Two Wrongs Nearly Make a Right in de Finetti's Qualitative Probability Conjecture

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Abstract

Bruno de Finetti falsely conjectured about conditions sufficient for a finite propositional ordering to be probability-agreeing. In that paper, he also misstated his own key axiom for the weaker distinction of being a qualitative probability. Something formally similar to his misstated axiom suffices for probability agreement if its essential constraint is applied to ordered multisets of propositions rather than to ordering assertions about pairs of propositions. The results reported here satisfy de Finetti's goal of stating necessary and sufficient conditions for probability agreement using only ordinal primitives, and affirm the fundamental soundness of de Finetti's original intuitions about finite-domain probability agreement.

Keywords: Qualitative probability, quasi-additivity, monotonicity, de Finetti's Conjecture, Scott's Theorem, probability agreement

1 Introduction

In a 1949 special issue of *Dialectica*, George Polya [6] strenuously criticized numerical calculi in general, and numerical subjective probability in particular, as methods for managing nondemonstrative inference. Polya did not dispute with any subjective probabilist by name, but left little doubt that he was thinking of the theories of his friend, Bruno de Finetti, who had also contributed an article to the same special issue. In his article, unsurprisingly, de Finetti favored numerical subjective probability for nondemonstrative reasoning.

Reacting to Polya's provocative critique, de Finetti revived some work from the 1930's [1] that he had put aside to concentrate on his famous gambling semantics for subjective probability. This earlier work had emphasized orderings of propositions as sources from which numerical probabilities might be derived. The revival looked at the orderings as vehicles for uncertain reasoning in their own rights, without any reference to numbers.

De Finetti's propositional orderings were transitive, bounded (*i.e.* the tautology is strictly more credible than the contradiction, and no proposition is more credible than the tautology nor less credible than the contradiction), complete (any two propositions A and B in the domain can be compared ordinally), and definite (exactly one of $A = B$, $A > B$, or $A < B$ obtains).

Throughout the current paper, any ordering of propositions is assumed to display those properties. Any propositional domain discussed here is finite. Further, to avoid discussion of some trivial cases, we assume throughout that every proposition besides the contradiction is ordered strictly superior to the contradiction.

De Finetti required a further property in the 1930's, now often called *quasi-additivity*.

Definition. An ordering of propositions is *quasi-additive* just when for all propositions *X*, *Y*, and *Z* where $XZ = YZ = \varnothing$,

$$X \vee Z \geq Y \vee Z \Leftrightarrow X \geq Y$$

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De Finetti intended in 1949 to base his answer to Polya on the claim that quasi-additive orderings of propositions were always probability-agreeing.

Definition. An ordering of propositions is *probability-agreeing* just when there exists some probability distribution *p()* on the propositional domain, such that for all propositions *X* and *Y*,

$$X \geq Y \Leftrightarrow p(X) \geq p(Y)$$

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Quasi-additivity is clearly necessary for probability agreement. If de Finetti's sufficiency claim were correct, then he could devise qualitative reasoning systems based on quasi-additivity to draw inferences similar to those of numerical probabilistic systems. Nevertheless, the new systems would dispense with all numerical aspects of probability, and also display other features Polya held desirable.

De Finetti was in a hurry to make his reply, which he presented at an Italian conference [2] before the end of 1949. He found that he was unable to prove that quasi-additivity sufficed for finite-domain probability agreement, and so he offered his claim as a conjecture.

In 1959, Kraft, Pratt, and Seidenberg [5] presented a counterexample to de Finetti's conjecture. They also offered and proved their own necessary and sufficient conditions for finite-domain probability agreement.

Those conditions sometimes required that the original domain be expanded to include a finite number of auxiliary propositions. Scott [9] presented a succinct restatement of the conditions which used only the original propositions.

Scott's Theorem (1965). An ordering of propositions is probability agreeing **unless** there is some finite set of ordinal assertions of the same weak sense, at least one of which is strict, where every atomic proposition in the domain appears the same number of times on the left side of the inequalities as on the right side.

