安田正實教授退官記念集

2012 年 3 月

The publications of Professor Masami Yasuda
Congratulations on his retirement from Chiba University,
Department of Mathematics and Informatics
March, 2012

はじめに

歴史的な天災、東日本大地震から早いもので１年が経ちました。この未曾有の大惨事をどう克服していくかということは国民一人一人が真剣に考えなければなりません。そんな中、世間では「リスク」、「収束」ということばが頻繁にとびかっておりますが、はたして数学的にはそれはきちんと定義されているのでしょうか？収束は数学的にはご存知のとおりイプシロンデルタ論法により厳密に定義できますが「リスク」に関しては、未だ発展途上にあるように思われます。東京大学教授の楠岡先生は雑誌「数理科学」の中で、２０世紀は確率に関する研究で大きな成功をおさめることができたがはたして２１世紀はリスクに関する研究の成功をなしとげることができるだろうか、といった趣旨のことを述べられておりました。

安田研究室でも安田正實教授、蔵野正美名誉教授を中心にこの３，４年間「リスク」という言葉に一番ふさわしい数学的定義は何なのか？という研究を続けてきました。

このような社会の注目を浴びる最先端の研究室に所属できたことを私は本当に幸せに、そして誇りに思います。

最後に、お忙しい年度末にもかかわらず、すばらしい論文を寄稿していただいた国内外の先生方に心から感謝申し上げます。

影山正幸（統計数理研究所、４月より名古屋市立大学赴任予定）

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Publication List of Professor Masami Yasuda

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共同研究者と学生より

安田先生とのこと 城西大学理学部数学科講師　岩村覚三

安田先生と初めてお会いしたのは 5thBC（Bellman　Continuum）　Hawaii　の少なくても 1 年以上前のはずだから 19991-1992 年前後です。その頃の DP 研究部会は日本科学技術連盟千駄ヶ谷のセミナールームで開催されてました。そのセミナールームで初めてお会いしました。私は、1970 年から 1985 年ごろまで整数計画法のアルゴリズムの研究で貴重な時間を浪費してしまい、matroid や greedoid 上のアルゴリズムは全て離散 DP アルゴリズムであることを確認して DP アルゴリズムの偉大さにわが身を鞭打たれ、当時は小田中敏夫先生が事務方をやっていた日本オペレーションズ・リサーチ学会 DP 研究部会に帰ってきたばかりでした。Stopping problem を研究してこられたとのことでしたが、既に蔵野先生や中神先生方とファジイ過程（ファジイ Markov Decision Process、ファジイ MDP）の構成とその諸特徴の研究を始めていました。岩村は小田中モデルが上手く数学に乗らないでもがいていましたが、安田先生方の方法は上手くいって Fuzzy Sets and Systems に共著論文が次々に掲載されていました。結局小田中モデルは目的関数を積分形式のファジイ DP とでも呼べるような DP として、中国科学院から日本に来た Baoding Liu により大方の決着がつきました。その結果は Liu-Esogbue として英語の本にもなっています。ファジイ過程の研究はそこで蔵野・安田・中神（時には吉田）のファジイ MDP、Liu-Esogbue の Fuzzy criterion set and fuzzy criterion dynamic programming、それに九州大学岩本先生のファジイ決定過程モデルの 3 通りがあるわけです。その中で、安田先生がたのモデルは Hausdorff space を使ったもっとも難しいモデルになっていますが、ほとんどの論文には例題が付けられていてわかりやすく述べられているといえます（だが、岩村は細部の全ての証明を後付け終わったわけではない！）。1994 年の 6thBC Hachiouji では蔵野先生方と論文審査を分担されたり、また岩村勤務先の城西大学数学科でファジイ MDP の基礎に関する講演をしていただいたりしました。岩村は日本数学会に所属していないので日本数学会では安田先生との関係はありません。そうでした、つい最近の DP 研究会では（2008 − 20010）主査・副主査で研究部会の裏方をご一緒しました。千葉大学理系総合研究棟一階まで安田先生が数学・情報科学科棟からポットや PC や電源コード類を持ってきて参加者にコーヒーやお茶の準備をしてくれたことを今でも感謝しています。岩村はひそかに千葉大学統計数学三人衆と呼んで安田、蔵野、中神の三人の共同研究を称賛の目で見つめ、若干羨ましくも思っていました。共同研究は意外に遂行が難しく、些細な好みでせっかくの共同研究をやめてしまったり、最悪の場合は相互に非難中傷をしてしまったりとなかなか難しいものです。そういう風にはならずに、ファジイ MDP モデル構築を成し遂げられた安田先生におめでとうございますとお祝いの言葉を退官を記念して差し上げたいと思います。（終わり）

安田先生との共同研究　千葉大学名誉教授　蔵野正美

安田先生！千葉大学でのお仕事、永い間御苦労さまでした。無事定年退職を迎えられ、まずは心よりお祝い申し上げます。

「ひとたび経て、再びは来ない野中の道…」(三好達治詩集　旅人)

千葉大での安田先生を中心とした共同研究を思い起こすことは私にとって喜びでありまた有意義なことです。

私は昭和 47 年に千葉大教育学部に赴任しました。その後安田先生が鹿児島大から教養部にこられました。少し後れて中神先生が理学部にこられました。これで三人組がそろいました。毎週月曜日の午後、定例で安田先生の研究室（セミナー室付きの広い部屋）に集まり、最適停止問題の教科書

Y. S. Chow and H. Robbins, Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin, 1971

を勉強しました。黄色く変色した原本のコピー冊子の目次をみますと、例えば第 4 章では §1 – §3(中神)、§4 – §6(蔵野)、§7 – §9(安田) とセミナーの発表者の名前が記されていました。(なんと素晴らしいことか！私の財産！私の青春！）

このセミナーで停止ルールの単調性（ one-stage lookahead policy）の重要性と応用性の広さを学び、これを基にして共著の論文を書きました。懐かしい響き：多数決ルール、あるいは単調論理関数の多人数停止問題（OR 学会誌 1980, 1982)。[その後 10 年を経てその連続版（Dynamic Games, 1999）を書きました。]

これにより我々三人組の結び付きは不動のものとなり、私は共同研究について貴重な体験をしました。その後、北九州大学の吉田先生が加わってのファジー数学の共同研究、さらに和歌山大学の門田先生が加わっての一般効用関数の MDPs の共同研究を進んで行きました。これらはすべて安田先生を真ん中にすえた共同研究でした。

私のとって共同研究の良い点は、人それぞれ理解の仕方や感じ方、イメージの作り方、また発想の仕方などに微妙あるいは大いなる違いや独自性があり、それをお互いに付き合わせることにより新しい課題の発見や新しいアイディアが生まれてくることでした。また、スランプに落ちいって論文が全く書けない時など共同研究の有難さが身にしみました。

さて共同研究をしたくてもその場がなければ成立しません。私に場を提供して下さったのは安田先生です。感謝感謝！安田先生の人を寛大に受けいれる包容力、ふところの広さには尊敬に値いするところです。それに決断力と積極性！

退職後には楽しいことが沢山待っているらしいです。健康第一にして、楽しく過ごされんことを祈っています。

私が尊敬する安田正實先生

大阪府立大学名誉教授・近畿大学経営学部教授

寺岡義伸

いよいよ、この3月末でご退官、長い大学教員生活、お疲れ様でした。私の時もそうでしたが、このような場合、皆さんは「おめでとう」とお祝いの言葉を下さるのですが、見方を変えれば、年を取って身体にガタがきたので、大学としては用済みとも考えられます。「この齢まで無事に勤めることが出来ました」という意味で、お祝いなのでしょうか。しかし、現在では65歳はまだ青年の域、20代の娘と結婚して子供を作る人物も出てきています。お互いに、健康に注意して「あの年寄り、いつまでも元気で、しつこいな」と言われるよう、大いにのさばりましょう。

私が、安田先生と初めてお会いしたのは、昭和45年春の「統計数学若手研究会」の時でした。それから、統計サマーセミナー・OR学会・数学会・計画数学シンポジウム・RIMS研究集会等々を通じて、随分長いお付き合いとなりました。

安田先生の研究業績は“世界の安田”として、その道のプロから高く評価され、質は勿論、量もすごく、大変な業績、私のような者がとやかく批評するのは失礼と言うものですので、差し控えさせていただきます。

安田先生の良い所は、すごい研究業績を出されながらも、ちっとも偉ぶらず、他人にも優しく、学生の面倒見も細やかな、その素晴らしい人柄だと思います。私よりも3歳年下ですが、昔から尊敬しておりました。そんな訳で、先日、影山先生から、記念論文集の原稿を頼まれた時は、是非原稿を提出させていただきたいと思いましたが、私も、この3月末、2度目の勤務先である近畿大学を定年退職となり、学年末の雑用に加え、研究室の整理で、とても新しい研究成果を論文の形に纏められる環境にありません。更に加えて、1年半前から足や腰の痛みや痺れを感じるようになり、昨年から歩行に杖を使う身となってしまいました。そこで、この3月、学校の仕事が一段落した頃を見計らって、徹底的に身体の検査と治療できるよう、予定を入れてしまいました。

申し訳ありませんが、安田正實先生退官記念講演会と懇親会には、欠席させていただきます。また、記念論文ですが、研究室も整理してしまい、計画数学に関しての新しい結果を出せる環境にありませんので、計画数学とは距離のある分野の研究結果の要旨のみを提出させていただきます。

この研究は、私が研究者となって最初に取り組んだ“光の相互反射”に関する研究で、先日、ある縁から電気学会の照明分野で発表する機会を得ました。その講演に先立ち、相互反射に関するその後40年の発展を調べましたところ、相互反射の数学的基礎に関しては、一応の完成状態として大きな変化が無い状態でした。その当時の理

論は、光源は拡散光源で壁面は拡散面を前提に作られていました。ところが、近年、指向性光源の開発や非拡散性壁面の増加に伴い、拡散性を仮定しない相互反射の計算式の定式化が求められていることを知りました。拡散性を仮定せず、単に光束の授受としての相互反射を眺めて観ると、マルコフ連鎖の極限分布として、表現できることに気がつきました。この考えを纏めたのが下記の要旨です。

一見、経済現象とは無関係と見える照明工学の分野に相互反射と呼ばれる現象がある。光源を有する室内で各壁面での明るさを算定する場合、周囲に光を反射しない壁面から構成された室内であれば、直射照度を求めれば十分であるが、実際われわれが体験する照明環境では、多数の面で光の反射が無限に繰り返され、一つの平衡状態に到達する。このような光の相互反射を考慮した解析なしでは、正しい照明設計は行えないといえる。ところで、この光（光束）の相互反射を注意深く観察してみると、経済現象における財の流れと意外な類似性があることに驚かされる。

次に発表する論文は、室内にガラス戸やカーテンのような反射性のみならず、透過性と吸収性を持つ膜で仕切られた光学系を仮定し、単に面と面との間の光束の授受だけを前提として、透過性と反射性を持つ面間の相互反射の近似方程式を、推移行列を用いて表現する。この方程式で、各壁面を経済主体、光束を財で置き換えると、一般化された多重相互反射の方程式は、経済現象における財の均衡方程式に通じることに驚かされる。

一般化された多重相互反射の近似方程式

はじめに　近年、指向性光源の開発に伴い、これまでの均等性光源や拡散性照明器具を前提にした照明計算に修正の必要性が求められている。しかしながら、照明計算の出発は光束の移動の数学的記述であり、特別な場合として光源の均等性や壁面の拡散性が仮定された結果が示されているに過ぎない。本報告では、光源の均等性や壁面の拡散性を仮定せず、単に面と面との間の光束の授受だけを前提として、透過性と反射性を持つ面間の相互反射の近似方程式を推移行列を用いて表現する。指向性光源や非拡散性反射面・透過面を含む形式の明確化を目的とする。

モデルと定式化　一つの光学系を考える。その光学系は、ほぼ一様な光束発散度を持つようなn個の面に都合よく分割されているとする。各面では入射した光は反射されるか透過するか、吸収されるものとする。この系内に光源が置かれると、発生した光は各面で反射・透過・吸収を無限に繰り返し一つの平衡状態に到達する。相互反射を記述するには、次の２つの原理が基礎となる。

光束保存の原理

“相互反射系の各面素において、その面素の入射する光束は、その部分に吸収される光束とそれから外へ発散（反射または透過）される光束との和に等しい”。

相互反射系における全光束の法則

“相互反射系内のある面素から発散される光束は、その面素から直接発散される光束と、その面素から見える系内の他の部分からの相互反射による光束の増加分、およびその面素に透過してくる光束の和に等しい”。

ここでは全て各面から発散される光束を基本に各面の状態を記述する。

系内での光束の移動を観察し次の2種の係数を定義する。

空間的な位置関係および面の反射特性と透過特性により、面iから発散された単位光束のうち面jへ入射される光束を$F(i,j)$と置き**空間移動係数**と呼ぶことにする。

この時

$$0 \le F(i,j) \le 1 \quad for \quad i,j=1,---,n \quad ; \quad \sum_{j=1}^{n} F(i,j)=1 \quad for \quad i=1,---,n$$

が成立する。

また、反射や透過もi面から面jへの非空間的な光束の移動と考え、$\tau(i,j)$で示し**面内推移係数**と呼ぶことにする。この係数に関しては

$$0 \le \tau(i,j) \le 1 \quad for \quad i,j=1,---,n \quad ; \quad \sum_{j=1}^{n} \tau(i,j) < 1 \quad for \quad i=1,---,n$$

