Fuzzy Modal–Like Approximation Operations Based on Residuated Lattices

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Abstract

In many applications we have a set of objects together with their properties. Since the available information is often incomplete and/or imprecise, the true knowledge about subsets of objects can be determined approximately only. In this paper we present a fuzzy generalisation of two relation–based operations suitable for fuzzy set approximations. Main properties of these operations are presented. We show that under specific assumptions these operations coincide with the fuzzy rough approximation operators.

Keywords: Fuzzy Sets, Fuzzy Modal Operators, Rough Sets, Fuzzy Logical Connectives.

1 Introduction and motivation

In many applications the available information has the form of a set of objects and a set of properties of these object. Relationships between objects and their properties can be naturally modelled by a binary relation connecting objects with their properties. For a subset $A$ of objects, which might be viewed as an expert decision, the real knowledge about $A$ results from two sources: the selected elements of the set $A$ itself, and the properties of objects. Since explicit information is usually insufficient for a precise description of objects of $A$, some approximation techniques are often applied. The theory of rough sets provides methods for set approximation basing on similarities between objects – relationships among objects determined by properties of these objects (see, e.g., [13],[15],[16],[25],[26],[31]).

A more general approach has been recently proposed by Düntsch et al. ([5],[6]). Sets of objects are approximated by means of specific operators based only on object–property relations, without referring to similarities among objects.

Both these approaches were developed under the assumption that the available information, although incomplete, is given in a precise way. However, it is often more meaningful to know to what extent an object has some property than to know that it has (or does not have) this property. For example, it is usually more significant to know to what extend an agent can speak English, rather than just know whether he/she speaks English or not. Moreover, degrees to which objects have some properties need not be comparable (e.g., having two agents who speak English fluently, it often cannot be decided which one speaks English better). When imprecision of data is admitted, a natural solution seems to be a fuzzy generalisation ([30]) of the respective methods. In addition, if incomparability is to be involved, some lattice–based fuzzy approach seems to be adequate ([8]). Hybrid fuzzy–rough techniques were widely discussed in the literature (see [3],[4],[12],[14],[17],[18],[19],[27],[28]).

In this paper, we continue our investigations
on fuzzy approximation operations ([17],[18], [19],[20],[21],[22]). More precisely, we propose a fuzzy generalisation of set approximation operators proposed in [5] and [6] and present their main formal properties. As a basic algebraic structure we take any complete and commutative residuated lattice (e.g., [2],[10],[29]). The importance of these structures mainly results from the role they play in fuzzy logics ([7],[9]). Moreover, taking these algebras as a basis, many traditional structures can be straightforward generalise and many important properties of the original structures are preserved. We will show that these operators, while based on residuated lattices, are a fuzzy closure and a fuzzy interior topological operators. It will be indicated that in general case these operators give a better set approximation than fuzzy rough set–style methods. However, under some restrictive assumptions, both these techniques coincide.

2 Preliminaries

2.1 Algebraic foundations

A monoid is a structure \((M, \otimes, \varepsilon)\), where \(M\) is a non–empty set, \(\otimes\) is an associative operation in \(M\) (i.e. \(a \otimes (b \otimes c) = (a \otimes b) \otimes c\) for all \(a, b, c \in M\)), and \(\varepsilon \in M\) is a distinguished element such that \(a \otimes \varepsilon = \varepsilon \otimes a = a\) for any \(a \in M\). A monoid is called commutative iff \(\otimes\) is commutative, i.e. \(a \otimes b = b \otimes a\) for all \(a, b \in M\).

Typical examples of monoidal operations are triangular norms and triangular conorms ([24]). Recall that a triangular norm (t–norm) is any mapping \(t : [0,1] \times [0,1] \to [0,1]\), associative and commutative, non–decreasing in both arguments, and satisfying the condition \(t(1,a) = a\) for any \(a \in [0,1]\). The most popular t–norms are: (i) the Zadeh’s t–norm \(t_Z(a, b) = \min(a, b)\), (ii) the algebraic product \(t_P(a, b) = a \cdot b\), (iii) the Lukasiewicz t–norm \(t_L(a, b) = \max(0, a + b - 1)\). A triangular conorm (t–conorm) is a mapping \(s : [0,1] \times [0,1] \to [0,1]\), associative, commutative, non–decreasing in both arguments, and satisfying \(s(0, a) = a\) for every \(a \in [0,1]\). Well–known t–conoms are: (i) the Zadeh’s t–conorm \(s_Z(a, b) = \max(a, b)\), (ii) the bounded sum \(s_P(a, b) = a + b - a \cdot b\), (iii) the Lukasiewicz t–conorm \(s_L(a, b) = \min(1, a + b)\).