Example. The main Kraft, Pratt, and Seidenberg counterexample illustrates the exceptional condition mentioned in Scott's Theorem. Any ordering containing the following assertions is *not* probability-agreeing:

$$a \vee c \vee d > b \vee e \qquad a \vee e \geq c \vee d$$

$$b \vee e \geq a \vee d \qquad d \geq a \vee e$$

Among the four assertions, *a* appears twice on the left and twice on the right; *b* appears once on the left and once on the right, and so on.

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Scott's Theorem has some attractive normative interpretations [10]. Nevertheless, its reliance on counting atoms distances it from de Finetti's original stated intention, to secure probability agreement based only upon what is logically implied by the ordinal axioms which define the "≥" relation.

In the sections to come, it turns out that de Finetti came very close to achieving his goal in 1949. De Finetti made an elementary mistake in his paper, but one which suggests that his intuition about the conditions for probability agreement was altogether sound.

Perhaps because of his haste in writing his 1949 paper, de Finetti misstated the condition for quasi-additivity. The second section considers the condition he actually wrote, something weaker than quasi-additivity, but also of arguably broader intuitive appeal.

The third section introduces multiset structures which are naturally partially ordered whenever the elements (of any kind, not just propositions) within the structures are transitively ordered. Thus, the existence of the structures' ordering is a necessary consequence of ordering the elements.

The fourth section considers how the essential intuitive content of the weaker condition that de Finetti mistakenly wrote in 1949 can be applied to the partial orderings of structures whose elements are ordered propositions. When it is, probability agreement for the propositional ordering is ensured.

2 De Finetti's Faulty Statement of Quasi-additivity

What de Finetti intended to require of his 1949 orderings was quasi-additivity. What he actually wrote instead in the published version of his paper imposed on his orderings only a consequence of quasi-additivity, what is often called *monotonicity*. It is also convenient to define a second consequence (in complete and definite orderings) of quasi-additivity at this point.

Definitions. Let *A*, *A'*, *A"* and *B*, *B'*, *B"* be propositions, where $A \Leftrightarrow A' \vee A"$, $B \Leftrightarrow B' \vee B"$, *A'* excludes *A"*, and *B'* excludes *B"*. An ordering of propositions is *monotonic* just when for all such *A*, *A'*, *A"*, *B*, *B'* and *B"*,

$$A' \geq B' \text{ and } A" \geq B" \Rightarrow A \geq B$$

An ordering of propositions displays *weak-dominance* just when for all such *A*, *A'*, *A"*, *B*, *B'*, and *B"*,

$$A' > B' \text{ and } A" \geq B" \Rightarrow A > B$$

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It is easily seen that monotonicity is indeed strictly weaker than quasi-additivity. For example, the familiar possibility calculus is well-known to be monotonic, and not to be quasi-additive. It is obvious, therefore, that monotonicity is *not* sufficient for probability agreement.

The combination of both monotonicity and weak-dominance implies quasi-additivity in a complete, definite ordering. Monotonicity implies one direction of the dual implication of the quasi-additive definition, that for $XZ = YZ = \emptyset$,

$$X \geq Y \Rightarrow X \vee Z \geq Y \vee Z$$

One simply makes the substitutions *X* for *A'*, *Y* for *B'*, *Z* for *A" and B"*, $X \vee Z$ for *A*, and $Y \vee Z$ for *B*, and notes that

$$X \geq Y \Leftrightarrow X \geq Y \text{ and } Z \geq Z$$

Achieving the converse is straightforward from the contrapositive of weak-dominance:

$$B \geq A \Rightarrow B' \geq A' \text{ or } B" > A"$$

One makes the same substitutions as before, and notes that $Z > Z$ is necessarily false.

Given the close resemblance of form between monotonicity and weak-dominance, it is entirely possible that de Finetti was thinking of both when he wrote down only one.