が要請される。

この時、$\tau(i,j)>0$ならば面iから面jへこの率の光束が透過するということである。これは、面iに入射した光がそこで光電池で電流に変換されて面jへ移動し、再び光に変換されて面jから発散する場合も含める。特別な場合として

$$\tau(i,i)=\rho_i \text{ は 面} i \text{での反射率}$$

を意味することは言うまでも無い。

無限回の反射と透過・吸収の繰り返しの後平衡状態になったときの面iにおける光束発散度をL_i、相互反射以前の面iにおける初期光束発散度をL_{0i}と置く。

上記二つの原理から、

$$[L_i]=[L_{0i}]+[\tau(i,j)]^T[F(i,j)]^T[L_i]$$

（一般化された多重相互反射の近似方程式）

が成立する。

ここに、上付きのTは転置行列を意味する。　また、$[L_i]$と$[L_{0i}]$は$n\times 1$列ベクトル、$[\tau(i,j)]$と$[F(i,j)]$は$n\times n$正方行列であることは言うまでもない。

照明工学の書物では空間移動係数は使われず、全ての面は完全拡散面と仮定され、光束授受の順番も逆（すなわち、行と列が逆）になっており、形態係数、交換係数、固有光束入射係数等と呼ばれる係数を用いている[1,2,3]。

指向性光源に対しては、各面への直射照度から初期光束発散度を計算して上式に代入、非拡散性面に対しては面iから面jへの光束の移動量を反射特性や透過特性と幾何学的位置関係を考慮して$F(i,j)$ を計算して代入すればよい。

参考文献

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Exact fraction for the probability of run by A. de Moivre using Fibonacci, Tribonacci sequence

岩本誠一　　　木村寛　　　安田正實

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概要

情報理論、ハイパーキューブの部分列や株価変化のチャート理論、さまざまな分野において、フィボナッチ数列が応用されることはよく知られている。ここでは、1717 年（初版）に A.de Moivre (ドモアブル)「偶然の学理 (The doctrine of Chances)」がベルヌーイ列における連（生起継続回数）の計算で求めた式であることを述べる。彼の計算結果はこのフィボナッチ数、トリボナッチ数、さらにテトラナッチ数など一般化フィボナッチ数による関係で示されるものである。さらにそれを拡張したと主張する、積分の台形公式で知られている T.Simpson(シンプソン) の 1740 年の結果との関連も紹介する。これらのことから、de Moivre による連の計算、Simpson による拡張が分数で表現できる再帰関係式を示す。

1　フィボナッチ数とトリボナッチ数

1.1　フィボナッチ列

フィボナッチ数 (Fibonacci numbers: F(n) = F(n-1) + F(n-2) with F(0) = 0 and F(1) = 1) は非常に多くの分野で、さまざまな形による結果としてよく知られている。この数列は $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots\}$ は Lamé's sequence ともよばれるが、とくにここでは

F(n+2) = number of binary sequences of length n that have no consecutive 0's.

F(n+2) = number of subsets of 1,2,...,n that contain no consecutive integers.

を取り上げる。つまり、フィボナッチ数は、0 と 1 からなるすべての文字列のなかで、部分列 "11" を含まないものの総数と一致することが知られている。たとえば、フィボナッチ数列を入力して検索できる The OEIS(On-Line Encyclopedia of Integer Sequences) Foundation WEB page に記述され、よく知られている。1963 年からフィボナッチ協会発行の The Fibonacci Quarterly には数多くの結果が報告されている。

上記の命題としては、

> The probability of not getting two heads in a row in n tosses of a coin is $F_(n+2)/2^n$ (Honsberger 1985, pp. 120-122). Fibonacci numbers are also related to the number of ways in which n coin tosses can be made such that there are not three consecutive heads or tails.

と述べられている。

文献 [Hons]: Honsberger, R. "A Second Look at the Fibonacci and Lucas Numbers." Ch. 8 in Mathematical Gems III. Washington, DC: Math. Assoc. Amer., 1985.

文献 [ChWe]: Chandra, Pravin and Weisstein, Eric W. "Fibonacci Number." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/FibonacciNumber.html`

また

Tetranacci numbers: a(n) = a(n-1) + a(n-2) + a(n-3) + a(n-4) with a(0)=a(1)=a(2)=0, a(3)=1. について

$$a(n+4) = \text{number of}\, 0-1\, \text{sequences of length}\, n\, \text{that avoid}\, 1111.$$

- David Callan (callan(AT)stat.wisc.edu), Jul 19 2004. The On-Line Encyclopedia of Integer Sequences!

文献 [MiSe]:「フィボナッチ列の O(1) 時間生成について」三河 賢治, 仙波 一郎、「An O(1) Time Algorithm for Generating Fibonacci Strings」Kenji MIKAWA and Ichiro SEMBA, 電子情報通信学会論文誌 Vol. J85-D-I, No. 2, pp. 116-121, Feb. 2002. では、このリストを効率よく生成するアルゴリズムが研究されている。またハイパーキューブの部分列、ハミルトンパスの再帰的構成、また株価変化に関するチャート理論などにもフィボナッチ列の応用が知られている。入試問題などにも多く見受けられる (2011 年大阪教育大後期、2010 年千葉大前期、2008 年横浜国立大後期)。

一方、数論の研究者では組合せ数論のなかでより深い研究がなされ、このうち、つぎの論文などが、フィボナッチ数列での結果、0と1の数列で"11"を表れない場合数の数え上げがフィボナッチ数となること、の拡張を論じている。

文献 [Ca06]: D.Callan: Permutations avoiding a Nonconsecutive Instance of a 2- or 3-Letters Pattern, 2006. `www.stat.wisc.edu/~callan/notes/nonconsec.../nonconsec_pattern.ps`

文献 [Ca09]: D. Callan, Pattern avoidance in "flattened" partitions, Discrete Math., 309 (2009), 4187-4191.

ここでは上記の結果に関連して、17 世紀の数学者ドモアブル、文献 [deM]: Abraham de Moivre(1667-1754) による "「The principle of Chance(偶然の学理)」の第 LXXIV(74) 問(連の確率計算)"(1738年第2版)、(ただし 1756 年では第 LXXXVIII(88) 問) の解がトリボナッチ数列による分数表現で与えられることを述べる。この文献は `doctrineof chance00moiv.pdf`(size 18M) で検索すれば入手できる。さらにより高次の再帰関係数列式、テトラナッチ数列への拡大、更なる発展、定積分の近似計算式で有名なシンプソン、文献 [Sim]:Thomas Simpson(1710-1761) による「偶然の性質と法則」(The Nature and Laws of Chance)(1792 年) での剽窃的な記述の結論に触れる。彼自身による序文では、「皆のために、ド・モアブルのような偉大な人のあとで、このような題目を解説しようと企てるのは、ずうずうしいと思われるかも知れないが、それでも私は満足だ」と述べている。

本論の目的は、動的計画法の理論でわれわれが議論していた事実 [岩本ほか 2006 年、2007 年 etc] が、既にド・モアブルによる母関数を一種の多項式にもちいた確率計算には、数列の再帰関係式が表れていることを注意したい。

Abraham de Moivre(1718, 1738, 1756); The principle of Chance, 1738 第88問、1756 第74問 (The probability of a run of given length)。

Isac Todhunter(1865); A History of the Mathematical Theory of Probability from the ime of Pacsal to that of Laplace, Macmillan, London. Reprinted by Chelsea, New York, 1949. 「確率論史」(安藤洋美訳) 現代数学社、(1975) 第9章ドモアブル。

Thomas Simpson(1740); The Nature and Laws of Chance. The Whole after a new, general, and conspicuous Manner, and illustrated with a great Variety of Examples. Cave, London. Reprinted 1792.

Pierre-Simon de Laplace,(1812); Théorie Analytique des Probabilitiés. Paris. 2nd. ed. 1814; 3rd.ed.1820. Reprinted in Oeuvres, Vol.7,1886.

安藤洋美:「確率論の生い立ち」現代数学社、1992。

Anders Hald(1989): A History of Probability & Statistics and their Applications before 1750, John Wiley & Sons, 第 22 章 de Moivre and the Doctorine of Chances,1718,1738, and 1756, 6 節 The Theory of Runs.

Julian Havil; IMPOSSIBLE? Surprising Solutions to Counterintuitive Conundrums, Princeton Univ Press, 2008.

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S.Iwamoto and A.Kira; On Golden inequalities, RIMS 1504 (2006), 168-176.

S.Iwamoto and Y.Kimura; Alternate Da Vinci Code, Journal of Political Economy, Kyushu University, 76(4) 2010, 1-19.

S.Iwamoto and M.Yasuda; Golden Optimal value in discrete-time dynamic optimization processes, RIMS 1559 (2007), 56-66.

S.Iwamoto and M.Yasuda; Golden Optimal path in Discrete-time Dynamic Optimization Processes, Advanced Studies in Pure Math., Volume 53, 2009, Pages 99-108.

Theorem 1.1 n 桁の $\{0, 1\}$ からなるすべての列 2^n のうち、部分列 11 を含まないもの (a sequence of avoiding "11") は、$F(n+2)$ 個ある。ここで $F(n) = F_n$ はフィボナッチ数:

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$F(n) = F_n$	1	1	2	3	5	8	13	21	34	55	$\cdots$

上記の命題を確かめるために、2つの例を挙げる。

Example 1 $n = 3$ 桁の $\{0, 1\}$ からなるすべての列 $2^3 = 8$ のうち、部分列 11 を含まないものは、$2^3 = 8$ 個のうち、$F(3+2) = F(5) = 5$ 個ある。

$x_1\ x_2\ x_3$	included/none
0 0 0	none
0 0 1	none
0 1 0	none
0 1 1	included
1 0 0	none
1 0 1	none
1 1 0	included
1 1 1	included

$x_1\ x_2\ x_3\ x_4$	$\sum x_i x_{i+1}$		
0 0 0 0	0	1 0 0 0	0
0 0 0 1	0	1 0 0 1	0
0 0 1 0	0	1 0 1 0	0
0 0 1 1	1	1 0 1 1	1
0 1 0 0	0	1 1 0 0	1
0 1 0 1	0	1 1 0 1	1
0 1 1 0	1	1 1 1 0	2
0 1 1 1	2	1 1 1 1	3

Example 2 $n = 4$ 桁では $F(4+2) = F(6) = 8$ 個ある。この場合における表では, *included/none* の代わりに、$\sum_i x_i x_{i+1}$ を計算した。明らかに

$$\text{"11" are included} \leftrightarrow \text{"}\textstyle\sum_i x_i x_{i+1} \neq 0\text{"}$$

$$\text{"11" are none"} \leftrightarrow \text{"}\textstyle\sum_i x_i x_{i+1} = 0\text{"}$$

が成り立つから、この計算 $\sum_i x_i x_{i+1}$ によって判別を下すことができる。つまりこの値が 0 かどうかによって数え上げれば、つぎの結果が得られる。

Theorem 1.2 (同値命題の定理) n 個の独立なベルヌーイ列、$X_i \sim \mathrm{Binom}\left(1, \frac{1}{2}\right), i = 1, 2, \cdots, n$ において、

$$P\left(\sum_{i=1}^{n-1} X_i X_{i+1} = 0\right) = \frac{F(n+2)}{2^n} \tag{1}$$

が成り立つ。ここで $F(n) = F_n$ は n-th フィボナッチ数とする。

(注意) 論理関数として、積和 $\vee_{i=1}^{n-1}(X_i \wedge X_{i+1}) = 0$ をもちいても同じ。

Lemma 1.1 数列 $\{a_n, b_n\}_{n=1,2,\cdots}$ を、$a_1 = b_1 = 1$ とし、また $n \geq 1$ について

$$\begin{cases} a_{n+1} &= a_n + b_n \\ b_{n+1} &= a_n \end{cases} \tag{2}$$

とおくと、

$$F_{n+2} = a_n + b_n, \quad F_n = b_n \tag{3}$$

が成り立つ。

Lemma 1.2

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \qquad n = 1, 2, \cdots \tag{4}$$

定理 1.2 の証明: n 桁のフィボナッチ列を考える。このうち、

(1) n 桁目の値が 0 の個数を a_n とし、
(2) n 桁目の値が 1 の個数を b_n とする。

すなわちフィボナッチ列の総数は $a_n + b_n$ である。つぎに続く $n+1$ 桁目の数字を考えると、a_n として数えたものは 0 と 1 の 2 種類がつぎ桁に位置づけられる。しかし b_n で数えたものは 0 のみしか位置づけられない。このことから、(2) が成り立つ。

したがって

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}$$

n 番目では

$$\therefore a_n + b_n = F_{n+1} + F_n = F_{n+2}, \quad b_n = F_n$$

(QED)

Corollary 1.1 フィボナッチ列（情報理論の意味で）において

(1) n 桁目の値が 0 の個数は $a_n = F(n+1)$.
(2) n 桁目の値が 1 の個数は $b_n = F(n)$.

が成り立つ。

たとえば、$n = 3$ では、(1) は $a_3 = F(4) = 3$ で、$\{000, 010, 100\}$, (2) は $b_3 = F(3) = 2$ で、$\{001, 101\}$. $n = 4$ では、(1) は $a_4 = F(5) = 5$ で、$\{0000, 0010, 0100, 1000, 1010\}$, (2) は $b_4 = F(4) = 3$ で、$\{0001, 0101, 1001\}$.

