subseteq E_i \land H_i & x_\alpha^i = I(E_i|H_i) \oplus \bigoplus C_r \\
\bigoplus C_r \subseteq H_i & \text{if } x_\alpha^{i-1} = +\infty \\
\bigoplus C_r \subseteq H_0 & x_\alpha^i = 0 \\
x_\alpha^i \in [0, +\infty] 
\end{cases}
\]
where $x^{\alpha-1}$ (with $r$-th component $x_r^{\alpha-1}$) is solution of the system $S_{\alpha-1}$ and $x_r^{-1} = +\infty$ for any $C_r \in C_\alpha$. Moreover

$$H_0^\alpha = \{ \bigvee H_i : \bigcap C_i \subseteq H_i, x_i^{\alpha-1} = +\infty \}.$$ 

Proof: We only sketch the proof. It is easy to prove that the $\odot$-information measures obtained starting from the solutions of systems $S_\alpha$ are a nested class (and so $iii \Rightarrow ii$). The proof of implication $ii \Rightarrow i$ has been essentially given in the previous Proposition. To prove $i \Rightarrow iii)$ consider that coherence implies the existence of a $(\odot, \oplus)$-decomposable information measure on $A \times B$ where $A$ is the algebra spanned by events $E_i, H_i$ $(i=1,\ldots,n)$ and $B$ the additive set spanned by the events $H_i$ $(i=1,\ldots,n)$. Since $B$ contains all the events $H_0^\alpha$, the solution of any system $S_\alpha$ is given by $x_k^\alpha = P(H_0^\alpha)$.

The previous result implies that the coherence of a given assignment $I$ can be proved by finding a nested class agreeing with it, i.e. checking the compatibility of a sequence of systems $S_\alpha$ (with $\alpha = 0, \ldots, k$).

References