Let \((W, \leq)\) be a poset and let \(\circ\) be a binary operation in \(W\). The residuum \(\rightarrow\) of \(\circ\) is a binary operation in \(W\) defined for all \(a, b \in W\) as follows:

\[ a \rightarrow b = \sup\{c \in W : a \circ c \leq b\}. \]

This operation is an algebraic counterpart of a residual implication ([11]). Three well–known residual implications are: (i) the Gödel implication (the residuum of \(t_Z\)) \(a \rightarrow_Z b = 1\) iff \(a \leq b\) and \(a \rightarrow_Z b = b\) otherwise, (ii) the Gaines implication (the residuum of \(t_P\)) \(a \rightarrow_P b = 1\) iff \(a \leq b\) and \(a \rightarrow_P b = \frac{b}{a}\) otherwise, and (iii) the Lukasiewicz implication (the residuum of \(t_L\)) \(a \rightarrow_L b = \min(1, 1 - a + b)\).

**Definition 1** A residuated lattice is a structure \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) such that

(i) \((L, \land, \lor, 0, 1)\) is a bounded lattice with the top element 1 and the bottom element 0,

(ii) \((L, \otimes, 1)\) is a monoid,

(iii) \(\rightarrow\) is the residuum of \(\otimes\).

The operation \(\otimes\) is called a product. □

A residuated lattice \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) is called complete iff the underlying lattice \((L, \land, \lor, 0, 1)\) is complete; it is called commutative iff its product \(\otimes\) is commutative.

Given a residuated lattice \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\), the precomplement operation is defined for any \(a \in L\) as:

\[ \neg a = a \rightarrow 0. \]

The operations \(\otimes\) and \(\rightarrow\) of a commutative residuated lattice are algebraic counterparts of a t–norm and a residual implication based on \(\otimes\), while \(\neg\) corresponds to a fuzzy negation ([11]).

One can easily observe that the operation \(\neg\) is a generalisation of the pseudo–complement in a lattice ([23]). If \(\land = \otimes\), then \(\rightarrow\) is the relative pseudo–complement and the lattice \((L, \land, \lor, \rightarrow, \neg, 0, 1)\) is a Heyting algebra.

Some basic properties of commutative residuated lattices are given in the following proposition.
Proposition 1 In every residuated lattice \((L, \land, \lor, \otimes, \to, 0, 1)\) the following statements hold for all \(a, b, c \in L\):

(i) \(\otimes\) is isotone in both arguments
(ii) \(\to\) is antitone in the 1st and isotone in the 2nd argument
(iii) \(\neg\) is antitone
(iv) \(a \otimes b \leq a \land b\)
(v) \(a \to a = a \to 1 = 0 \to a = 1\) and \(1 \to a = a\)
(vi) \(a \to b = 1\) iff \(a \leq b\)
(vii) \(a \to (b \to c) = (a \otimes b) \to c\)
(viii) \(a \otimes (a \to b) \leq b\)
(ix) \(a \leq \neg\neg a\)
(x) \((a \to b) \otimes (b \to c) \leq (a \to c)\).

If in addition \(L\) is complete, then for every \(a \in L\) and for every indexed family \((b_i)_{i \in I}\) of elements of \(L\),

(xii) \(a \otimes \sup_{i \in I} b_i = \sup_{i \in I} (a \otimes b_i)\)
(xiii) \((\sup_{i \in I} b_i) \to a = \inf_{i \in I} (b_i \to a)\)
(xiv) \(a \to (\inf_{i \in I} b_i) = \inf_{i \in I} (a \to b_i)\).

Some recent results on residuated lattices can be found, for example, in [10].