1.2 トリボナッチ数列

前節で述べたフィボナッチ列が 2 項の和で定めたことに対して、3 項の和としたものが、トリボナッチとよばれる。さらに 4 項へと拡張した場合を後節で考える。

Definition 1.1 トリボナッチ数列の定義:

$$T_0 = T_1 = 0,\, T_2 = 1,\;\; T_{n+3} = T_n + T_{n+1} + T_{n+2}, \quad (n \geq 0)$$

この定義によるいくつかの項を列挙してみるとつぎのように計算できる。

n	0	1	2	3	4	5	6	7	8	9	10
T_n	0	0	1	1	2	4	7	13	24	44	81

	11	12	13	14	15	16	⋯	20	21	22	23
	149	274	504	924	1705	3136	⋯	35890	66012	121415	223317

フィボナッチ数と比べると、急激に増加する。フィボナッチ列の考え方が兎（うさぎ）の「親番（つがい)」と「子番」の２属性から生成された。親（◎印）と子（○印）が、

規則（1）◎→◎と○、規則（2）○→◎

つまり、各世代での合計 ”F_n = ◎ + ○” であった。親番を A, 子番を B とした文字列の代入 (StringReplace) とその個数数え上げ (StringCount) による数式処理をおこなうと、`StringReplace (1) "A" -> "AB", (2) "B" -> "A"` 入力とその結果表示： Input, Rule, Initial, number-of-repeat: `NestList[StringReplace[#,{"A"->"AB","B"->"A"}]&,"A",5]` 数え上げ：`StringCount[%,"A"]` でフィボナッチ列 $F_1, F_2, \cdots, F_6$ が６個並んで出てくる。

この場合には３種の属性：◎、△、○を考え、３つの規則

規則（1）◎→◎と△、
規則（2）△→◎と○、
規則（3）○→◎

として各世代での合計 ”T_n = ◎ + △ + ○” とする。この場合も数式処理
`StringReplace (1) "A" -> "AB", (2) "B" -> "AC", (3) "C" -> "A"`
で同様となる。

一般項の計算は数式処理（Mathematica）により計算するが、そのままでは冗長で多少手を加えてみる。まず３世代の変換行列における固有方程式

$$x^3 = 1 + x + x^2 \tag{5}$$

を求めると、

```
CharacteristicPolynomial[{{1, 1, 1}, {1, 0, 0}, {0, 1, 0}}, x]
Solve[1 + x + x^2 - x^3 == 0, x]
{
a_1 = {x -> 1/3 (1 + (19 - 3 Sqrt[33])^(1/3) + (19 + 3 Sqrt[33])^(1/3))},
a_2 = {x -> 1/3 - 1/6 (1 + I Sqrt[3]) (19 - 3 Sqrt[33])^(1/3)
      - 1/6 (1 - I Sqrt[3]) (19 + 3 Sqrt[33])^(1/3)},
a_3 = {x ->  1/3 - 1/6 (1 - I Sqrt[3]) (19 - 3 Sqrt[33])^(1/3)
      - 1/6 (1 + I Sqrt[3]) (19 + 3 Sqrt[33])^(1/3)}}
```

であり、この解

$$a_1 = \tfrac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right)$$
$$a_2 = \tfrac{1}{3}\left(1 + \omega\sqrt[3]{19 - 3\sqrt{33}} + \varpi\sqrt[3]{19 + 3\sqrt{33}}\right)$$
$$a_3 = \tfrac{1}{3}\left(1 + \varpi\sqrt[3]{19 - 3\sqrt{33}} + \omega\sqrt[3]{19 + 3\sqrt{33}}\right)$$

ただし、$\omega = \dfrac{-1+\sqrt{3}i}{2}$ とする。したがって $1+\omega+\overline{\omega}=0$ などより、つぎを得る。

$$a_1+a_2+a_3=1, \quad a_1a_2+a_2a_3+a_3a_1=-1, \quad a_1a_2a_3=1 \tag{6}$$

これらを用いて、べき行列により表現する。(参考：ja.wikipedia.org/wiki/フィボナッチ数)

MatrixPower[mat, n] の計算とその成分結果:

$$\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\end{pmatrix}^n = \begin{pmatrix}a^n_{(1,1)}&a^n_{(1,2)}&a^n_{(1,3)}\\a^n_{(2,1)}&a^n_{(2,2)}&a^n_{(2,3)}\\a^n_{(3,1)}&a^n_{(3,2)}&a^n_{(3,3)}\end{pmatrix} = \begin{pmatrix}T_{n+2}&T_{n+1}+T_n&T_{n+1}\\T_{n+1}&T_n+T_{n-1}&T_n\\T_n&T_{n-1}+T_{n-2}&T_{n-1}\end{pmatrix}$$

$a^n_{(1,1)}=T_{n+2}$:

$$\frac{a_1^{n+2}}{(a_1-a_2)(a_1-a_3)}-\frac{a_2^{n+2}}{(a_1-a_2)(a_2-a_3)}+\frac{a_3^{n+2}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(1,2)}=T_{n+1}+T_n$:

$$-\frac{a_1^{n+2}(a_2+a_3)}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^{n+2}(a_1+a_3)}{(a_1-a_2)(a_2-a_3)}-\frac{(a_1+a_2)a_3^{n+2}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(1.3)}=T_{n+1}$:

$$\frac{a_1^{n+2}a_2a_3}{(a_1-a_2)(a_1-a_3)}-\frac{a_1a_2^{n+2}a_3}{(a_1-a_2)(a_2-a_3)}+\frac{a_1a_2a_3^{n+2}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(2,1)}=T_{n+1}$:

$$\frac{a_1^{n+1}}{(a_1-a_2)(a_1-a_3)}-\frac{a_2^{n+1}}{(a_1-a_2)(a_2-a_3)}+\frac{a_3^{n+1}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(2.2)}=T_n+T_{n-1}$:

$$-\frac{a_1^{n+1}(a_2+a_3)}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^{n+1}(a_1+a_3)}{(a_1-a_2)(a_2-a_3)}-\frac{(a_1+a_2)a_3^{n+1}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(2.3)}=T_n$:

$$\frac{a_1^{n+1}a_2a_3}{(a_1-a_2)(a_1-a_3)}-\frac{a_1a_2^{n+1}a_3}{(a_1-a_2)(a_2-a_3)}+\frac{a_1a_2a_3^{n+1}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(3.1)}=T_n$:

$$\frac{a_1^{n}}{(a_1-a_2)(a_1-a_3)}-\frac{a_2^{n}}{(a_1-a_2)(a_2-a_3)}+\frac{a_3^{n}}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(3.2)}=T_{n-1}+T_{n-2}$:

$$-\frac{a_1^{n}(a_2+a_3)}{(a_1-a_2)(a_1-a_3)}+\frac{a_2^{n}(a_1+a_3)}{(a_1-a_2)(a_2-a_3)}-\frac{a_3^{n}(a_1+a_2)}{(a_1-a_3)(a_2-a_3)},$$

$a^n_{(3.3)}=T_{n-1}$:

$$\frac{a_1^{n}a_2a_3}{(a_1-a_2)(a_1-a_3)}-\frac{a_1a_2^{n}a_3}{(a_1-a_2)(a_2-a_3)}+\frac{a_1a_2a_3^{n}}{(a_1-a_3)(a_2-a_3)}$$

これらのコンピュータ結果からも

$$T_{n+3}=T_n+T_{n+1}+T_{n+2}, \quad (n\geq 0) \tag{7}$$

という関係が検証され、初期値から、

$$\begin{aligned}T_{n+1} &= a^n_{(3,1)} + a^n_{(3,2)} \\ &= \frac{a_1^n(a_2a_3 - a_2 - a_3)}{(a_2 - a_1)(a_3 - a_1)} + \frac{a_2^n(a_1a_3 - a_1 - a_3)}{(a_1 - a_2)(a_3 - a_2)} + \frac{a_3^n(a_1a_2 - a_1 - a_2)}{(a_1 - a_3)(a_2 - a_3)}\end{aligned}$$

と得られる。またフィボナッチ、トリボナッチ、テトラナッチの番は永遠に不滅の生命力を有する。したがって増加の状況が

$$F_n/F_{n-1} \sim \frac{1+\sqrt{5}}{2} = 1.62\cdots \quad (\text{黄金比})$$

と評価され、$n \geq 6$ でも $F_n \sim 13 \times \left(\frac{1+\sqrt{5}}{2}\right)^{n-6}$, $(n \geq 6)$ であり、

$$\lim_n \frac{T_{n+1}}{T_n} = a_1 = a_1 = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right)$$

が成り立つことが知られている。

2 テトラナッチ数列とド・モアブルの結果

つぎにテトラナッチ数列 $\{Q_n; n = 0, 1, 2, \cdots\}$:

$$\begin{aligned}&Q_0 = Q_1 = Q_2 = 0,\ Q_3 = 1, \\ &Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n\end{aligned} \tag{8}$$

に対しては

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$	25	$\cdots$
Q_n	0	0	0	1	1	2	4	8	15	29	56	$\cdots$	1055026	$\cdots$

ここで、後で必要となる $Q_{25} = 1055026$ を取り上げておく。

前と同様にこのような行列のベキ計算によれば、項の計算はコンピュータによる数式処理のくり返し線形再帰命令 (LinearRuccurensive) で簡単にできる。StringReplace での rule は (1) `"A" -> "AB"`, (2) `"B" -> "AC"`, (3) `"C" -> "AD"`, (4) `"D" -> "A"` とすれば前と同様に計算できる。

さらにつぎも成り立つ。

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}^n = \begin{pmatrix}Q_n & Q_{n-1}+Q_{n-2}+Q_{n-3} & Q_{n-1}+Q_{n-2} & Q_{n-1}\\ Q_{n-1} & Q_{n-2}+Q_{n-3}+Q_{n-4} & Q_{n-2}+Q_{n-3} & Q_{n-2}\\ Q_{n-2} & Q_{n-3}+Q_{n-4}+Q_{n-5} & Q_{n-3}+Q_{n-4} & Q_{n-3}\\ Q_{n-3} & Q_{n-4}+Q_{n-5}+Q_{n-6} & Q_{n-4}+Q_{n-5} & Q_{n-4}\end{pmatrix} \tag{9}$$

実は, これに関連した結果はド・モアブル (A.de Moivre,1667-1754) が一種の母関数（ベキ級数）の "有意な部分" の計算式をもちいて計算してものが知られている。

> 第 LXXIV(74) 問: 与えられた試行回数のなかで、途切れることなく続いた回数の反復数を達成する確率を求めること。
>
> 「偶然の学理」A.de Moivre, "The Doctrine of Chances" (1718, 1738, 1756)
>
> (参考 ; Impossible? Surprising Solutions to Counterintuitive Conundrums by Julian Havil, Princeton Univ Press, 2008)

その結果の一つは、

【1】ベルヌーイ試行において、10 回試行したときに 3 回以上表の出ることが続く確率 $P(10,3)$ は？

答えとして、

$$\frac{1}{8}\left(1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{7}{16}+\frac{13}{32}+\frac{24}{64}+\frac{44}{128}\right)=\frac{65}{128}\approx 0.508 \tag{10}$$

【2】「計算をもっと簡単にする方策を検討」し続け、24 回の試行で、4 回以上表の出ることが続く確率 $P(24,4)$ は？

答えとして（一部計算間違いが指摘されているが、確認してみると間違いではない）この場合の計算として導いた結果, $P(24,4)\approx 0.497$ が得られている。

現代のコンピュータを用いない時代にこのような素晴らしい計算には感嘆するばかりといわれる。

Example 3 このドモアブルの問題はコンピュータがあれば、極めて容易に確率計算となる。これがベルヌーイ試行におけるフィボナッチ、トリボナッチ、テトラナッチ数列とつぎのように関係付けられる。

記号 $P(n,k)$ の定義：

$$P(n,k)=P\left(\sum_{i=1}^{n-k+1} X_iX_{i+1}\cdots X_{i+k-1}>0\right)$$

とおく。したがってドモアブルの結果は、これを用いて表すと

$$P(10,3)=\frac{65}{128}=1-\frac{63}{128}=1-\frac{T(10+3)}{2^{10}}=1-\frac{T_{13}}{2^{10}}=1-\frac{504}{2^{10}}\approx 0.508$$

であった。4 項式の初期値の関係から, $25=21+4$, $n=21$ として

$$P(25,4)=1-\frac{Q(21+4)}{2^{21}}=1-\frac{Q_{25}}{2^{21}}=1-\frac{1055026}{2097152}\approx 1-0.503076=0.496924$$

とコンピュータで求められ、答えはドモアブルの結果と一致する。

以上の結果から、つぎの定理としてまとめられる。

Theorem 2.1 n 個の独立なベルヌーイ列、$X_i\sim \mathrm{Binom}\left(1,\frac{1}{2}\right), i=1,2,\cdots,n$ において、

(1)

$$P\left(\sum_{i=1}^{n-2} X_iX_{i+1}X_{i+2}=0\right)=\frac{T(n+3)}{2^n} \tag{11}$$

が成り立つ。ここで $T(n)=T_n$ は n-th 拡張トリボナッチ数とする。

(2)

$$P\left(\sum_{i=1}^{n-3} X_iX_{i+1}X_{i+2}X_{i+3}=0\right)=\frac{Q(n+4)}{2^n} \tag{12}$$

が成り立つ。ここで $Q(n)=Q_n$ は n-th テトラナッチ数列 (quadruplet sequence) とする。

多くの研究論文で、このようなベルヌーイ試行での成功が継続して出ることの解析は研究されている。たとえば、Mark Schilling; The longest run of heads, College Mathematics Journal, 1990, pp.196-207. `http://www.stat.wisc.edu/~callan/notes/nonconsec_pattern/nonconsec_pattern.pdf`

よく知られている様々な関係式のうち、この行列を用いるとそれぞれの数列に関して、フィボナッチ $F_n=F_{n-1}+F_{n-2}$, トリボナッチ $T_n=T_{n-1}+T_{n-2}+T_{n-3}$ テトラナッチ $Q_n=Q_{n-1}+Q_{n-2}+Q_{n-3}+Q_{n-4}$ を定める非線形な関係式;(カッシーニ、シムソンの定理) $F_{n-1}F_{n+1}-F_n^2=(-1)^2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

両辺の行列式を求めれば、$(-1)^n = F_{n+1}F_{n-1} - F_n^2$ から、つぎの式が得られる。

$$F_{n+1} = \frac{F_n^2 + (-1)^n}{F_{n-1}} \tag{13}$$

同様に

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} T_{n+2} & T_{n+1} + T_n & T_{n+1} \\ T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \end{pmatrix}$$

をもちいて、行列式の計算で $1 = T_n^3 - 2T_{n+1}T_nT_{n-1} + T_{n+2}T_{n-1}^2 + T_{n+1}^2T_{n-2} - T_{n+2}T_nT_{n-2}$

$$T_{n+2} = \frac{1 - T_n^3 + 2T_{n+1}T_nT_{n-1} - T_{n+1}^2T_{n-2}}{T_{n-1}^2 - T_nT_{n-2}} \tag{14}$$

