Linear recurrence sequences of DP generated by the partial Bell polynomial and Faá di Bruno's Formula

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December 8, 2018

Summary of this talk:

One of basic properties in the Dynamic Programming is the general recursiveness and so it has been known to have the close relation with the several famous sequences. These are Fibonacci, Lucas sequence and Chebyshev's Polynomials, etc. Much of these could be founded in OEIS Web and the FQ journal.

Here we will show another representation for the relation of Perrin(Padovan) , Tribonacci sequences by using the partial Bell polynomial and Faá di Bruno's Formula.

本講演の概要

フィボナッチ数列はパスカル三角形での 2 項係数の斜めの和 (Slant sum) として表されることがよく知られている。ここではそれに関連して、

ルーカス数列、ペリン数列 (パドバン数列) など線形再帰数列

を 2 項係数の和として表し、さらにトリボナッチ数列を含み、その理由を部分 Bell 多項式や Faá di Bruno's Formula から考える。

Faá di Bruno's formula

W.P. Johnson; The curious History of Faá di Bruno's Formula, Math. Asso. Amer. Monthly, vol.109 (2002)

$$
\begin{aligned}
(f \circ g)^{(m)}(t) &= \frac{d^m}{dt^m} f(g(t)) \\
= \sum \frac{m!}{b_1! b_2! \cdots b_m!} f^{(k)}(g(t)) & \left(\frac{g'(t)}{1!}\right)^{b_1} \left(\frac{g''(t)}{2!}\right)^{b_2} \cdots \left(\frac{g^{(m)}(t)}{m!}\right)^{b_m} \\
= \sum f^{(k)}(g(t)) \cdot B_{m,k} & \left(g'(t), \cdots, g^{(m)}(t)\right)
\end{aligned}
$$

where the sum is over the non-negative integers $\{b_1, b_2, \ldots, b_m\}$:
$b_1 + 2b_2 + \cdots + mb_m = m$,
$k := b_1 + b_2 + \cdots + b_m$.

Example of Faá di Bruno's formula

[Example](cf. WikipediaA)

$$(f \circ g)^{(4)} = f^{(4)} \{g'g'g'g'\} + f^{(3)} \{6g''g'g'\} +$$

$$+f'' \{3g''g''\} + f'' \left\{4g^{(3)}g'\right\} + f' \left\{g^{(4)}\right\}$$

Partition counting: Coefficients for each term

$$\begin{array}{lcl} 1+1+1+1 & \longleftrightarrow & 1 \\ 2+1+1 & \longleftrightarrow & 6 \\ 2+2 & \longleftrightarrow & 3 \\ 3+1 & \longleftrightarrow & 4 \\ 4 & \longleftrightarrow & 1 \end{array}$$

Briefly $n = \underbrace{1+\cdots+1}_{b_1} + \underbrace{2+\cdots+2}_{b_2} + \underbrace{3+\cdots+3}_{b_3} + \cdots$

The number is $\dfrac{n!}{b_1!b_2!\cdots(1!)^{b_1}(2!)^{b_2}\cdots}$

Wolfram Mathematica

Command [BellY] in Mathematica(Wolfram) :

BellY$[n, k, \{x_1, \ldots, x_{n-k+1}\}]$

gives the partial Bell Polynomial $Y_{n,k}(x_1, \ldots, x_{n-k+1})$

Example is as follows.

$>$ In[1] $:=$ BellY[4,2,$\{x_1, x_2, x_3\}$]

$>$ Out[1] $=$ $3x_2^2 + 4x_1x_3$

This means that, by the above direct example shows,

$$(f \circ g)^{(4)}(x) = \cdots + f''\{3g''g''\} + f''\left\{4g^{(3)}g'\right\} + \cdots ,$$

the coefficient f'' gives $3g''g'' + 4g^{(3)}g'$ corresponding $3x_2^2 + 4x_1x_3$.

Fibonacci and Lucas sequences

Now firstly it begin to show you the well known results on Fibonacci sequence.