Monotonicity is also an interesting normative property in its own right. That may further help to explain why his mistake did not set off mental 'alarm bells' when de Finetti prepared his manuscript for publication.

For example, monotonicity implies another normative property discussed at length by de Finetti in 1949, that for all propositions *A* and *B*,

$$\text{If } A \Rightarrow B, \text{ then } B \geq A$$

In recent times, this property is sometimes taken as the key part of the definition of a plausibility ordering [8]. De Finetti's discussion of the property and its normative significance, whether or not it appears as part of a probability-agreeing ordering, or even as part of a complete ordering, is among the earliest expositions of its kind.

And, of course, monotonicity restates an axiom of a now-popular class of generally non-probability-agreeing belief-modeling calculi, triangular co-norms and norms. This suggests that there is widespread agreement about the intuitive and normative appeal of monotonicity, and not just among probabilists.

So, de Finetti would have spoke for many others had he wrote in 1949 that monotonicity

> ...expresses, I think, a peculiar requirement which a logic of probable inference or of plausible inference needs to have; it is difficult to be able to have interest in a prospective theory which lacks it.

Surely, more would agree with him about the attractiveness of monotonicity than of quasi-additivity, the intended subject of his praise in the quoted passage.

In the remainder of the paper, we shall use the following special case of monotonicity. Let $\{ a, b, ...z \}$ be any exhaustive set of plural exclusive propositions, and let A and B be any two propositions. Then

$$(Aa \geq Ba) \wedge (Ab \geq Bb) \wedge \quad \wedge (Az \geq Bz) \Rightarrow A \geq B$$

It is easily verified that this form is implied by the version of monotonicity used by de Finetti.

One advantage of this form is that it can be interpreted as a statement about conditional beliefs determining unconditional ones, as might be written

$$(A \mid a \geq B \mid a) \wedge (A \mid b \geq B \mid b) \wedge ... \wedge (A \mid z \geq B \mid z) \Rightarrow A \geq B$$

based upon the idea that, for $s > \emptyset$

$$(A \mid s \geq B \mid s) \Leftrightarrow As \geq Bs \qquad (1)$$

Like monotonicity, the idea expressed in (1) enjoys some endorsement from outside the probabilistic community [3]. Although conditional belief is not much treated in his 1949 paper, de Finetti used a principle similar to (1) in his qualitative work which the 1949 paper revives [1].

So, the alternative form of monotonicity can be explained and motivated based upon the various interpretations of the conditional $A \mid s$. Those interpretations include what we would believe about A if we were to learn that s is true, or if we were to add s to our other premises as we reflect upon our beliefs about A.

We know that one and only one of $\{ a, b, ..., z \}$ is true, or that at most one can consistently be included among our premises. Regardless of which one element that is, we might find that we would conclude that A is no less credible than B if we were to learn the truth or if we were to adopt *any* one of $\{ a, b, ..., z \}$ as a premise.

We could not learn the truth, nor strengthen our premises using any element from the set of alternatives, except that we would come to assert that "*A* is no less credible than *B*." In contemplating that circumstance, we may feel a sense of inevitability about asserting $A \geq B$ now, or feel disquiet about asserting $A < B$ now.

3 Partially Ordered Bags of Elements

A *multiset*, also known as a *bag*, is a standard data structure in which an element may appear more than once within the structure, but like a set, there is no ordering of the elements within the structure.

Two bags are equal just when they contain the same elements, each element being represented the same number of times in each bag. The *size* of a bag is the number of elements it contains.

Any transitive ordering of elements imposes a partial order on same-sized bags of elements, based on ordering assertions about pairs of elements of the same weak sense. For example, the ordering assertions

$$a = b, \ c = d, \ \text{and } e = f$$

may be said to order the bags $[a, c, e]$ and $[b, d, f]$, so that $[a, c, e] = [b, d, f]$. If $a > b$ instead of $a = b$, then we may say $[a, c, e] > [b, d, f]$. Formally,

Definition. For any finite transitively ordered domain of objects D, the *object-matching partial order* asserts, for same-size bags A and B of the objects in D,

> that $A > B$ just when there is a bijection $f()$ from A to B in which for each element a in A, $a \geq f(a)$ in the ordering of D, and for some pair of elements the ordering is strict,

and

> that $A = B$ just when there is a bijection $f()$ from A to B in which for each element a in A, $a = f(a)$ in the ordering of D.