を得る。

さらに同様に

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} Q_{n+3} & Q_{n+2} + Q_{n+1} + Q_n & Q_{n+2} + Q_{n+1} & Q_{n+2} \\ Q_{n+2} & Q_{n+1} + Q_n + Q_{n-1} & Q_{n+1} + Q_n & Q_{n+1} \\ Q_{n+1} & Q_n + Q_{n-1} + Q_{n-2} & Q_n + Q_{n-1} & Q_n \\ Q_n & Q_{n-1} + Q_{n-2} + Q_{n-3} & Q_{n-1} + Q_{n-2} & Q_{n-1} \end{pmatrix}$$

をもちいて、行列式の計算で

$$\begin{aligned} (-1)^n &= Q_{n+3}\left(Q_{n-1}^3 - 2Q_nQ_{n-1}Q_{n-2} + Q_{n+1}Q_{n-2}^2 + Q_n^2Q_{n-3} - Q_{n+1}Q_{n-1}Q_{n-3}\right) \\ &\quad -Q_n^4 + 3Q_{n+1}Q_n^2Q_{n-1} - Q_{n+1}^2Q_{n-1}^2 - 2Q_{n+2}Q_nQ_{n-1}^2 \\ &\quad -2Q_{n+1}^2Q_nQ_{n-2} + 2Q_{n+2}Q_n^2Q_{n-2} + 2Q_{n+2}Q_{n+1}Q_{n-1}Q_{n-2} \\ &\quad -Q_{n+2}^2Q_{n-2}^2 + Q_{n+1}^3Q_{n-3} - 2Q_{n+2}Q_{n+1}Q_nQ_{n-3} + Q_{n+2}^2Q_{n-1}Q_{n-3} \end{aligned}$$

したがって

$$Q_{n+3} = \frac{ENUM}{DNUM} \tag{15}$$

where

$$\begin{aligned} ENUM &= (-1)^n + Q_n^4 - 3Q_{n+1}Q_n^2Q_{n-1} + Q_{n+1}^2Q_{n-1}^2 + 2Q_{n+2}Q_nQ_{n-1}^2 + 2Q_{n+1}^2Q_nQ_{n-2} \\ &\quad -2Q_{n+2}Q_n^2Q_{n-2} - 2Q_{n+2}Q_{n+1}Q_{n-1}Q_{n-2} + Q_{n+2}^2Q_{n-2}^2 - Q_{n+1}^3Q_{n-3} \\ &\quad +2Q_{n+2}Q_{n+1}Q_nQ_{n-3} - Q_{n+2}^2Q_{n-1}Q_{n-3} \\ DNUM &= Q_{n-1}^3 - 2Q_nQ_{n-1}Q_{n-2} + Q_{n+1}Q_{n-2}^2 + Q_n^2Q_{n-3} - Q_{n+1}Q_{n-1}Q_{n-3} \end{aligned}$$

3 階差数列を考える

フィボナッチ数列 $\{F_n; F_{n+2} = F_{n+1} + F_n\}$ において、階差 $\{F_n - F_{n-1}\}$ は再びフィボナッチ数列となる。すなわち $F_{n+3} - F_{n+2} = (F_{n+2} - F_{n+1}) + (F_{n+1} - F_n)$, よって

$$F_{n+3} = 2F_{n+2} - F_n \tag{16}$$

が成り立つ。

最近の FQ 誌に掲載された

Howard, F. T.; Cooper, Curtis: Some identities for r-Fibonacci numbers. Fibonacci Quart. 49 (2011), no. 3, 231-242.

においては、この関係を階差とは考えずに直接解いている (Thm 2.2.,2.3,2.4)。後者の著者は、FQ の main editor である。

ここでは、この関係式をもちいると、両辺を２のべき乗で割って、順次次数を下げていくとつぎの関係式が得られることを述べておく。差分演算子 $\Delta H_n = H_{n+1} - H_n$ を用いると、

$$
\begin{array}{l}
\Delta F_n = \Delta F_{n-1} + \Delta F_{n-2} \quad (n \geq 2) \\
\Delta T_n = \Delta T_{n-1} + \Delta T_{n-2} + \Delta T_{n-3} \quad (n \geq 3) \\
\Delta Q_n = \Delta Q_{n-1} + \Delta Q_{n-2} + \Delta Q_{n-3} + \Delta Q_{n-4} \quad (n \geq 4)
\end{array}
\tag{17}
$$

を解くことに帰着される。

帰納法をもちいても当然得られる。したがって次の定理は明らか。

Theorem 3.1 フィボナッチ数列では

$$\frac{F_{r+3}}{2^{r+1}} = 1 - \frac{1}{2^2}\left(\frac{F_1}{2^0} + \frac{F_2}{2^1} + \frac{F_3}{2^2} + \cdots + \frac{F_r}{2^{r-1}}\right) \tag{18}$$

が示される。同様にトリボナッチ数列では $T_{n+4} = 2T_{n+3} - T_n$ より、

$$\frac{T_{r+4}}{2^{r+1}} = 1 - \frac{1}{2^3}\left(\frac{T_1}{2^0} + \frac{T_2}{2^1} + \frac{T_3}{2^2} + \cdots + \frac{T_r}{2^{r-2}}\right) \tag{19}$$

またテトラナッチ数列では

$$\frac{Q_{r+5}}{2^{r+1}} = 1 - \frac{1}{2^4}\left(\frac{Q_1}{2^0} + \frac{Q_2}{2^1} + \frac{Q_3}{2^2} + \cdots + \frac{Q_r}{2^{r-3}}\right) \tag{20}$$

を得る。

フィボナッチ数列では、階差数列から得られる関係式：$F_{n+3} = 2F_{n+2} - F_n$ を用いたが、トリボナッチ数列の階差の関係式；$T_{n+4} = 2T_{n+3} - T_n$ 、さらにテトラナッチ数列の階差の関係式；$Q_{n+5} = 2Q_{n+4} - Q_n$ から同様にして得られる。フィボナッチ数列の関係式 (18) はよく知られているもののひとつである。

数値例 $(r = 9)$ で確かめると

$$
\begin{array}{rcl}
\dfrac{F_{12}}{2^{9+1}} & = & \dfrac{144}{1024} = 1 - \dfrac{1}{2^2}\left(\dfrac{F_1}{2^0} + \dfrac{F_2}{2^1} + \dfrac{F_3}{2^2} + \cdots + \dfrac{F_9}{2^8}\right) = 1 - \dfrac{1}{4} \cdot \dfrac{880}{256} \\
\dfrac{T_{13}}{2^{9+1}} & = & \dfrac{504}{1024} = 1 - \dfrac{1}{2^3}\left(\dfrac{T_2}{2^0} + \dfrac{T_3}{2^1} + \cdots + \dfrac{T_9}{2^7}\right) = 1 - \dfrac{1}{8} \cdot \dfrac{520}{128} \approx 0.4921875 \\
\dfrac{Q_{14}}{2^{9+1}} & = & \dfrac{773}{1024} = 1 - \dfrac{1}{2^4}\left(\dfrac{Q_3}{2^0} + \dfrac{Q_4}{2^1} + \cdots + \dfrac{Q_9}{2^6}\right) = 1 - \dfrac{1}{16} \cdot \dfrac{251}{128}
\end{array}
\tag{21}
$$

この数値例 (21) 式の値は、ドモアブルが求めた (10) 式の計算結果

$$
\begin{array}{rl}
 & \dfrac{1}{2^3}\left(\dfrac{T_2}{2^0} + \dfrac{T_3}{2^1} + \cdots + \dfrac{T_9}{2^7}\right) \\
= & \dfrac{1}{2^3}\left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{4}{8} + \dfrac{7}{16} + \dfrac{13}{32} + \dfrac{24}{64} + \dfrac{44}{128}\right) \\
\approx & 0.5078125
\end{array}
$$

に他ならない。したがってこれと同様な関係式がより高次の場合について成立することが予想され、これは２，３、４次の場合で、彼は２０次程度に関する結果を求めた（前述)。素晴らしさを超えた凄まじい計算力に驚嘆する。

4 偉大な人の後で

T. シンプソン「The Nature and Laws of Chance」(偶然の性質と法則,1740 年) の連 (第 XXIV(24) 問) の問題：「皆のため、ドモアブルのような偉大な人の後で、このような題目を解説しようと企てるのは、ずうずうしいと思われるかも知れないが、それでも私は満足だ」(序文より)

ベルヌーイ試行での連の計算を行っている。提起された事象の起る確率を p, 反対の事象の確率を $q = 1 - p$ とする。

「継続の法則は明らかである」として直ちに一般公式を書き下す。

$$Z_n = a\left\{1 - \ddot{n}x + \binom{\dddot{n}}{2}x^2 - \binom{\ddddot{n}}{3}x^3 + \cdots\right\} + \dot{n}x - \binom{\ddot{n}}{2}x^2 + \binom{\dddot{n}}{3}x^3 - \cdots$$

ここでシンプソンの略記号 $\dot{n} = n - r,\ \ddot{n} = n - 2r,\ \dddot{n} = n - 3r,\ \cdots$

$p = 2/3, q = 1/3, r = 3, n = 10$ ならば、

$$a = (2/3)^3 = 8/27, x = qp^3 = 8/81$$

$$Z_{10} = \frac{8}{27}\left(1 - 4 \times \frac{8}{81}\right) + 7 \times \frac{8}{81} - \binom{4}{2}\left(\frac{8}{81}\right)^2 = \frac{592}{729}$$

としている。

この結果を検証するために階差数列を考えれば、もっと簡潔であり、結果の正しいことを確認でき、より一般な形の結果が求められる。これが主な主張である。実際、つぎの定理で述べる関係式を用いて計算すると、

n	1	2	3	4	5	6	7	8	9	10
Z_n	0	0	$\frac{8}{27}$	$\frac{32}{81}$	$\frac{40}{81}$	$\frac{16}{27}$	$\frac{1448}{2187}$	$\frac{4736}{6521}$	$\frac{1688}{2187}$	$\frac{592}{729}$
W_n	$\frac{9}{2}$	$\frac{9}{4}$	$\frac{57}{8}$	$\frac{147}{16}$	$\frac{369}{32}$	$\frac{891}{64}$	$\frac{2217}{128}$	$\frac{5475}{256}$	$\frac{13473}{512}$	$\frac{33291}{1024}$

確かにシンプソンの結果：“$Z_{10} = \dfrac{592}{729} \approx 0.812071$” と一致した。メデタシめでたし！ 途中での過程では 4 桁にもなるときがあるが、１０番目では 3 桁ずつの分数になり、ここで結果を示しているのは、やはり計算の達人の表れか。

いままでのドモアブルの結果、シンプソンの結果は、現代流の記号で表現するとつぎのように書き表すことができる。つまり確率に関する「再帰関係式」(recurresive relation) である。

Theorem 4.1 n 個の独立ばベルヌーイ列, $X_i \sim \mathrm{Binom}(1, p), i = 1, 2, \cdots, n$ ただし $(0 < p < 1)$ に対して、

$$P(n, k) = P\left(\sum_{i=1}^{n-k-1} X_i X_{i+1} \cdots X_{i+k-1} > 0\right)$$

とおくとき、つぎの関係式が成り立つ。

$$P(n, k) = P(n-1, k) + \{1 - P(n-k-1, k)\}qp^k, \quad k \le n \tag{22}$$

境界条件は

$$P(0, k) = P(1, k) = \cdots = P(k-1, k) = 0, \quad P(k, k) = qp^{k-1}$$

さらに k を固定して、$W_n = \dfrac{1}{qp^n}(1 - Z_n)$, ここで 正となる確率を

$$Z_n = P\left(\sum_{i=1}^{n-k+1} X_i X_{i+1} \cdots X_{i+k-1} > 0\right)$$

とし、ゼロとなる確率から $W_n = \dfrac{1}{qp^n}P(\sum_{i=1}^{n-k+1} X_i X_{i+1} \cdots X_{i+k-1} = 0)$ であり、

$$W_n = \frac{1}{p}W_{n-1} - \frac{q}{p}W_{n-k-1} \tag{23}$$

$$\Delta W_n = \frac{q}{p}\left(\Delta W_{n-1} + \Delta W_{n-2} + \cdots + \Delta W_{n-k}\right) \tag{24}$$

W_n の ゼロとなる確率は、“$11\cdots1$” を含まない、言いかえると避ける (avoiding sequence) 確率であり、$p=q=1/2$ という場合はドモアブルの場合であり、係数がすべて 1 となる、この線形差分方程式の解は、フィボナッチ、トリボナッチ、テトラナッチ数列に他ならない。これを用いると、確率は分数で表現される。

Golden optimal primal-dual control processes

Masami YASUDA and Seiichi IWAMOTO

Department of Mathematical Science
Faculty of Science, Chiba University
Chiba 263-8522, Japan
tel&fax. +81(43)290-3662, email: yasuda@math.s.chiba-u.ac.jp
and
Professor emeritus
Kyushu University
Fukuoka 813-0012, Japan
tel&fax. +81(92)672-1665 email: iwamotodp@kyudai.jp

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Abstract

In this paper we discuss two discrete-time control (primal) processes from the viewpoints of duality and Golden optimality. At first we derive an associated dual process. We show that it has a Golden optimal path. Then we find the Golden optimal solution for both primal and dual processes through three approaches — (i) evaluation-optimization, (ii) dynamic programming, and (iii) variational method —.

1 Introduction

In a class of optimization problems there arises the question of whether an optimal solution is Golden or not. This question is partly resolved for a class of static optimization problems [10–12, 14]. Recently it has been shown that a Golden path/trajectory is optimal in discrete/continuous-time control processes [13, 18]. It is also obtained by solving a corresponding Bellman equation for dynamic programming [1, 2, 9, 17, 22].