Example 1 Let \(t\) be a continuous triangular norm and let \(i_t\) be the residual implication determined by \(t\). The structure \(([0,1], \min, \max, t, i_t, 0, 1)\) is a commutative residuated lattice.

2.2 Fuzzy sets and fuzzy relations

Let \((L, \land, \lor, \otimes, \to, 0, 1)\) be a residuated lattice and let \(X\) be a non–empty universe. An \(L\)-fuzzy set in \(X\) is any mapping \(F : X \to L\). For any \(x \in X\), \(F(x)\) in the degree of membership of \(x\) to \(F\). By a fuzzy set we mean an \(L\)-fuzzy set for any residuated lattice \(L\).

Two specific \(L\)-fuzzy sets, \(\emptyset\) and \(X\), are defined as: \(\emptyset(x) = 0\) and \(X(x) = 1\) for any \(x \in X\). The family of all \(L\)-fuzzy sets in \(X\) will be denoted by \(\mathcal{F}_L(X)\).

Typical operations on \(L\)-fuzzy sets are:

\[\begin{align*}
(A \cup_L B)(x) &= A(x) \lor B(x) \\
(A \cap_L B)(x) &= A(x) \land B(x) \\
(A \cap_L B)(x) &= A(x) \otimes B(x) \\
(\neg_L A)(x) &= \neg A(x).
\end{align*}\]

For all \(A, B \in \mathcal{F}_L(X)\), we will write \(A \subseteq_L B\) to denote that \(A(x) \leq B(x)\) for every \(x \in X\).

Let \(f\) and \(g\) be two mappings of the form \(f, g : \mathcal{F}_L(X) \to \mathcal{F}_L(X)\). We say that \((f, g)\) is a pair of \(L\)-fuzzy approximation operations iff for every \(A \in \mathcal{F}_L(X)\),

\[f(A) \subseteq_L A \subseteq_L g(A)\]

An \(L\)-fuzzy relation from \(X\) to \(Y\) is any mapping \(R : X \times Y \to L\), i.e. this is an \(L\)-fuzzy set in \(X \times Y\). For any \(x \in X\) and for any \(y \in Y\), \(R(x, y)\) is the degree, to which \(x\) is \(R\)-related with \(y\). The class of all \(L\)-fuzzy relations from \(X\) to \(Y\) will be denoted by \(\mathcal{R}_L(X, Y)\). Given \(R \in \mathcal{R}_L(X, Y)\), we will write \(R^\sim\) to denote the converse of \(R\), that is \(R^\sim \subseteq \mathcal{R}_L(Y, X)\) is such that \(R^\sim(y, x) = R(x, y)\) for every \(x \in X\) and for every \(y \in Y\). An \(L\)-fuzzy relation \(R \in \mathcal{R}_L(X, Y)\) is called serial iff \(\sup_{y \in Y} R(x, y) = 1\) for any \(x \in X\).

If \(X = Y\), then any \(R \in \mathcal{R}_L(X, Y)\) is called an \(L\)-fuzzy relation on \(X\). The family of all \(L\)-fuzzy relations on \(X\) will be denoted by \(\mathcal{R}_L(X)\). Recall that \(R \in \mathcal{R}_L(X)\) is (i) reflexive iff \(R(x, x) = 1\) for any \(x \in X\), (ii) symmetric iff \(R(x, y) = R(y, z)\) for all \(x, y \in X\), (iii) \(L\)-transitive iff \(R(x, y) \otimes R(y, z) \leq R(x, y)\) for all \(x, y, z \in X\).

2.3 Fuzzy topological operators

In this section we recall the definitions of a fuzzy closure and a fuzzy interior operators (see, e.g., [1]). Let \((L, \land, \lor, \otimes, \to, 0, 1)\) be a residuated lattice and let \(int_L, cl_L : \mathcal{F}_L(X) \to \mathcal{F}_L(X)\) be two mappings. We say that \(int_L\) is an \(L\)-fuzzy interior operator iff for all \(A, B \in \mathcal{F}_L(X)\) it satisfies:

(I.1) \(int_L(X) = X\)
(I.2) \(int_L(A) \subseteq_L A\)
(I.3) \(int_L(int_L(A)) = int_L(A)\)
(I.4) \(A \subseteq_L B\) implies \(int_L(A) \subseteq_L int_L(B)\).