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Note that the object-matching partial order is definite whenever the domain of elements is definitely ordered.

One might be concerned that there could be two incompatible ways to pair up the elements of two ordered bags. There is not. In [10], the following proposition was proven.

Proposition. For bags A and B of the preceding definition, at most one of $A > B$, $B > A$, or $A = B$ holds if the ordering of objects in D is definite.

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With that proposition secure, it is straightforward that the object-matching partial order is transitive.

There may, of course, be several distinct *compatible* ways to pair up the elements in two bags. That does not concern us here.

Alternatively, there may be no way to pair up the elements so as to produce any ordering between two bags. The object-matching partial order really is a partial order, that is, some bags are ordinally incomparable with other bags. If the ordering of elements is complete and definite, however, then all bags of size one are obviously completely and definitely ordered.

4 A Parallel of Monotonicity for Bags of Propositions

Begin by defining an operation for bags that is analogous to propositional conjunction.

Definition. The *projection* of a bag of propositions X and a proposition a, denoted Xa, is the bag which contains an element $x \wedge a$ for every proposition x which is an element of X.

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With that in hand, consider an assumption with similar form to monotonicity, but applicable to bags of propositions, rather than to pairs of individual propositions.

Bag Monotonicity. Let A and B be same-sized bags of propositions, and let $\{ a, b, ..., z \}$ be a set of exclusive and exhaustive propositions, and "≥" be the relational operator of the object-matching partial order. Then *bag monotonicity* requires that:

$$(\mathrm{Aa} \geq \mathrm{Ba}) \wedge (\mathrm{Ab} \geq \mathrm{Bb}) \wedge \quad \wedge (\mathrm{Az} \geq \mathrm{Bz}) \Rightarrow \neg (\mathrm{B} > \mathrm{A})$$

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The form of the consequent reflects that the bag ordering is partial, so we should be mindful of the possibility that A and B might be unordered. However, if and A and B are ordered, then of course $\neg (B > A)$ is equivalent to $A \geq B$.

Bag monotonicity is easily seen to imply propositional monotonicity, since what the assumption says about bags of size one constrains propositional ordering assertions. Bag monotonicity is incompatible with possibility, however, which does exhibit propositional monotonicity.

Incompatibility example. Let $a > b$ in possibility; $a \vee b = a$, and $b > \varnothing$, so $[a \vee b, \varnothing] < [a, b]$ in the object-matching partial order. Note that $[a \vee b, \varnothing]b = [b, \varnothing] = [a, b]b$ and $[a \vee b, \varnothing]\neg b = [a, \varnothing] = [a, b]\neg b$. Thus, if bag monotonicity obtained, then it could not be the case that $[a, b] > [a \vee b, \varnothing]$, but it is.

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Bag monotonicity is a necessary condition for probability agreement. If *p()* is an agreeing probability distribution for the ordering of the propositions, then the sum of the probabilities assigned to a bag's elements equals the sum of the sums of the probabilities of those elements projected onto each proposition of $\{ a, b, ... z \}$.

$$\Sigma_{\text{x in bag}}\, \mathrm{p}(\mathrm{x}) = \Sigma_{\text{s in } \{ \text{a,..., z} \}} \Sigma_{\text{x in bag}}\, \mathrm{p}(\mathrm{x} \wedge \mathrm{s})$$

A necessary condition for $B > A$ in the object-matching partial order is obviously that

$$\Sigma_{\text{x in bag B}}\, \mathrm{p}(\mathrm{x}) > \Sigma_{\text{y in bag A}}\, \mathrm{p}(\mathrm{y})$$