In this paper we discuss a typical dynamic optimization from the two veiwpoints of duality and Golden optimality. The question is whether duality transmits Golden optimality or not. We present two discrete-time control (primal) processes. Then we derive associated dual processes. We show that the dual processes have also a Golden optimal path. Further we find the Golden optimal solution of both primal and dual processes through three approaches — (i) evaluation-optimization, (ii) dynamic programming, and (iii) variational method —. Here (i) evaluates the total cost and optimizes it among the stationary policy class. The evaluation compresses an infinite-sequence problem into one-variable one for primal process and two-variable for dual. This is possible under the stationary reward-accumulation and state-dynamics. (ii) and (iii) solve Bellman equation and Euler equation, respectively.

Let us consider a typical type of criterion — quadratic — in a deterministic optimization. We minimize quadratic criteria

$$\begin{aligned} I(x) &= \sum_{n=0}^{\infty} \left[x_n^2 + (bx_n - x_{n+1})^2 \right] \\ J(x) &= \sum_{n=0}^{\infty} \left[(bx_n - x_{n+1})^2 + x_{n+1}^2 \right] \end{aligned}$$

where $-\infty < b < \infty$. There exists a difference between I and J:

$$J(x) \;=\; I(x) - x_0^2.$$

This difference helps us to derive optimal solution of the primal problem. However how does the difference affects the form of dual?

2 Golden Paths

A real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is called *Golden number* [3,6,23]. It is the larger of the two solutions to quadratic equation

$$x^2 - x - 1 = 0. \tag{1}$$

Sometimes (1) is called *Fibonacci.* This has two real solutions: ϕ and its *conjugate* $\overline{\phi} := 1 - \phi$. We note that

$$\phi + \overline{\phi} = 1, \qquad \phi \cdot \overline{\phi} = -1.$$

Further we have

$$\begin{aligned} \phi^{-1} &= \phi - 1, \quad & \phi^{-2} &= 2 - \phi, \quad & \phi^{-1} + \phi^{-2} &= 1 \\ \phi^2 &= 1 + \phi, \quad & \overline{\phi}^2 &= 2 - \phi, \quad & \phi^2 + \overline{\phi}^2 &= 3. \end{aligned}$$

A point $\phi^{-2}x$ splits an interval $[0, x]$ into two intervals $[0, \phi^{-2}x]$ and $[\phi^{-2}x, x]$. A point $\phi^{-1}x$ splits the interval into $[0, \phi^{-1}x]$ and $[\phi^{-1}x, x]$. In either case, the length constitutes the Golden ratio $\phi^{-2} : \phi^{-1} = 1 : \phi$. Thus both divisions are the *Golden section* [3,23].

Definition 2.1 ([18]) *A sequence* $x : \{0, 1, \ldots\} \to R^1$ *is called Golden if and only if either*

$$\frac{x_{t+1}}{x_t} = \phi^{-1} \qquad or \qquad \frac{x_{t+1}}{x_t} = \phi^{-2}.$$

Lemma 2.1 ([18]) *A Golden sequence* x *is either*

$$x_t = x_0\phi^{-t} \qquad or \qquad x_t = x_0\phi^{-2t}.$$

We remark that

$$\phi^{-t} = (\phi - 1)^t, \qquad \phi^{-2t} = (2 - \phi)^t = (1 + \phi)^{-t}$$

where

$$\phi - 1 = \phi^{-1} \approx 0.618, \qquad 2 - \phi = (1 + \phi)^{-1} = \phi^{-2} \approx 0.382$$

Let us introduce a controlled linear dynamics with real parameter b as follows.

$$x_{t+1} = bx_t + u_t \qquad t = 0, 1, \ldots \tag{2}$$

where $u : \{0, 1, \ldots\} \to R^1$ is called *control.* If $u_t = px_t$ (resp. $px_t + q$), the control u is called *proportional (resp. linear)*, where p, q are real constants. A sequence x satisfying (2) is called *path.* We say that a quadratic function $w(x) = ax^2$ is *Golden* if $a = \phi$. It is called *inverse-Golden* if $a = \phi^{-1}$.

Definition 2.2 ([18]) *A proportional control* u *on dynamics (2) is called Golden if and only if it generates a Golden path* x.

Lemma 2.2 ([18]) *A proportional control* $u_t = px_t$ *on (2) is Golden if and only if*

$$p = -b + \phi^{-1} \qquad or \qquad p = -b + \phi^{-2}. \tag{3}$$

Definition 2.3 *A sequence* $x : \{0, 1, \ldots\} \to R^1$ *is called alternately Golden if and only if either*

$$\frac{x_{t+1}}{x_t} = -\phi^{-1} \qquad or \qquad \frac{x_{t+1}}{x_t} = -\phi^{-2}.$$

Lemma 2.3 *An alternately Golden sequence* x *is either*

$$x_t = x_0(-1)^t\phi^{-t} \qquad or \qquad x_t = x_0(-1)^t\phi^{-2t}.$$

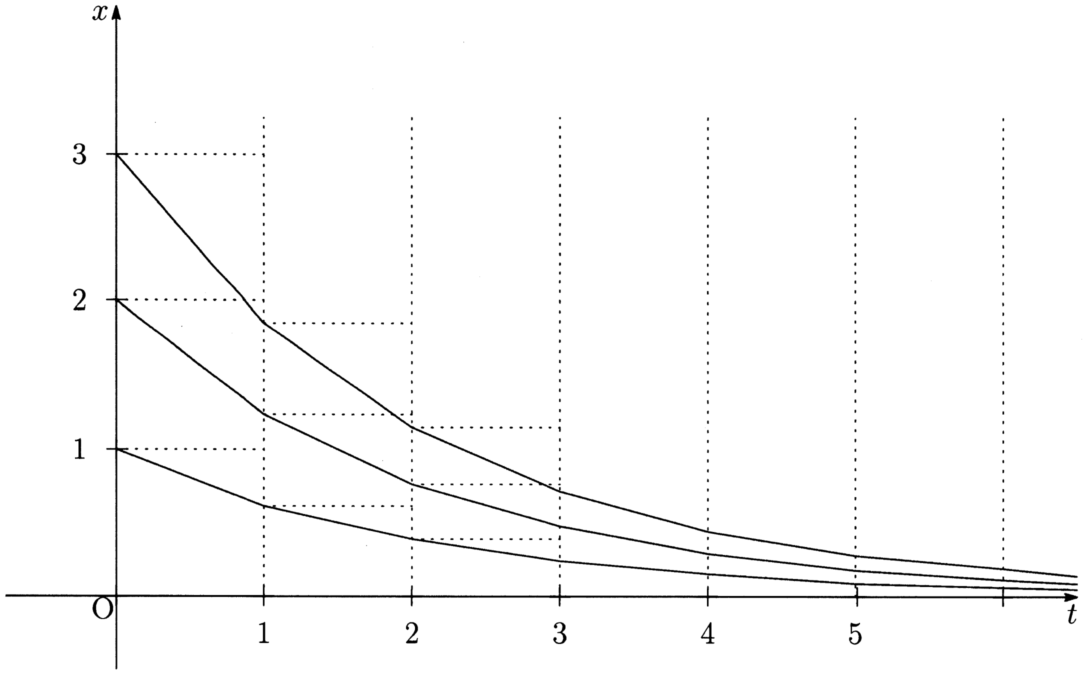

Fig. 1 Golden paths $x = c\phi^{-t}$ $c = 1, 2, 3$

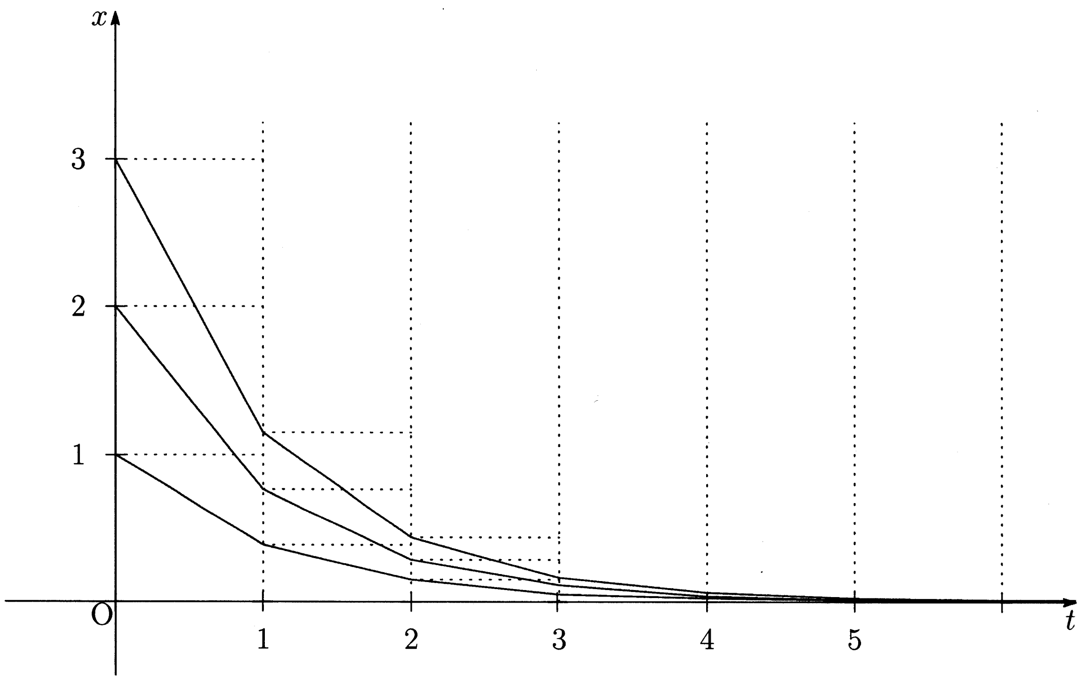

Fig. 2 Golden paths (c) $x = c\phi^{-2t}$ $c = 1, 2, 3$

3 Discrete Euler equation

Let b be any given real constant. Let a function $k : R^1 \rightarrow R^1$ and a sequence of functions $f_n : R^2 \rightarrow R^1$ $(n \geq 0)$ be C^1-class.

3.1 Fixed initial cost

First we evaluate any sequence $x = \{x_n\}_{n\geq 0}$ by

$$J_1(x) \;=\; k(x_0) + \sum_{n=0}^{\infty} f_n(x_n,\, x_{n+1} - bx_n).$$

Let D^1 be the set of x such that $J_1(x)$ takes a finite value. We consider a discrete-type extremal problem

$$\mathrm{EP}_1 \qquad \text{extremize} \;\; J_1(x) \;\; \text{subject to} \;\; \text{(i)}\; x \in D^1.$$

This has not an initial condition $x_0 = c$ but an initial cost function $k(x_0)$.

Let $g = g(x_n, y_n)$ be any two-variable C^1-function. Then we define

$$g_1 \;=\; \frac{\partial g}{\partial x_n}(x_n, y_n), \quad g_2 \;=\; \frac{\partial g}{\partial y_n}(x_n, y_n) \tag{4}$$

where $y_n \;=\; x_{n+1} - bx_n$.

Lemma 3.1 *Let $x = \{x_n\}_{n\geq 0}$ be an extremal. Then x satisfies a system of variational equations – discrete type Euler equation and two transversality conditions –*

$$\begin{array}{lll} \text{(EE)} & f_{n1} - (bf_{n2} - f_{n-12}) \;=\; 0 & n \geq 1 \\ \text{(TC)}_0 & k'(x_0) + f_{01} - bf_{02} \;=\; 0 & \\ \text{(TC)}_\infty & \lim_{n\to\infty} f_{n2} \;=\; 0. & \end{array} \tag{5}$$

Proof. Formally three equations are derived as follows (see [4, 5, 7, 8, 20]). Let $\eta = \{\eta_n\}_{n\geq 0}$ be any sequence. Then $y := x + \epsilon\eta$ is feasible for any $\epsilon \in R^1$. Let us define

$$J(\epsilon) \;:=\; k(y_0) + \sum_{n=0}^{\infty} f_n(y_n,\, y_{n+1} - by_n).$$

Then $J(\cdot)$ must take a minimum value at $\epsilon = 0$ for any such η. This implies $J'(0) = 0$.