We say that \(cl_L\) is an \(L\)-fuzzy closure operator iff for all \(A, B \in \mathcal{F}_L(X)\) it satisfies:

(C.1) \(cl_L(\emptyset) = \emptyset\)
(C.2) \(A \subseteq_L cl_L(A)\)
(C.3) \(cl_L(cl_L(A)) = cl_L(A)\)
(C.4) \(A \subseteq_L B\) implies \(cl_L(A) \subseteq_L cl_L(B)\).
3 Modal–like operators based on residiuated lattices

Let a complete and commutative residiuated lattice \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) be given and let \(X\) and \(Y\) be two non–empty sets. Define two operators \(\{R\}_L, (R)_L : \mathcal{R}_L(X, Y) \times \mathcal{F}_L(Y) \rightarrow \mathcal{F}_L(X)\) as follows: for any \(R \in \mathcal{R}_L(X, Y)\), for any \(A \in \mathcal{F}_L(Y)\), and for any \(x \in X\),

\[
\begin{align*}
[R]_LA(x) &= \inf_{y \in Y} (R(x, y) \rightarrow A(y)) \quad (1) \\
(R)_LA(x) &= \sup_{y \in Y} (R(x, y) \otimes A(y)). \quad (2)
\end{align*}
\]

Observe that the above operators are generalisations of the fuzzy modal operators of fuzzy necessity and fuzzy possibility, respectively.

Assume that \(X\) is the set of objects, \(Y\) is the set of properties of these objects, and \(R(x, y)\), \(x \in X\) and \(y \in Y\), is the degree, to which the object \(x\) has the property \(y\). Then for any \(A \in \mathcal{F}_L(Y)\) and for any \(x \in X\), \([R]_L A(x)\) is the degree, to which all properties of the object \(x\) are in \(A\), while \((R)_L A(x)\) is the degree, to which some property of \(x\) is in \(A\).

**Definition 2** Let \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) be a complete and commutative residiuated lattice, let \(X \neq \emptyset\) and \(Y \neq \emptyset\), and let \(R \in \mathcal{R}_L(X, Y)\). Define the following two operators \(\triangle_L, \nabla_L : \mathcal{F}_L(X) \rightarrow \mathcal{F}_L(X)\) as: for every \(A \in \mathcal{F}_L(X)\) and for every \(x \in X\),

\[
\begin{align*}
\triangle_L(A)(x) &= (R)_L[R^{-1}]_L A(x) \quad (3) \\
\nabla_L(A)(x) &= [R]_L[R^{-1}]_L A(x) \quad (4)
\end{align*}
\]

(3) is called the \(L\)–lower bound of \(A\), while (4) is called the \(L\)–upper bound of \(A\).

Following the interpretation of the operators (1) and (2) given above, we can interepret the operators \(\triangle_L\) and \(\nabla_L\) as follows. For every \(L\)–fuzzy set \(A\) of objects and for every object \(x \in X\),

- \(\triangle_L(A)(x)\) is the degree, to which all objects having some property of \(x\) are in \(A\),
- \(\nabla_L(A)(x)\) is the degree, to which every property of \(x\) is possessed by some object from \(A\).

Main properties of the operators (3)–(4) are given in the following theorem.

**Theorem 1** Let \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) be a complete and commutative residiuated lattice, let \(X\) and \(Y\) be two non–empty sets, and let \(R \in \mathcal{R}_L(X, Y)\). Then

\[
\begin{align*}
\text{(i) if } R \text{ is serial then } \triangle_L(X) &= X \text{ and } \nabla_L(0) = 0 \\
\text{(ii) for every } A \in \mathcal{F}_L(X), \\
\triangle_L(A) & \subseteq_L A \subseteq_L \nabla_L(A) \\
\text{(iii) for every } A, B \in \mathcal{F}_L(X), \text{ if } A \subseteq L B, \text{ then} \\
\triangle_L(A) & \subseteq_L \triangle_L(B) \\
\nabla_L(A) & \subseteq_L \nabla_L(B) \\
\text{(iv) for every } A \in \mathcal{F}_L(X), \\
\triangle_L(\triangle_L(A)) & = \triangle_L(A) \\
\nabla_L(\nabla_L(A)) & = \nabla_L(A). \ \square
\end{align*}
\]

Theorem 1 immediately implies the following.