If for every proposition s in $\{ a, b, ..., z \}$,

$$\Sigma_{\text{x in bag B}}\, \mathrm{p}(\mathrm{x} \wedge \mathrm{s}) \leq \Sigma_{\text{y in bag A}}\, \mathrm{p}(\mathrm{y} \wedge \mathrm{s})$$

then

$$\Sigma_{\text{x in bag B}}\, \mathrm{p}(\mathrm{x}) \leq \Sigma_{\text{y in bag A}}\, \mathrm{p}(\mathrm{y})$$

which conflicts with a necessary condition for $B > A$.

Bag monotonicity is also sufficient for probability agreement. Choose the set of atoms of the propositional domain for $\{ a, b, ..., z \}$. Suppose there are two same-sized bags, A and B, in which every atom of the domain appears the same number of times in bag B as it does in bag A.

If so, then for every atom s in $\{ a, b, ..., z \}$

$$\mathrm{As} = \mathrm{Bs}$$

because As and Bs are the same bag. If the ordering displays bag monotonicity, then $B > A$ and $A > B$ are excluded.

Scott's Theorem says that a propositional ordering is probability agreeing unless there is an exceptional condition. An exceptional condition in Scott's sense would correspond with a bag representation like the B and A as just discussed, except that $B > A$ or $A > B$. Since bag monotonicity excludes the exceptional conditions identified by Scott, any propositional ordering which displays bag monotonicity must be probability-agreeing.

Although bag monotonicity expresses an idea which is formally similar to propositional monotonicity, we may be concerned about whether there are normative parallels between the two ideas.

One way of looking at bags that might have appealed to de Finetti in 1949 is as portfolios of bets. One can imagine the bag assertion $B > A$ arising from a series of choices, each between two propositions, with the believer to be given \$1 for each of his or her choices that comes true.

Suppose bag B becomes the repository of the propositions selected by the believer, while A holds those rejected. Perhaps some of the choices involved ties, and so the believer chose arbitrarily in such cases. Assuming at least one choice was strict, however, the believer would seem to be committed to agreeing that B should be more lucrative than A, and be strictly preferred as an investment.

If a Scott-style violation of bag monotonicity has occurred, then the two portfolios B and A always pay identical amounts. How much they pay depends on which atom comes true, but regardless of which one that is, B pays the same amount as A, whatever that amount happens to be. B is not, cannot be, and can be seen in advance not to be, more lucrative than A.

This kind of argument is likely what de Finetti had in mind when he said in [2] that probability-disagreeing orderings were subject to "contradictions." It is also a kind of argument that could be made in favor of monotonicity or quasi-additivity for propositional pairwise ordering assertions.

On the other hand, de Finetti did not offer any explicit gambling arguments in his 1949 rebuttal paper. He hoped that quasi-additivity would prove sufficient for finite probability agreement, and that quasi-additivity would immediately appeal to Polya's intuition. Polya had not indicated any particular interest in gambling arguments.

So even without the dramatic element of money changing hands through wagering, de Finetti could well have imagined that Polya would find violations of monotonicity interesting. Perhaps de Finetti even rehearsed asking Polya to explain how a reasonable person could assert a distinction based on credibility between two structures, knowing that the person would deny that there is *any* distinction between them if a is true, or if b is true, ..., or if z is true. One of those, after all, *is* true.

If there was such a rehearsal, then the structures in de Finetti's mind would have been disjunctions. Nevertheless, the same question might be interesting in the case of bags as well.

It is, of course, possible to motivate propositional monotonicity in ways that do not generalize to justify bag monotonicity. One approach is to pay attention to the precise kind or aspect of "credibility" in question, to what one means when saying that one proposition is "more plausible" than another.