(1) $F_{n+1} = \sum_{i+j=n} \binom{i}{j}$

(2) Tchebyshev polynomial including the imaginary number i.

Then consider presentaion on the tribonacci sequence.
This uses the Faá di Bruno's formula and its generating function(Birmajer/Gil/Weuner(2015))

Binomial Coefficients creates Fibonacci sequences

Fibonacci Sequence

Slanted sum:

$$1 = 1 = \binom{0}{0}$$

$$1 = 1 = \binom{1}{1}$$

$$2 = 1 + 1 = \binom{1}{0} + \binom{2}{2}$$

$$3 = 2 + 1 = \binom{2}{1} + \binom{3}{3}$$

$$5 = 1 + 3 + 1 = \binom{2}{0} + \binom{3}{2} + \binom{4}{4}$$

Repeat !!

Fibonacci Quartary(FQ), OEIS; A000045

$$F_5 = 1 + 3 + 1 = \binom{2}{2} + \binom{3}{1} + \binom{4}{0}$$

$$F_6 = 3 + 4 + 1 = \binom{3}{2} + \binom{4}{1} + \binom{5}{0}$$

$$F_7 = 1 + 6 + 5 + 1 = \binom{3}{3} + \binom{4}{2} + \binom{5}{1} + \binom{6}{0}$$

$$\cdots \quad \cdots$$

In generally

$$F_{n+1} = \sum_{i+j=n} \binom{i}{j}$$

Tribonacci Sequence

Möbius axis

		b^2		
	$2ab$		$2bc$	
a^2		$2ac$		c^2
1	2	$1+2$	2	1

			b^3			
		$3ab^2$		$3b^2c$		
	$3a^2b$		$6abc$		$3bc^2$	
a^3		$3a^2c$		$3ac^2$		c^3
1	3	$3+3$	$1+6$	$3+3$	3	1

Feinberg; “New Slant”, Fibo.Quarterly 1964

	a	b	c				
$1st:$	1	1	1				
	a^2	ab	b^2, ac	bc	c^2		
$2nd:$	1	2	$1+2=3$	2	1		
	a^3	a^2b	ab^2, a^2c	b^3, abc	b^2c, ac^2	bc^2	c^3
$3rd:$	1	3	$3+3=6$	$1+6=7$	$3+3$	3	1

Contiuing to 4,5,6,7th $\cdots$

What kind of this relation in the table?

$4th$:	1	4	10	16	19	16	10	4	1		
$5th$:	1	5	15	30	45	51	45	30	15	5	1
$6th$:	1	6	21	50	90	126	141	126	90	50	$\cdots$
$7th$:	1	7	28	77	161	266	357	393	357	266	$\cdots$

$a_{i,j}$; i-th row and j-th column:

Example ; $a_{6,5} = 126$,

$$a_{5,2} + a_{5,3} + a_{5,4} = 30 + 45 + 51 = 126$$

The meaning from Feinberg's sum is

For examples, letting $n = 3$ and $a = 1, b = x, c = x^2$, that is ,

$$(a, b, c) = (1, x, x^2), \quad a + b + c = 1 + x + x^2$$

this imply that

Three terms

comparing between each coefficients,

$$\begin{array}{ll} a^3 = 1, \quad 3a^2b = 3x, & 3ab^2 + 3a^2c = 6x^2, \\ b^3 + 6abc = 7x^3, & 3b^2c + 3ac^2 = 6x^4, \\ 3bc^2 = 3x^5, & c^3 = x^6 \end{array}$$

by corresponding Möbius triangle axis.