**Corollary 1** For every complete and commutative residiuated lattice \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) and for every \(L\)–fuzzy relation from \(X\) to \(Y\), \((\triangle_L, \nabla_L)\) is the pair of \(L\)–fuzzy approximation operations. \square

These operators might be interpreted as (some kind of) fuzzy necessity and fuzzy possibility, respectively. Specifically, for any \(A \in \mathcal{F}_L(X)\) and for any \(x \in X\), \(\triangle_L(A)(x)\) might be viewed as the degree, to which the object \(x\) certainly belongs to \(A\), whereas \(\nabla_L(A)(x)\) is the degree, to which \(x\) possibly belongs to \(A\).

In view of Theorem 1 we easily get:

**Theorem 2** For every complete and commutative residiuated lattice \((L, \land, \lor, \otimes, \rightarrow, 0, 1)\) and for every serial \(L\)–fuzzy relation from \(X\) to \(Y\), \(\triangle_L\) is an \(L\)–fuzzy interior operator and \(\nabla_L\) is an \(L\)–fuzzy closure operator. \square

In classical case modal operators of possibility and necessity are dual in the sense that the necessity (resp. possibility) of the fact \(A\)
is exactly the impossibility (resp. lack of certainty) of \( \neg A \). In the fuzzy case, however, duality does not hold in general. In particular, when complete residuated lattices are taken as a basis, fuzzy necessity and fuzzy possibility are only weakly dual (see [20]) in the following sense:

\[
[R]_L A \subseteq_L \neg (R)_{L} \neg A
\]

\[
(R)_{L} A \subseteq_L \neg [R]_{L} \neg A.
\]

The following theorem shows that the operators (3)–(4) share the same property.

**Theorem 3 (Weak duality)** Let a complete and commutative residuated lattice \((L, \land, \lor, \otimes, \neg, 0, 1)\) be given and let \(R \in \mathcal{R}_L(X, Y)\). Then for every \(A \in \mathcal{F}_L(X)\),

\[
\Delta_L(A) \subseteq_L \neg_L \nabla_L(\neg_A L) \quad \nabla_L(A) \subseteq_L \neg_L \Delta_L(\neg_A L). \quad \Box
\]

**Example 2** Let \(P\) be a non–empty set of patients and let \(S\) be a non–empty set of symptoms of these patients. For each patient \(p \in P\) and for each symptom \(s \in S\), we know the degree, to which \(p\) suffers from \(s\). This information is represented by an \(L\)-fuzzy relation from \(P\) to \(S\). A specialist, who investigates a new disease, say \(d\), decided for each patient to what extent his/her symptoms fit to the disease \(d\). This decision is represented by an \(L\)-fuzzy set \(D \in \mathcal{F}_L(P)\).

Now, we have the following interpretation of \(\Delta_L(D)\) and \(\nabla_L(D)\). Namely, for any patient \(p \in P\),

- \(\Delta_L(D)(p)\) is the degree, to which some symptom of \(p\) characterises the disease \(d\)
- \(\nabla_L D(p)\) is the degree, to which all patients who have some symptom of \(p\), suffer from the disease \(d\).

Let \([0, 1], \min, \max, t_L, i_L, 0, 1\) be the residuated lattice such that \(t_L\) and \(i_L\) are the Lukasiewicz triangular norm and the Lukasiewicz implication, respectively. Next, let the relationships between patients and symptoms, represented by the fuzzy relation \(R\), and the decision \(D\), be given as in Table 1. By simple calculations one can easily obtain the \(\mathcal{L}\)-lower bound \(\Delta_{\mathcal{L}}(D)\) of the decision \(D\) and the \(\mathcal{L}\)-upper bound \(\nabla_{\mathcal{L}}(D)\) of \(D\) given in