An example of this careful attention to meaning can be found in Hamblin [4], in an early discussion of the possibility calculus (which obeys propositional, but not bag, monotonicity). He contrasted some meanings of possibilistic ordering assertions (*e.g.* relative surprise depending on which proposition were true) with the meanings of assertions in an ordinal probabilistic system he devised.

Both orderings could be described as 'plausibility' orderings. However, both might not be expected to exhibit bag monotonicity.

Is the falsehood of $a \vee b$ strictly more surprising than the falsehood of a alone? Someone confident of a and skeptical of b might be equally surprised at either one. The respondent does not deny that $a \vee b$ can be true when a is false, nor disparage other senses of plausibility which place $a \vee b$ strictly ahead of a. Those were not the questions asked.

If what one meant by *more plausible* had this character, then no more explanation is needed for why bag monotonicity is lacking. What a speaker means by affective words like *plausible* and *surprise*, and that a speaker might choose to speak about some aspects of his or her experience of uncertainty rather than others, would seem to be the speaker's prerogative.

Thus, there is no necessary "irrationality" or "inconsistency" in embracing propositional monotonicity while rejecting bag monotonicity. The two monotonicities may sometimes have similar normative motivation, but other times they simply may not.

5 The Aftermath

Polya [7, volume 2, pages 138-139] made a different kind of answer to de Finetti while the status of the conjecture remained unresolved. Polya considered the potential for an uncertainty calculus based on infinitesimals, which avoided some problems he saw in standard numerical probabilistic models of belief.

Polya's reply shifted the terms of the discussion away from anything in de Finetti's rebuttal. In particular, Polya could concede the attractions of probability agreement in finite domains, while still enjoying what he saw as the advantages of a non-probabilistic schema for belief change in transfinite or open-ended domains (*e.g.* a domain comprising a mathematical conjecture and whichever of its consequences that might be verified at any particular time).

Thinking back today, we benefit from a half-century's hindsight. We can now see that if Polya had pursued his proposal for infinitesimals a bit further, then he would have arrived at a belief representation similar to Hamblin's possibility calculus.

Moreover, it is by now well-known that some probability orderings (denote them using "≥") exist which are syntactically related to atomically-agreeing possibility orderings (denote them using "≥*") in the following way.

$$A \geq B \Leftrightarrow A\neg B \geq^* B\neg A$$

$$C \geq^* D \Leftrightarrow C \geq D\neg C$$

That is, some orderings which feature bag monotonicity exactly describe, and are exactly described by, other orderings which lack bag monotonicity. Bag monotonicity itself, then, cannot be a necessary feature of "rational" belief orderings, however useful the property is as a guide to deliberation in many situations.

By the same reasoning, no set of properties which is necessary and sufficient for probability agreement can be necessary for "rationality" in ordered belief. However provocatively expressed, Polya's chief point in 1949 was essentially irrebuttable.

6 Conclusions

Polya championed the view that some aspects of human plausible reasoning were best modeled without any recourse to numbers whatsoever. This posed a challenge for his friend, de Finetti, who was more comfortable with numerical representations of belief.

In rising to this challenge, de Finetti felt that he could craft a fully non-numeric motivation of a truly number-free counterpart of probability suitable for modeling plausible reasoning in finite propositional domains. In some haste, he set out to do that, and fell short.

On any fair reading, de Finetti came close to specifying what the axioms of one such motivation might be. He was correct that attention specifically to ordinal principles would suffice for the purpose. He was correct that one could take the domain as it was given, and not need to introduce new propositions. He was ironically correct that a very mild and widely attractive condition, monotonicity, could be the key principle in his motivation.

Where he went wrong is that monotonicity is obviously too weak to impose probability agreement when applied to pairs of propositions. De Finetti was, after all, thinking of a stronger principle when he wrote what he did.

As it happens, however, all transitive orderings imply partially ordered structures for which an analog of monotonicity might make sense, depending on the notion of credibility or plausibility being modeled. When the analog of monotonicity is imposed on the partially ordered structures, the underlying propositional ordering is indeed constrained to be probability-agreeing.

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