Expession for Tribonacci sequences:

$$
\begin{array}{lll}
T_0 & = 1 = 1 & = a_{0,0} \\
T_1 & = 1 = 1 & = a_{1,0} \\
T_2 & = 2 = 1 + 1 & = a_{2,0} + a_{1,1} \\
T_3 & = 4 = 1 + 2 + 1 & = a_{3,0} + a_{2,1} + a_{1,2} \\
T_4 & = 7 = 1 + 3 + 3 & = a_{4,0} + a_{3,1} + a_{2,2} \\
T_5 & = 13 = 1 + 4 + 6 + 2 & = a_{5,0} + a_{4,1} + a_{3,2} + a_{2,3} \\
\cdots & \cdots\cdots &
\end{array}
$$

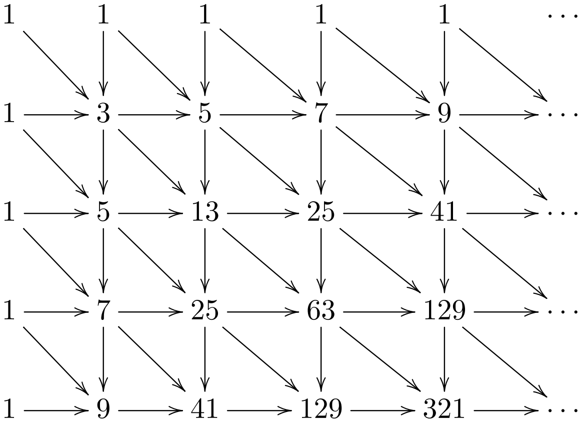
Wong/Maddocks (FQ 1975): Similar diagram as Finobacci slant sum in the Pascal triangle.

General form for $a_{n,j}$

Anatielleo and Vincenzi

$$a_{n,j} = \sum \binom{n}{j_1, j_2, j_3}; \quad j_1 + j_2 + j_3 = n, \; j_2 + 2j_3 = j$$

Path for the transition of three directions



Jump as Keima Tobi(桂馬 Shogi), Knight(Chess), so it becomes
$1 = T_0$, $1 = T_1$, $1 + 1 = 2 = T_2, 1 + 3 = 4 = T_3$,
$1 + 5 + 1 = 7 = T_4, 1 + 7 + 5 = 13 = T_5, \cdots$

Deleting the arrow, to get for the rule.

The matrix form

$$\begin{pmatrix} 1 & & & & & & & & \\ 1 & 1 & & & & & & & \\ 1 & 3 & 1 & & & & & & \\ 1 & 5 & 5 & 1 & & & & & \\ 1 & 7 & 13 & 7 & 1 & & & & \\ 1 & 9 & 25 & 25 & 9 & 1 & & & \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 & & \\ 1 & 13 & 61 & 129 & 129 & 61 & 13 & 1 & \\ 1 & 15 & 85 & 231 & 321 & 231 & 85 & 15 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

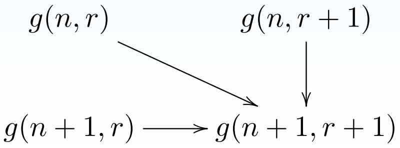
From the matrix to the sequence:

Sum of three terms

Alladi and Hogatt(1977)

$$g(n+1, r+1) = g(n+1, r) + g(n, r+1) + g(n, r)$$

where $g(n, 0) = g(0, r) = 1,\ n, r = 0, 1, 2, \cdots$



Then obtaining Tribonacci numbers

Each term of Tribonacci $\{T_n\}$ could be represented by $\{g(s,t); s,t \geq 0\}$ as follows:

$$\begin{aligned} T_n &= \sum_{s+t=n, s,t\geq 0} g(s,t) \\ &= g(n,0) + g(n-1,1) + g(n-2,2) + \cdots \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$T(n)$	1	1	2	4	7	13	24	44	81	149	274	$\cdots$