Let us now calculate $J'(0) \;=\; \lim_{h\to 0} \dfrac{J(h) - J(0)}{h}$. We note that

$$\frac{J(h) - J(0)}{h} \;=\; \frac{k(x_0 + h\eta_0) - k(x_0)}{h} + \sum_{n=0}^{\infty} g_n$$

where

$$g_n := g_n(h) \;=\; \frac{f_n(x_n + h\eta_n,\, x_{n+1} - bx_n + h(\eta_{n+1} - b\eta_n)\,) - f_n(x_n,\, x_{n+1} - bx_n)}{h}.$$

From the mean value theorem, there exists θ_0, θ $(0 \leq \theta_0,\ \theta \leq 1)$ satisfying

$$\begin{aligned} \frac{k(x_0 + h\eta_0) - k(x_0)}{h} &= k'(x_0 + \theta_0 h\eta_0)\eta_0 \\ g_n &= f_{n1}\eta_n + f_{n2}(\eta_{n+1} - b\eta_n) \end{aligned}$$

where

$$\begin{aligned} f_{n1} &= f_{n1}(x_n + \theta h\eta_n,\ x_{n+1} - bx_n + \theta h(\eta_{n+1} - b\eta_n)\,) \\ f_{n2} &= f_{n2}(x_n + \theta h\eta_n,\ x_{n+1} - bx_n + \theta h(\eta_{n+1} - b\eta_n)\,). \end{aligned}$$

Then it holds that

$$\begin{aligned} \sum_{n=0}^{N} g_n &= \sum_{n=0}^{N} [\, f_{n1}\eta_n + f_{n2}(\eta_{n+1} - b\eta_n)\,] \\ &= (f_{01} - bf_{02})\eta_0 + \sum_{n=1}^{N} [\, f_{n1} - (bf_{n2} - f_{n-12})\,]\,\eta_n + f_{N2}\eta_{N+1}. \end{aligned}$$

Then we have

$$\sum_{n=0}^{\infty} g_n = (f_{01} - bf_{02})\eta_0 + \lim_{N\to\infty} \sum_{n=1}^{N} [\, f_{n1} - (bf_{n2} - f_{n-12})\,]\,\eta_n + \lim_{N\to\infty} f_{N2}\eta_{N+1}.$$

Thus

$$\begin{aligned} J'(0) &= \lim_{h\to 0}[\, k'(x_0 + \theta_0 h\eta_0) + (f_{01} - bf_{02})\,]\eta_0 \\ &\quad + \lim_{h\to 0}\lim_{N\to\infty} \sum_{n=1}^{N} [\, f_{n1} - (bf_{n2} - f_{n-12})\,]\,\eta_n + \lim_{h\to 0}\lim_{N\to\infty} f_{N2}\eta_{N+1} \\ &= \lim_{h\to 0}[\, k'(x_0 + \theta_0 h\eta_0) + f_{01} - bf_{02}\,]\eta_0 \\ &\quad + \lim_{N\to\infty} \sum_{n=1}^{N} \lim_{h\to 0} [\, f_{n1} - (bf_{n2} - f_{n-12})\,]\,\eta_n + \lim_{N\to\infty}\lim_{h\to 0} f_{N2}\eta_{N+1}. \end{aligned}$$

Consequently it holds that

$$\begin{aligned} J'(0) &= (\, k'(x_0) + f_{01} - bf_{02}\,)\eta_0 \\ &\quad + \sum_{n=1}^{\infty}(f_{n1} - bf_{n2} + f_{n-12})\eta_n + \lim_{n\to\infty} f_{n2}\eta_{n+1} \end{aligned} \tag{6}$$

where

$$\begin{aligned} f_{n1} &= f_{n1}(x_n,\ x_{n+1} - bx_n) \\ f_{n2} &= f_{n2}(x_n,\ x_{n+1} - bx_n). \end{aligned}$$

Since $J'(0)$ must vanish for any D^1-sequence η, we have the desired system of variational equations. □

3.2 Fixed initial state

Second we evaluate any sequence $x = \{x_n\}_{n\geq 0}$ by

$$J_2(x) \;=\; \sum_{n=0}^{\infty} f_n(x_n,\, x_{n+1} - bx_n).$$

Let D^2 be the set of x such that $J_2(x)$ takes a finite value. We consider a discrete-type extremal problem

$$\mathrm{EP}_2(\mathrm{c}) \qquad \text{extremize} \;\; J_2(x) \;\; \text{subject to} \;\; \text{(i)}\; x_0 = c, \quad \text{(ii)}\; x \in D^2.$$

This has not an initial cost function $k(x_0)$ but an initial condition $x_0 = c$.

Lemma 3.2 *Let x be an extremal. Then x satisfies a system of variational equations – discrete type Euler equation and a transversality condition –*

$$\text{(EE)} \qquad f_{n1} - (bf_{n2} - f_{n-12}) \;=\; 0 \quad n \geq 1$$
$$\text{(TC)}_\infty \qquad \lim_{n\to\infty} f_{n2} \;=\; 0.$$

Proof. Let $\eta = \{\eta_n\}_{n\geq 0}$ be any sequence satisfying $\eta_0 = 0$. Then the same way as in proof of Lemma 3.1 leads

$$J'(0) \;=\; \sum_{n=1}^{\infty} (f_{n1} - bf_{n2} + f_{n-12})\eta_n + \lim_{n\to\infty} f_{n2}\eta_{n+1}. \tag{7}$$

This implies the desired system of variational equations. □

4 Primal Process I(P); quadratic in current state

This section minimizes a quadratic cost function

$$\sum_{n=0}^{\infty} \left[x_n^2 + (x_{n+1} - bx_n)^2\right]$$

where $b \in R^1$ is a given constant. This problem is also solved as a control process with criterion

$$\sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right)$$

under an additive dynamics

$$x_{n+1} = bx_n + u_n.$$

4.1 Evaluation-optimization I(P)

Let R^∞ be the set of all sequences of real values :

$$R^\infty = \{x = (x_0, x_1, \dots, x_n, \dots) \mid x_n \in R^1 \quad n = 0, 1, \dots\}.$$

First we evaluate any sequence x through the quadratic criterion

$$I(x) \;=\; \sum_{n=0}^{\infty} \left[x_n^2 + (x_{n+1} - bx_n)^2\right] \quad \text{on} \quad R^\infty.$$

Second we minimizes the evaluated value for any given initial value c :

$$\text{MP}_1(\text{c}) \qquad \text{minimize} \quad I(x) \quad \text{subject to} \quad \text{(i)}\ x \in R^\infty, \quad \text{(ii)}\ x_0 = c.$$

Then the case $b = 0$ gives

$$I(x) \;=\; \sum_{n=0}^{\infty} \left[x_n^2 + x_{n+1}^2\right] \;=\; x_0^2 + 2\sum_{n=1}^{\infty} x_n^2.$$

This attains a minimum c^2 at all but first nothing $y = (c,\ 0,\ 0,\ \dots,\ 0,\ \dots)$.

In the following we assume $b \neq 0$. Let us now evaluate a few special paths :

1. The $y = (c,\ 0,\ 0,\ \dots,\ 0,\ \dots)$ yields $I(y) \;=\; (1 + b^2)c^2$.
2. Always all $z = (c,\ c,\ \dots,\ c,\ \dots)$ yields

$$I(z) \;=\; \begin{cases} \infty & c \neq 0 \\ 0 & c = 0 \end{cases}$$

3. A proportional $w = c(1,\ \rho,\ \dots,\ \rho^n,\ \dots)$ yields

$$\begin{aligned} I(w) \;&=\; \left\{c^2 + (\rho - b)^2 c^2\right\} \left(1 + \rho^2 + \cdots + \rho^{2n} + \cdots\right) \\ &=\; \frac{1 + (\rho - b)^2}{1 - \rho^2} c^2 \qquad (0 < |\rho| < 1). \end{aligned}$$

Let us now minimize only the ratio part of the above evaluated value

$$f(\rho) = \frac{1 + (\rho - b)^2}{1 - \rho^2}$$

under $-1 < \rho < 1$.

$$\text{MP}_1 \qquad \text{minimize} \quad f(\rho) \quad \text{subject to} \quad -1 < \rho < 1.$$

Lemma 4.1 MP$_1$ *has the minimum value* $f(\alpha) \;=\; \dfrac{b^2 + \sqrt{b^4 + 4}}{2}$ *at* $\hat{\rho} \;=\; \alpha \;=\; \dfrac{b^2 + 2 - \sqrt{b^4 + 4}}{2b}$.

Proof. We get

$$\begin{aligned} f'(\rho) &= 2\frac{(\rho-b)(1-\rho^2)+\rho\{1+(\rho-b)^2\}}{(1-\rho^2)^2} \\ &= -2\frac{b\rho^2-(b^2+2)\rho+b}{(1-\rho^2)^2} \\ f''(\rho) &= -2\frac{\{2b\rho-(b^2+2)\}(1-\rho^2)+4\rho\{b\rho^2-(b^2+2)\rho+b\}}{(1-\rho^2)^3} \\ &= -2\frac{2b\rho^3-3(b^2+2)\rho^2+6b\rho-(b^2+2)}{(1-\rho^2)^3}. \end{aligned}$$

Letting the numerator of $f'(\rho)$ vanish, we have a quadratic equation

$$b\rho^2-(b^2+2)\rho+b=0. \tag{8}$$

This equation has two solutions

$$\alpha = \frac{b^2+2-\sqrt{b^4+4}}{2b}, \quad \beta = \frac{b^2+2+\sqrt{b^4+4}}{2b}.$$

Then

$$\alpha = \frac{2b}{b^2+2+\sqrt{b^4+4}}, \quad \beta = \frac{2b}{b^2+2-\sqrt{b^4+4}} = \frac{1}{\alpha}.$$

Eq.(8) is equivalent to

$$\rho^2+1 = \left(b+\frac{2}{b}\right)\rho. \tag{9}$$

Thus α is the smaller in absolute value of two cross points for a quadratic curve $y=\rho^2+1$ and a line $y=\left(b+\frac{2}{b}\right)\rho$ on ρ-y plane.

Let

$$B = b+\frac{2}{b} \quad \text{for} \quad b\neq 0.$$

Then

$$\alpha = \frac{B-\sqrt{B^2-4}}{2} = \frac{2}{B+\sqrt{B^2-4}}$$

where

$$-\infty < B \leq -2\sqrt{2}, \qquad 2\sqrt{2} \leq B < \infty.$$

Hence α takes

$$-(\sqrt{2}-1) \leq \alpha < 0, \qquad 0 < \alpha \leq \sqrt{2}-1.$$

The sign of left equality holds iff $b = -\sqrt{2}$. The sign of right equality holds iff $b = \sqrt{2}$.

Hence MP_1 has a minimum at

$$\hat{\rho} = \alpha$$

Then, from

$$(\alpha - b)(1-\alpha^2) + \alpha\{1 + (\alpha - b)^2\} = 0,$$

we have

$$\begin{aligned} \frac{1+(\alpha-b)^2}{1-\alpha^2} &= \frac{b}{\alpha} - 1 \\ &= \frac{b^2+2+\sqrt{b^4+4}}{2} - 1. \end{aligned}$$

Thus the minimum value turns out to be

$$\frac{1+(\alpha-b)^2}{1-\alpha^2} = \frac{b^2+\sqrt{b^4+4}}{2}.$$

□

Moreover let us investigate the behavior of function $y = f(\rho)$ on the entire domain $-\infty < \rho < \infty,\ \rho \neq 0$. This takes a local minimum $f(\alpha) = \dfrac{b^2+\sqrt{b^4+4}}{2}$ at $\rho = \alpha$ and a local maximum $f(\beta) = \dfrac{b^2-\sqrt{b^4+4}}{2}$ at $\rho = \beta$. We note that

$$1 \leq \frac{b^2+\sqrt{b^4+4}}{2} < \infty, \qquad f\left(\frac{b^2+2}{2b}\right) = -1 < \frac{b^2-\sqrt{b^4+4}}{2} < 0.$$

Let the numerator of $f''(\rho)$ denote by $p(\rho)$:

$$\begin{aligned} p(\rho) &= 2b\rho^3 - 3(b^2+2)\rho^2 + 6b\rho - (b^2+2) \\ &= \{2b\rho - (b^2+2)\}(1-\rho^2) + 4\rho\{\,b\rho^2 - (b^2+2)\rho + b\,\}. \end{aligned}$$

Then

$$p'(\rho) = 6\{\rho^2 - (b^2+2)\rho + b\}.$$

This has the two zero-points $\alpha,\ \beta$, too. Hence $p(\rho)$ has a local maximum at $\rho = \alpha$ and a local minimum at $\rho = \beta$. Further the local maximum value is

$$p(\alpha) = \{2b\alpha - (b^2+2)\}(1-\alpha^2) = -(1-\alpha^2)\sqrt{b^4+4} < 0.$$

Hence $p(\rho)$ has a zero-point at some point $\gamma\,(>\beta\,)$ for $b > 0$ and a zero-point at some point $\gamma\,(<\beta\,)$ for $b < 0$. Therefore $f(\rho)$ has only one stationary point at γ (for case $b = 1$, see Fig.3).

Therefore we have an optimal path $\hat{x} = c(1,\ \alpha,\ \alpha^2,\ \ldots,\ \alpha^n,\ \ldots\}$ in this class. This $\hat{x}$ yields a minimum value $\dfrac{b^2 + \sqrt{b^4+4}}{2}c^2$.

In case $b = 1$, we have $\alpha = \phi^{-2}$. Thus an optimal $\hat{x} = c(1,\ \phi^{-2},\ \phi^{-4},\ \ldots,\ \phi^{-2n},\ \ldots)$ yields

$$I(\hat{x}) \ = \ \phi c^2.$$

The optimal path $\hat{x}$ is *golden.* The quadratic minimum value function $I(\hat{x})$ is *golden.* Thus in the case a golden optimal path yields a golden value function.

On the other hand, case $b = -1$ leads $\alpha = -\phi^{-2}$. Then an optimal $\hat{x} = c(1,\ -\phi^{-2}, \phi^{-4},\ -\phi^{-6},\ \phi^{-8},\ \ldots,\ (-1)^n\phi^{-2n},\ \ldots)$ yields

$$I(\hat{x}) \ = \ \phi c^2.$$

The optimal path $\hat{x}$ is *alternately golden.* The quadratic minimum value function $I(\hat{x})$ is *golden.* Thus in this case an alternately golden optimal path yields a golden value function.

4.2 Dynamic programming I(P)

Let us now consider a control process with an additive transition $T(x, u) = bx + u$. Here b is a constant, which represents a *characteristics* of the process :

$$\mathrm{PC}_1(c) \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} (x_n^2 + u_n^2) & \\ \text{subject to} & \text{(i)} \ \ x_{n+1} = bx_n + u_n & \\ & \text{(ii)} \ \ -\infty < u_n < \infty & n \geq 0 \\ & \text{(iii)} \ \ x_0 = c. & \end{array}$$

Then the value function v satisfies Bellman equation :

$$v(x) \ = \ \min_{-\infty<u<\infty} \left[x^2 + u^2 + v(bx+u)\right], \quad v(0) = 0. \tag{10}$$

The initial condition is justified as follows. Let $x_0 = c = 0$. Then by selecting

$$u = (u_0,\ u_1,\ \ldots,\ u_n,\ \ldots) = (0,\ 0,\ \ldots,\ 0,\ \ldots)$$

we get

$$x = (x_0,\ x_1,\ \ldots,\ x_n,\ \ldots) = (0,\ 0,\ \ldots,\ 0,\ \ldots).$$

The pair (x, u) yields a minimum value 0. Eq.(10) has a quadratic form $v(x) = vx^2$, where $v \ \in R^1$.