<table>
<thead>
<tr>
<th></th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>0.4</td>
<td>0.9</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>(p_2)</td>
<td>0.5</td>
<td>0.8</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>(p_3)</td>
<td>0.8</td>
<td>0.2</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>(p_4)</td>
<td>0.5</td>
<td>0.4</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>(p_5)</td>
<td>0.7</td>
<td>0.8</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>(p_6)</td>
<td>0.3</td>
<td>0</td>
<td>0.9</td>
<td>0.6</td>
</tr>
<tr>
<td>(p_7)</td>
<td>1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 1: The relation \(R\) and the decision \(D\)

<table>
<thead>
<tr>
<th>(\Delta_{\mathcal{L}}(D))</th>
<th>(D)</th>
<th>(\nabla_{\mathcal{L}}(D))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>0.3</td>
<td>0.8</td>
</tr>
<tr>
<td>(p_2)</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>(p_3)</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>(p_4)</td>
<td>0.4</td>
<td>0.5</td>
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<tr>
<td>(p_5)</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>(p_6)</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>(p_7)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 2: The decision \(D\) and its approximations

Table 2. Note that the expert decision wrt the patient \(p_6\) is exact – the degree of certainty and possibility of \(p_6\) suffering from the disease \(d\) coincide with the expert opinion. Yet for \(p_1\) it seems that the decision is too strong – it coincides with the degree of possibility, but the degree of certainty is much lower. In contrast, for the patient \(p_5\) the expert’s opinion seems to be too weak, since his/her decision coincides with the degree of certainty of \(p_5\) having the disease \(d\) and is definitely lower than the degree of possibility. \(\Box\)

Concluding this section it is worth emphasizing that using other types of fuzzy implications in the definitions of operators (3) and (4) in general do not guarantee the desired approximation property. In particular, recall that \(S\)-implications ([11]), the second main class of fuzzy implications, are defined as follows: \(I(x, y) = s(n(x), y)\), where \(x, y \in [0, 1]\), \(s\) stands for a triangular conorm, and \(n\) is a fuzzy negation. Clearly, in the signature of residuated lattices we do not have a counterpart of a triangular conorm. In order to have these operations, extended residuated lattices ([18],[21]) could be used (the signature of these algebras is extended by an
antitone involution $\sim$ satisfying $\sim 1 = 0$ and $\sim 0 = 1$; this operation is an algebraic counterpart of a fuzzy involutive negation). Recall also that the Kleene–Dienes fuzzy implication $IKD(x, y) = \max(1 - x, y)$, $x, y \in [0, 1]$, is the $S$–implication. In Example 2, using this implication and, as before, the Lukasiewicz t–norm, we get

$$\nabla_L(D) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{pmatrix}.$$  

One can easily note that $D \not\subseteq \nabla(D)$, so indeed these class of fuzzy implications are not suitable in definitions (3) and (4).

4 Fuzzy rough approximation operators

In this section we present relations between fuzzy approximation operators discussed in the previous section and fuzzy rough approximation operators being fuzzy generalisations of Pawlak’s concepts ([16]).

Let a complete and commutative residuated lattice $(L, \land, \lor, \otimes, \to, 0, 1)$ be given, let $X$ be a non-empty domain, and let $R \in \mathcal{R}_L(X)$. Traditionally, $X$ is a set of objects and $R$ represents similarities (or indiscernibilities) between objects determined by the properties of these objects. A structure $\Sigma = (L, X, R)$ is called a fuzzy approximation space. The following two operators $\Sigma$, $\overline{\Sigma} : \mathcal{R}_L(X) \times \mathcal{F}_L(X) \to \mathcal{F}_L(X)$ are defined as: for every $A \in \mathcal{F}_L(X)$ and for every $x \in X$,

$$\Sigma(A)(x) = \inf_{y \in X} (R(x, y) \to A(y)),$$

$$\overline{\Sigma}(A)(x) = \sup_{y \in X} (R(x, y) \otimes A(y)).$$

For any $A \in \mathcal{F}_L(X)$, $\Sigma(A)$ is called the $L$–fuzzy lower rough approximation of $A$ and $\overline{\Sigma}(A)$ is the $L$–fuzzy upper rough approximation of $A$. One can immediately observe that

$$\Sigma(A) = [R]_L(A),$$

$$\overline{\Sigma}(A) = \langle R \rangle_L(A).$$

The following proposition presents some main properties of the operators (5)–(6) (see, for example, [17],[19],[20]).