Tchebyschev Polynomial; 1st, 2nd

Two types of Tchebyschev Polynomial:

$$\text{1st;} \quad T_n(x) = \frac{\omega(x)^n + \overline{\omega}(x)^n}{\omega(x)\cdot\overline{\omega}(x)}, \quad T_n(\cos\theta) = \cos(n\theta)$$

$$\text{2nd;} \quad U_n(x) = \frac{\omega(x)^{n+1} - \overline{\omega}(x)^{n+1}}{\omega(x) - \overline{\omega}(x)}, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin(\theta)}$$

where $\omega(x) = x + \sqrt{x^2 - 1}$, $\overline{\omega}(x) = x - \sqrt{x^2 - 1}$.

$$\begin{cases} \omega(x)\cdot\overline{\omega}(x) = 1, \\ \omega(x) - \overline{\omega}(x) = 2\sqrt{x^2 - 1} \end{cases}$$

Using Chebyshev Polynomials

Curiously the imaginary i appears in the real sequence.

Lucas number $\{L_n\}$ by 1st Chebyshev Pol.$\{T_n(x)\}$:

$$L_n = 2\, i^n\, T_n\left(\frac{-i}{2}\right)$$

Fibonacci number $\{F_n\}$ by 2nd Chebyshev Pol.$\{U_n(x)\}$:

$$F_n = i^n\, U_n\left(\frac{-i}{2}\right)$$

Using Chebyshev Polynomials

Recurrent relation

The 1st Chebyshev :

$$T_0(x) = 1,\, T_1(x) = x \quad T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

The 2nd Chebyshev :

$$U_0(x) = 1,\, U_1(x) = 2x \quad U_{n+1} = 2xU_n(x) - U_{n-1}(x)$$

Difference is only the initial values and the same relation.

Several sequrences

Comments by OEIS

Perrin: For $n \geq 3$, $a(n)$ is the number of maximal independent sets in a cycle of order n. - Vincent Vatter, Oct 24 2006
For $n \geq 3$, also the numbers of maximal independent vertex sets, maximal matchings, minimal edge covers, and minimal vertex covers in the n-cycle graph C_n. - Eric W. Weisstein, Mar 30 2017 and Aug 03 2017

Padovan: Number of compositions of n into parts congruent to 2 mod 3 (offset -1). - Vladeta Jovovic, Feb 09 2005,
$a(n) =$ number of compositions of n into parts that are odd and $>= 3$. - David Callan, Jul 14 2006
Equals the INVERTi transform of Fibonacci numbers prefaced with three 1's, - Gary W. Adamson, Apr 01 2011

A050443: Related to Perrin sequence. $a(p)$ is divisible by p for primes p. Wells states that Mihaly Bencze [Beneze] (1998).

Perrin sequence

Related in the field of Combinatorial Theory and Integer Theory(D.E.Knuth,2011)

Perrin:(OEIS:A1001608)

$Pe(n)$:

$$Pe(0) = 3,\ Pe(1) = 0,\ Pe(2) = 2,$$

$$Pe(n) = Pe(n-2) + Pe(n-3)$$

http://www.math.s.chiba-u.ac.jp/~yasuda/ippansug/fibo/2016hoso11.pdf

Perrin sequence

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$Pe(n)$	3	0	2	3	2	5	5	7	10	12	17	$\cdots$

OEIS:A001608

$Pe(n)$:

$$Pe(n) = \sum_{(k,j)} \binom{k}{j} \frac{n}{k}; \quad \{(k,j); 2k+j=n\}$$

Padovan sequence

OEIS: A000931

$Pa(n)$:

$$Pa(0) = 1,\ Pa(1) = 1,\ Pa(2) = 1$$

$$Pa(n) = Pa(n-3) + Pa(n-4)$$

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$Pa(n)$	1	1	1	2	2	3	4	5	7	9	12	$\cdots$

Relation to Perrin (by WikipediA, OEIS) is

$$Pe(n) = Pa(n+1) + Pa(n-10)$$

OEIS:A050443

OEIS:A050443

$Q(n)$:

$$Q(0) = 4,\ Q(1) = 0,\ Q(2) = 0,\ Q(3) = 3,$$

$$Q(n) = Q(n-3) + Q(n-4)$$

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$Q(n)$	4	0	0	3	4	0	3	7	4	3	10	$\cdots$