Theorem 4.1 *The control process* $\mathrm{PC}_1(c)$ *with characteristic value* b $(\in R^1)$ *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a quadratic minimum value function* $v(x) = vx^2$, *where*

$$v = \frac{b^2 + \sqrt{b^4+4}}{2}, \qquad p = -\frac{bv}{1+v} = \frac{2 - b^2 - \sqrt{b^4+4}}{2b}.$$

The proportional optimal policy f^∞ splits at any time an interval $[0,\ x]$ into $[0,\ (b+p)x] = \left[0,\ \frac{bx}{1+v}\right]$ and $\left[\frac{bx}{1+v},\ x\right]$. In particular, when $b = 1$, the quadratic coefficient v is reduced to the *Golden number*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Further the division of $[0,\ x]$ into $[0,\ \phi^{-2}x]$ and $[\phi^{-2}x,\ x]$ is Golden. That is, the ratio of length of two intervals constitutes the Golden ratio:

$$1 : \phi \ = \ \phi^{-2} : \phi^{-1}.$$

Corollary 4.1 *The process* $\mathrm{PC}_1(c)$ *with* $b = 1$ *has a Golden optimal policy* $f^\infty, f(x) = -\phi^{-1}x$, *and the Golden value function* $v(x) = \phi x^2$.

Corollary 4.2 *The process* $\mathrm{PC}_1(c)$ *with* $b = -1$ *has an alternately Golden optimal policy* $f^\infty, f(x) = \phi^{-1}x$, *and the Golden value function* $v(x) = \phi x^2$.

4.3 Euler equation I(P)

We solve

$$\begin{array}{llll} & \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \left[x_n^2 + (x_{n+1} - bx_n)^2 \right] & \\ \mathrm{PC}_1' & \text{subject to} & \text{(i)} \quad x_0 = c & \\ & & \text{(ii)} \quad x \in R^\infty & \end{array}$$

through variational approach. Let us apply Lemma 3.2. We take

$$f_n(x_n,\ x_{n+1} - bx_n) \ = \ x_n^2 + (x_{n+1} - bx_n)^2.$$

Then

$$f_{n1} \ = \ 2x_n, \qquad f_{n2} \ = \ 2(x_{n+1} - bx_n).$$

An extremal x satisfies

$$\begin{array}{lll} \text{(EE)} & x_n - [b(x_{n+1} - bx_n) - (x_n - bx_{n-1})] \ = \ 0 & n \geq 1 \\ \text{(TC)}_\infty & \displaystyle\lim_{n\to\infty} (x_{n+1} - bx_n) \ = \ 0. & \end{array}$$

Thus (EE) is reduced to

$$bx_{n+1} - (b^2+2)x_n + bx_{n-1} = 0.$$

Then the associated characteristic equation

$$bt^2 - (b^2+2)t + b = 0$$

has two solutions

$$\alpha = \frac{b^2+2-\sqrt{b^4+4}}{2b}, \quad \beta = \frac{b^2+2+\sqrt{b^4+4}}{2b}.$$

As is shown in the proof of Lemma 4.1, we have

$$-(\sqrt{2}-1) < \alpha = \frac{2b}{b^2+2+\sqrt{b^4+4}} < \sqrt{2}-1, \quad \beta = \frac{2b}{b^2+2-\sqrt{b^4+4}} = \frac{1}{\alpha}.$$

From the initial condition $x_0 = c$, we choose an solution of (EE)

$$x_n = c\alpha^n. \tag{11}$$

Then

$$\lim_{n\to\infty}(x_{n+1} - bx_n) = \lim_{n\to\infty} c(\alpha - b)\alpha^n = 0$$

Thus (TC) is satisfied. Thus we have obtained an optimal path (11).

Now we consider two special cases.

Case $b = 1$ yields $\alpha = \phi^{-2}$. Then the optimal path x, $x_n = c\phi^{-2n}$, is golden.

Case $b = -1$ yields $\alpha = -\phi^{-2}$. Then the optimal path x, $x_n = c(-1)^n\phi^{-2n}$, is alternately golden.

5 Dual Process I(D)

This section maximizes a quadratic cost function

$$c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty}\left[\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\right],$$

which is derived from the primal (minimization) problem at the end of section. This maximization problem is also solved as a control process with criterion

$$c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty}\left(\lambda_n^2 + \nu_n^2\right)$$

under an additive dynamics

$$b\lambda_{n+1} = \lambda_n + \nu_n.$$

5.1 Dynamic programming I(D)

Let us solve

$$
\mathrm{DC}_1(c) \qquad \begin{array}{ll} \text{Maximize} & x_0^2 + 2bx_0\lambda_0 - \sum_{n=0}^{\infty}(\lambda_n^2 + \nu_n^2) \\ \text{subject to} & \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n \qquad n \geq 0 \\ & \text{(ii)} \quad -\infty < \nu_n < \infty \\ & \text{(iii)} \quad x_0 = c \end{array}
$$

through dynamic programming. The dual process $\mathrm{DC}_1(c)$ generates a family of subprocesses :

$$
\mathrm{DC}_1'(\lambda) \qquad \begin{array}{ll} \text{minimize} & \sum_{n=0}^{\infty}(\lambda_n^2 + \nu_n^2) \\ \text{subject to} & \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n \qquad n \geq 0 \\ & \text{(ii)} \quad -\infty < \nu_n < \infty \\ & \text{(iii)} \quad \lambda_0 = \lambda. \end{array}
$$

Let $v(c)$ be the maximum value of $\mathrm{DC}_1(c)$ and $w(\lambda)$ be the minimum value of $\mathrm{DC}_1'(\lambda)$. Then the two value functions $v,\ w$ satisfy the Bellman equation (BE):

$$
\begin{cases} w(\lambda) = \min_{\nu \in R^1} \left[\lambda^2 + \nu^2 + w\left(\dfrac{\lambda+\nu}{b}\right)\right], & w(0) = 0 \\ v(c) = \operatorname{Max}_{\lambda \in R^1} \left[c^2 + 2bc\lambda - w(\lambda)\right], & v(0) = 0 \end{cases} \qquad \lambda, c \in R^1. \tag{12}
$$

This equation has a quadratic form $w(\lambda) = w\lambda^2,\ v(c) = vc^2$, where $w, v \in R^1$.

Theorem 5.1 *The control process $\mathrm{DC}_1'(\lambda)$ has a proportional optimal policy $g^\infty, g(\lambda) = q\lambda$, and a quadratic minimum value function $w(\lambda) = w\lambda^2$, where*

$$
w = \frac{2 - b^2 + \sqrt{b^4+4}}{2}, \qquad q = \frac{b^2 - \sqrt{b^4+4}}{2}.
$$

The process $\mathrm{DC}_1(c)$ has a proportional maximizer $\lambda^*, \lambda^*(c) = pc$, and a quadratic maximum value function $v(c) = vc^2$, where

$$
v = \frac{b^2 + \sqrt{b^4+4}}{2}, \qquad p = \frac{b^2 - 2 + \sqrt{b^4+4}}{2b}.
$$

Now let us consider case $b = 1$. Then Eq.(12) is reduced to a functional equation on an only v :

$$
\begin{cases} v(\lambda) = \min_{\nu \in R^1} \left[\lambda^2 + \nu^2 + v(\lambda+\nu)\right] \\ v(c) = \operatorname{Max}_{\lambda \in R^1} \left[c^2 + 2c\lambda - v(\lambda)\right] \end{cases} \qquad \lambda, c \in R^1, \quad v(0) = 0. \tag{13}
$$

Corollary 5.1 *The equation (13) has a proportional minimizer* $\hat{\nu}(\lambda) = -\phi^{-1}\lambda$, *a proportional maximizer* $\lambda^*(c) = \phi^{-1}c$, *and a quadratic value function* $v(c) = \phi c^2$.

On the other hand, case $b = -1$ leads the following equation.

$$\begin{cases} v(\lambda) = \min_{\nu \in R^1} \left[\lambda^2 + \nu^2 + v(-\lambda - \nu)\right] \\ v(c) = \operatorname{Max}_{\lambda \in R^1} \left[c^2 - 2c\lambda - v(\lambda)\right] \end{cases} \quad \lambda, c \in R^1, \quad v(0) = 0. \tag{14}$$

Corollary 5.2 *The equation (14) has a proportional minimizer* $\hat{\nu}(\lambda) = -\phi^{-1}\lambda$, *a proportional maximizer* $\lambda^*(c) = -\phi^{-1}c$, *and a quadratic value function* $v(c) = \phi c^2$.

5.2 Euler equation I(D)

Now let us solve

$$\mathrm{DC}_1' \quad \begin{array}{ll} \text{Maximize} & c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty} \left[\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\right] \\ \text{subject to} & \text{(i)} \quad \lambda \in R^\infty \end{array}$$

through variational approach. In order to apply Lemma 3.1, we set

$$\begin{aligned} k(\lambda_0) &= c^2 + 2bc\lambda_0 \\ f_n(\lambda_n,\ \lambda_{n+1} - \frac{1}{b}\lambda_n) &= -\lambda_n^2 - b^2\left(\lambda_{n+1} - \frac{1}{b}\lambda_n\right)^2. \end{aligned}$$

Then

$$k'(\lambda_0) = 2bc, \quad f_{n1} = -2\lambda_n, \quad f_{n2} = -2b^2\left(\lambda_{n+1} - \frac{1}{b}\lambda_n\right).$$

Thus

$$\begin{array}{ll} \text{(EE)} & f_{n1} - \left(\frac{1}{b}f_{n2} - f_{n-12}\right) = 0 \quad n \geq 1 \\ \text{(TC)}_0 & k'(\lambda_0) + f_{01} - \frac{1}{b}f_{02} = 0 \\ \text{(TC)}_\infty & \lim_{n\to\infty} f_{n2} = 0 \end{array}$$

are reduced to

$$\begin{array}{ll} \text{(EE)} & \lambda_n - \left[(b\lambda_{n+1} - \lambda_n) - b\,(b\lambda_n - \lambda_{n-1})\right] = 0 \quad n \geq 1 \\ \text{(TC)}_0 & bc - \lambda_0 + (b\lambda_1 - \lambda_0) = 0 \\ \text{(TC)}_\infty & \lim_{n\to\infty} (b\lambda_{n+1} - \lambda_n) = 0. \end{array}$$

, respectively. An extremal x satisfies above three equations. Then (EE) is

$$b\lambda_{n+1} - (b^2+2)\lambda_n + b\lambda_{n-1} = 0.$$

The associated characteristic equation

$$bt^2 - (b^2+2)t + b = 0$$

has two solutions

$$\alpha = \frac{b^2+2-\sqrt{b^4+4}}{2b}, \quad \beta = \frac{b^2+2+\sqrt{b^4+4}}{2b}. \tag{15}$$

Then

$$\alpha = \frac{2b}{b^2+2+\sqrt{b^4+4}}, \quad \beta = \frac{2b}{b^2+2-\sqrt{b^4+4}} = \frac{1}{\alpha}.$$

From $(\mathrm{TC})_0$ we have a boundary condition

$$(\mathrm{TC})'_0 \qquad bc - 2\lambda_0 + b\lambda_1 = 0.$$

As a solution of (EE) we select

$$\lambda_n = A\alpha^n$$

which should satisfy both transversality conditions. Then, $(\mathrm{TC})'_0$ becomes

$$(\mathrm{TC})'_0 \qquad bc - 2A + bA\alpha = 0.$$

This yields

$$A = c\frac{b}{2-b\alpha} = c(b-\alpha).$$

Then

$$\lambda_n = c(b-\alpha)\alpha^n. \tag{16}$$

This gives

$$b\lambda_{n+1} - \lambda_n = c(b-\alpha)(b\alpha-1)\alpha^n = -c\alpha^{n+1},$$

which implies that

$$\lim_{n\to\infty}(b\lambda_{n+1} - \lambda_n) = 0.$$

Thus both the conditions are satisfied. Hence we obtain an optimal path (16).

We take two special cases.

Case $b=1$ yields $\alpha = \phi^{-2}$. Then the optimal path λ, $\lambda_n = c\phi^{-1}\phi^{-2n}$, is golden.

Case $b=-1$ yields $\alpha = -\phi^{-2}$. Then the optimal path λ, $\lambda_n = -c\phi^{-1}(-1)^n\phi^{-2n}$, is alternately golden.