**Proposition 2** Let $(L, \land, \lor, \otimes, \to, 0, 1)$ be a complete and commutative residuated lattice and let $\Sigma = (L, X, R)$ be a fuzzy approximation space. Then

(i) for every $A, B \in \mathcal{F}_L(X), A \subseteq_L B$ implies $\Sigma(A) \subseteq_L \Sigma(B)$ and $\overline{\Sigma}(A) \subseteq_L \overline{\Sigma}(B)$

(ii) $R$ is reflexive

$$\iff \Sigma(A) \subseteq_L A \text{ for every } A \in \mathcal{F}_L(X)$$

$$\iff A \subseteq_L \overline{\Sigma}(A) \text{ for every } A \in \mathcal{F}_L(X)$$

(iii) $R$ is symmetric

$$\iff \overline{\Sigma}(\Sigma(A)) \subseteq_L A \text{ for every } A \in \mathcal{F}_L(X)$$

$$\iff A \subseteq_L \overline{\Sigma}(\overline{\Sigma}(A)) \text{ for every } A \in \mathcal{F}_L(X)$$

(iv) if $R$ is an $L$–equivalence relation, then

$$\Delta_L(A) = \Sigma(A)$$

$$\nabla_L(A) = \overline{\Sigma}(A)$$

(v) if $R$ is an $L$–equivalence relation, then $\Sigma$ and $\overline{\Sigma}$ are an $L$–fuzzy interior and an $L$–fuzzy closure operators

(vi) for every $A \in \mathcal{F}_L(X)$,

$$\Sigma(A) \subseteq_L \neg_L \overline{\Sigma}(\neg_L A)$$

$$\overline{\Sigma}(A) \subseteq_L \neg_L \Sigma(\neg_L A). \square$$

In view of the property (ii) of the above proposition, it is easy to see that the operators (5)–(6) are actually $L$–fuzzy approximation operators if $R$ is reflexive. In contrast, the operators $\Delta_L$ and $\nabla_L$ give fuzzy set approximations regardless of the properties of the underlying $L$–fuzzy relation. Furthermore, if $R \in \mathcal{R}_L(X)$ is symmetric, then clearly $R = R^-$, so by Proposition 1(iii) we have for every $A \in \mathcal{F}_L(X)$,

$$\overline{\Sigma}(\Sigma(A)) = \Delta_L(A)$$

$$\Sigma(\overline{\Sigma}(A)) = \nabla_L(A).$$

Hence, for symmetric fuzzy relations the above compositions of $L$–fuzzy rough approximations (5)–(6) coincide with the operators (3)–(4). If $R$ is both reflexive and symmetric, then we get for every $A \in \mathcal{F}_L(X)$,

$$\Sigma(A) \subseteq_L \Sigma(\Sigma(A)) \subseteq_L A$$

$$A \subseteq_L \Sigma(\overline{\Sigma}(A)) \subseteq_L \overline{\Sigma}(A).$$
It follows then that $\Delta_L$ and $\nabla_L$ give tighter approximations of fuzzy sets than $\Sigma$ and $\bar{\Sigma}$. Moreover, the operators (3) and (4) coincide respectively with (5) and (6) for the case when $X = Y$ and the underlying relation is an $L$–fuzzy equivalence only.

Note finally that by Corollary 2, $\Delta_L$ and $\nabla_L$ are $L$–fuzzy topological operators (assuming only seriality of the underlying relation), whereas $\Sigma$ and $\bar{\Sigma}$ are $L$–fuzzy topological operators if the relation $R$ is an $L$–equivalence only.

Conclusions

In this paper we have considered a fuzzy generalisation of two relation–based operators discussed by Düntsch and Gediga. It has been shown that, while taking an arbitrary complete and commutative residuated lattice $L$ as a basis, these operators are an $L$–fuzzy interior and an $L$–fuzzy closure topological operators. For more specific fuzzy relations these operators coincide with the fuzzy counterparts of Pawlak’s rough approximation operators. In more general case, however, they give tighter approximations of fuzzy sets. We have also pointed out that basing on (algebraic counterparts of) other class of fuzzy implications the discussed operators are not approximation operators – this justifies our choice of the underlying algebraic structure.

References