Binomial coefficient and Fibonacci sequence

Fibonacci sequence

$F(n)$:

$$F(n) = \sum_{(k,j)} \binom{k}{j}; \quad \{(k,j) : k + j = n - 1\}$$

Binomial coefficient and Lucas sequence

Lucas sequence

$L(n)$:

$$L(n) = \sum_{(k,j)} \binom{k}{j} \frac{n}{k}; \quad \{(k,j) : k + j = n\}$$

Binomial coefficient and Perrin sequence

Perrin sequence

$P_e(n)$:

$$P_e(n) = \sum_{(k,j)} \binom{k}{j} \frac{n}{k}; \quad \{(k,j) : 2k + j = n\}$$

Binomial coefficient and A050443(OEIS)

OEIS:A050443

$Q(n)$:

$$Q(n) = \sum_{(k,j)} \binom{k}{j} \frac{n}{k}; \quad \{(k,j) : 3k + j = n\}$$

Why does it changed from the simple binomial to the added coefficient form and its sum condition from $\{(k,j); k+j=n\}$ to $\{(k,j); 2k+j=n\}$ or $\{(k,j); 3k+j=n\}$

$$\begin{array}{lcl} \text{Fibonacci} & & \text{Lucas, Perrin, A050443} \\ \binom{k}{j} & \Longrightarrow & \binom{k}{j}\frac{n}{k} \end{array}$$

In order to consider this reason, we'll pick up the Extended binomial coefficient by L.H.Lambert (1758) in the followings.

Extended binomial coefficient by J.H.Lambert

J.H.Lambert (1758)

$\mathcal{B}_t(z)$: Extended binomial coefficient

$$\mathcal{B}_t(z) = \sum_{k \geq 0} (t\,k)^{\underline{k-1}} \frac{z^k}{k!}$$

$x^{\underline{k}} = x(x-1)\cdots(x-k+1)$,
$x^{\underline{-k}} = \{(x+1)(x+2)\cdots(x+k)\}^{-1}$; $k \geq 0$

The notations by Graham, Knuth, Patashnik; Concrete Math (1994).

Relation with the binomial coefficient

$\mathcal{B}_1(z), \mathcal{B}_2(z), \mathcal{B}_{-1}(z)$: (Grahum/Knuth/Patashnik)

$$\begin{aligned}
\mathcal{B}_1(z) &= 1/(1-z) \\
\mathcal{B}_2(z) &= \textstyle\sum_k \binom{2k}{k} \frac{z^k}{1+k} \\
&= \textstyle\sum_k \binom{2k+1}{k} \frac{z^k}{1+2k} \\
&= \frac{1-\sqrt{1-4z}}{2z} \\
&= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots
\end{aligned}$$

Relation with the binomial coefficient

$\mathcal{B}_2(-1)$ and $\mathcal{B}_{-1}(z)$: (Grahum/Knuth/Patashnik)

$$\begin{aligned}
\mathcal{B}_2(-1) &= \frac{\sqrt{5}-1}{2} = 1/\phi, \quad \phi = \frac{\sqrt{5}+1}{2} \\
\mathcal{B}_{-1}(z) &= \textstyle\sum_k \binom{1-k}{k} \frac{z^k}{1-k} \\
&= \textstyle\sum_k \binom{2k-1}{k} \frac{(-z)^k}{1-2k} \\
&= \frac{1+\sqrt{1+4z}}{2} \\
&= 1 + z - z^2 + 2z^3 - 5z^4 + 14z^5 - 42z^6 + \cdots
\end{aligned}$$

Generating function:

$$\sum \binom{j}{k} z^k = \frac{(w_+)^{n+1} - (w_-)^{n+1}}{w_+ - w_-}$$

where the sum is $\{(j,k); j+k=n\}$ and

$$w_+ = \frac{1+\sqrt{1+4z}}{2}, \quad w_- = \frac{1-\sqrt{1+4z}}{2}.$$

Its generating function could imply

$$\sum \binom{j}{k} \frac{n}{j} z^k = w_+^n + w_-^n$$

where the sum is $\{(j,k); j+k=n\}$.

Lucas sequence

Therefore, Lucas number reduces to the follows

$$
\begin{aligned}
& \sum \binom{j}{k} \frac{n}{j} \\
= \quad & \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \\
= \quad & \{B_{-1}(1)\}^n + \{-B_2(-1)\}^n \\
= \quad & L_n \quad (LucasNumber)
\end{aligned}
$$

where the above sum is $\{(j,k); j+k=n\}$.

Binet formula

Lucas number

Definition

$$L_n = \alpha^n + \beta^n, \quad n = 0, 1, 2, \cdots$$

where $\alpha + \beta = 1, \alpha\beta = -1$, $(\alpha > \beta)$are two solution of the quardratic equation.

Binet formula

$$F_n = \frac{1}{5}L_n + \frac{2}{5}L_{n-1}$$

Similar to the Binet formula in this case

Similar as Binet form

Definition

$$A_n = \alpha^n + \beta^n + \gamma^n$$

where $\alpha + \beta + \gamma = 1, \alpha\beta + \beta\gamma + \gamma\alpha = -1, \alpha\beta\gamma = 1$ is the solution (one real number and two adjoint complex number)

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
A_n	3	1	3	7	11	21	39	71	131	241	443	$\cdots$

For a few numbers are $A_n = A_{n-3} + A_{n-2} + A_{n-1}, n = 3, 4, \cdots$. By OEIS, it is found the four case that A001644, etc. One of these is the avoiding sequences and it includs Iwamoto/Kimura/Yasuda(2013).

Tribinacci case

By the generating function

Tribinacci $\{T_n; n = 3, 4, 5 \cdots\}$ OEIS: A000073 has the next representation: A_n (OEIS: A001644)

$$
\begin{aligned}
T_n &= \frac{1}{22}\left[6\,A_n - 3\,A_{n-1} + A_{n-2} - 4\,A_{n-3}\right] \\
&= \frac{1}{22}\left[3\,A_{n-1} + 7\,A_{n-2} + 2\,A_{n-3}\right] \\
&= \frac{1}{22}\left[2\,A_n + A_{n-1} + 5\,A_{n-2}\right], \quad (n \geq 3)
\end{aligned}
$$

Partial Bell polynomial and Faá di Bruno's Formula

Source:
D.Birmajer, J.B.Gil and M.Weiner: Linear Recurrence Sequences and Their Convolutions via Bell Polynomials, Journal of Integer Sqeuences, Vol.18(2015)
For explicit forms, sequences are expressed by the Binomial coefficients.

(I) Fibonacci sequence ;

By Bell polynomial $B_{n-1,k}(1,2,0,\cdots) = \frac{(n-1)!}{k!}\binom{k}{n-1-k}$,

$$F_n = \sum_{j=0}^{n-1} \binom{n-1-j}{j} \qquad (1)$$

(II) Padovan sequence ;

By $B_{n-3,k}(0, 2!, 3!, 0, \cdots) = \frac{(n-3)!}{k!}\binom{k}{n-3-2k}$,

$$Pa(n) = \sum_{k=0}^{n-3}\binom{k}{n-3-2k} \quad n \geq 3 \tag{2}$$

(III) Tribonacci sequence ;

By $B_{n,k}(1!,2!,3!,0,\cdots) = \frac{n!}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}\binom{k-l}{n+\ell-2k}$,

$$T_n = \sum_{j=0}^{n-2}\sum_{k=0}^{j}\binom{k}{j-k}\binom{j-k}{n-2-j} \quad n \geq 2 \tag{3}$$

Thank you for the attendance.
Also for Prof.M. Tamaki giving me this chance to talk.
ご清聴ありがとうございました。
ご参加ありがとうございました。