5.3 Two-variable maximization I(D)

Now let us solve

$$\mathrm{DC}_1' \quad \begin{array}{ll} \text{Maximize} & c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty} \left[\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2 \right] \\ \text{subject to} & \text{(i)} \quad \lambda \in R^{\infty} \end{array}$$

through two-variable optimization method. Let $\lambda = \{\lambda_n\}_{n \geq 0}$ take the form of $\lambda_n = A\rho^n$, where $A \in R^1$, $-1 < \rho < 1$ are constants. Let $f(A, \rho)$ be the evaluated value of λ :

$$f(A, \rho) = c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty} \left[\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2 \right]$$

It is easily shown that

$$f(A, \rho) = c^2 + 2bcA - \frac{1 + (b\rho - 1)^2}{1 - \rho^2} A^2.$$

Then

$$\begin{aligned} f_A/2 &= bc - \frac{1 + (b\rho - 1)^2}{1 - \rho^2} A \\ f_\rho/2 &= -\frac{b(b\rho - 1)(1 - \rho^2) + \rho\{1 + (b\rho - 1)^2\}}{(1 - \rho^2)^2} = \frac{b\rho^2 - (b^2 + 2)\rho + b}{(1 - \rho^2)^2}. \end{aligned}$$

From $f_A = f_\rho = 0$, we have

$$\begin{cases} A = cb\dfrac{1 - \rho^2}{1 + (b\rho - 1)^2} \\ b\rho^2 - (b^2 + 2)\rho + b = 0. \end{cases}$$

This yields

$$\begin{cases} A = c(b - \alpha) \\ \rho = \alpha \end{cases} \quad \text{where} \quad \alpha = \frac{b^2 + 2 - \sqrt{b^4 + 4}}{2b}.$$

Thus we obtain an optimal path

$$\lambda = \{\lambda_n\}_{n \geq 0} : \lambda_n = c(b - \alpha)\alpha^n. \tag{17}$$

5.4 A derivation of dual process I(D)

We show how a dual process is derived from the primal process

$$\mathrm{PC}_1(c) \quad \begin{array}{lll} \text{minimize} & \sum_{n=0}^{\infty} (x_n^2 + u_n^2) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = bx_n + u_n & n \geq 0 \\ & \text{(ii)} \quad -\infty < u_n < \infty & \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

Let $x = \{x_n\},\ u = \{u_n\}$ satisfy the above conditions and $I(x,u)$ denote the value of objective function :

$$I(x,u) \;=\; \sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right).$$

Then we have for any *Lagrange multiplier sequence* $\lambda = \{\lambda_n\}$

$$I(x,u) \;=\; \sum_{n=0}^{\infty} \left[x_n^2 + u_n^2 - 2\lambda_n \left(x_{n+1} - bx_n - u_n\right)\right].$$

Here we take $-2\lambda_n$ as a Lagrange multiplier for equality condition (i) for brevity of notation [15, 16, 19, 21]. By rearranging terms, we have

$$\begin{aligned} I(x,u) \;&=\; x_0^2 + 2bx_0\lambda_0 - \sum_{n=0}^{\infty}\left[\,\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\,\right] \\ &\quad + \sum_{n=1}^{\infty}\left[x_n - (\lambda_{n-1} - b\lambda_n)\right]^2 + \sum_{n=0}^{\infty}(u_n + \lambda_n)^2 \\ &\geq\; x_0^2 + 2bx_0\lambda_0 - \sum_{n=0}^{\infty}\left[\,\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\,\right]. \end{aligned}$$

Letting

$$J(\lambda) \;=\; x_0^2 + 2bx_0\lambda_0 - \sum_{n=0}^{\infty}\left[\,\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\,\right],$$

we have an inequality

$$I(x,u) \geq J(\lambda)$$

for any feasible (x,u) and any λ. The sign of equality holds iff

$$\begin{aligned} x_n \;&=\; \lambda_{n-1} - b\lambda_n \qquad n \geq 1 \\ u_n \;&=\; -\lambda_n \qquad n \geq 0. \end{aligned}$$

Thus we have derived a dual problem

$$\begin{aligned} &\text{Maximize} \quad c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty}\left[\,\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\,\right] \\ &\text{subject to} \quad \text{(i)} \quad \lambda \in R^{\infty}. \end{aligned}$$

Introducing a control variable $\nu_n := b\lambda_{n+1} - \lambda_n$, this problem is formulated as a control process

$$\begin{aligned} &\text{Maximize} \quad x_0^2 + 2bx_0\lambda_0 - \sum_{n=0}^{\infty}\left(\lambda_n^2 + \nu_n^2\right) \\ &\text{subject to} \quad \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n \qquad n \geq 0 \\ &\qquad\qquad\quad \text{(ii)} \quad -\infty < \nu_n < \infty \\ &\qquad\qquad\quad \text{(iii)} \quad x_0 = c. \end{aligned}$$

This is the desired dual process $\mathrm{DC}_1(c)$.

An optimal path x of primal process $\mathrm{PC}_1(c)$ and an optimal path λ of dual process $\mathrm{DC}_1(c)$ are transformed through

$$\begin{aligned} x_n &= \lambda_{n-1} - b\lambda_n \qquad n \geq 1 \\ \lambda_n &= bx_n - x_{n+1} \qquad n \geq 0. \end{aligned}$$

6 Primal Process II(P); quadratic in next state

This section minimizes the second quadratic cost function

$$\sum_{n=0}^{\infty} \left[(x_{n+1} - bx_n)^2 + x_{n+1}^2 \right].$$

This problem is also solved as a control process with criterion

$$\sum_{n=0}^{\infty} (u_n^2 + x_{n+1}^2)$$

under the additive dynamics

$$x_{n+1} = bx_n + u_n.$$

6.1 Evaluation-optimization II(P)

Second we take the following quadratic criterion

$$J(x) = \sum_{n=0}^{\infty} \left[(x_{n+1} - bx_n)^2 + x_{n+1}^2 \right].$$

We consider

$$\mathrm{MP}_2(\mathrm{c}) \qquad \text{minimize} \quad J(x) \quad \text{subject to} \quad \text{(i) } x \in R^{\infty}, \quad \text{(ii) } x_0 = c.$$

Since

$$J(x) = I(x) - c^2,$$

$\mathrm{MP}_2(\mathrm{c})$ has the minimum value

$$J(\hat{x}) = \frac{b^2 - 2 + \sqrt{b^4 + 4}}{2} c^2$$

at the path

$$\hat{x} = c(1, \alpha, \alpha^2, \ldots, \alpha^n, \ldots)$$

where

$$\alpha = \frac{b^2 + 2 - \sqrt{b^4 + 4}}{2b}.$$

Hence we have an optimal path $\hat{x} = c(1,\ \alpha,\ \alpha^2,\ \ldots,\ \alpha^n,\ \ldots\}$ in this class. This $\hat{x}$ yields a minimum value $\dfrac{b^2 - 2 + \sqrt{b^4 + 4}}{2}c^2$.

In case $b = 1$, we have $\alpha = \phi^{-2}$. Thus an optimal $\hat{x} = c(1,\ \phi^{-2},\ \phi^{-4},\ \ldots,\ \phi^{-2n},\ \ldots)$ yields

$$I(\hat{x}) = \phi^{-1}c^2.$$

The optimal path $\hat{x}$ is *golden*. The quadratic minimum value function $I(\hat{x})$ is *inverse-golden*. Thus in the case the golden optimal path yields an inverse-golden value function.

In fact, a proportional $w = (c,\ \rho c,\ \ldots,\ \rho^n c,\ \ldots)$ yields

$$\begin{aligned} J(w) &= \left\{\rho^2 c^2 + (1-\rho)^2 c^2\right\}\left(1 + \rho^2 + \cdots + \rho^{2n} + \cdots\right) \\ &= \frac{\rho^2 + (1-\rho)^2}{1-\rho^2}c^2 \qquad (0 < \rho < 1). \end{aligned}$$

Fig. 3 shows that

$$\min_{0 \le x < 1} \frac{x^2 + (1-x)^2}{1-x^2}$$

is attained at $\hat{x} = \phi^{-2}$ with the minimum value

$$\frac{(\phi^{-2})^2 + \{1 - (\phi^{-2})\}^2}{1 - (\phi^{-2})^2} = \phi^{-1}.$$

6.2 An Illustrative Graph $x = f(u)$

Let us now describe a graph which has dual Golden extremum points.

$$f(u) = \frac{u^2 + (1-u)^2}{1-u^2}$$

$$f'(u) = (-2)\frac{u^2 - 3u + 1}{(u^2-1)^2}$$

$$f''(u) = 2\frac{2u^3 - 9u^2 + 6u - 3}{(u^2-1)^3}$$

The golden ratio

$$\frac{\phi^2}{-(-\phi)} = \frac{\phi^{-1}}{\phi^{-2}} = \frac{\phi}{1}$$

$$\frac{-(-\phi)}{1} = \frac{1}{\phi^{-1}} = \frac{\phi}{1}$$

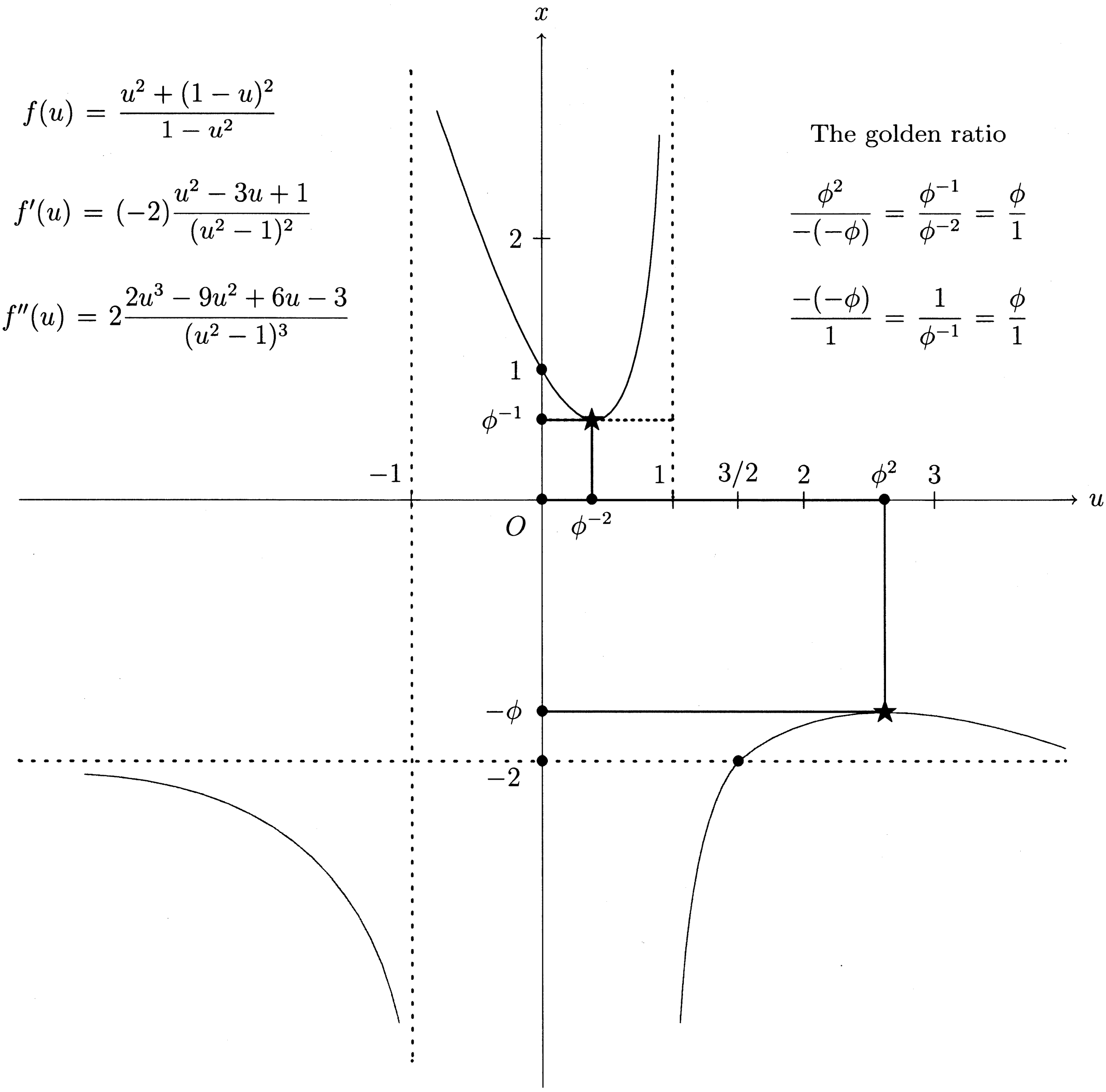

Fig. 3 Curve $x = f(u)$ has dual golden extremum points ★

We have the inequality

$$f(u) \geq \phi^{-1} \quad \text{on} \quad (-1, 1)$$
$$f(u) \leq -\phi \quad \text{on} \quad (-\infty, -1) \cup (1, \infty).$$

The first equality attains iff $\hat{u} = \phi^{-2}$, and the second equality attains iff $u^* = \phi^2$.

6.3 Dynamic programming II(P)

Here we consider the cost function $r : X \times U \to R^1$ which is quadratic in current control and next state :

$$r(x,u) \;=\; u^2 + (bx+u)^2.$$

Then a control process is represented by the following sequential minimization problem :

$$\mathrm{PC}_2(c) \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} (u_n^2 + x_{n+1}^2) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = bx_n + u_n & \\ & \text{(ii)} \quad -\infty < u_n < \infty & n \ge 0 \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

The value function v satisfies Bellman equation :

$$v(x) \;=\; \min_{-\infty<u<\infty} \left[u^2 + (bx+u)^2 + v(bx+u)\right]. \tag{18}$$

Eq. (18) has a quadratic solution $v(x) = vx^2$, where $v \in R^1$.

Theorem 6.1 *The control process* $\mathrm{PC}_2(c)$ *with characteristic value* b $(\in R^1)$ *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a quadratic minimum value function* $v(x) = vx^2$, *where*

$$v \;=\; \frac{b^2 - 2 + \sqrt{b^4+4}}{2}, \qquad p \;=\; -\frac{1+v}{2+v}b \;=\; \frac{2 - b^2 - \sqrt{b^4+4}}{2b}.$$

The policy f^∞ splits an interval $[0,\, x]$ into $\left[0,\, \frac{bx}{2+v}\right]$ and $\left[\frac{bx}{2+v},\, x\right]$. When $b = 1$, the coefficient v is reduced to the *inverse Golden number*

$$\phi^{-1} = \phi - 1 = \frac{-1+\sqrt{5}}{2} \approx 0.618$$

Further the division of $[0,\, x]$ into $[0,\, \phi^{-2}x]$ and $[\phi^{-2}x,\, x]$ is Golden :

$$\phi^{-2} : \phi^{-1} \;=\; 1 : \phi.$$

Corollary 6.1 *The process* $\mathrm{PC}_2(c)$ *with* $b = 1$ *has a Golden optimal policy* $f^\infty, f(x) = -\phi^{-1}x$, *and the inverse Golden value function* $v(x) = \phi^{-1}x^2$.

Corollary 6.2 *The process* $\mathrm{PC}_2(c)$ *with* $b = -1$ *has a Golden optimal policy* $f^\infty, f(x) = \phi^{-1}x$, *and the inverse Golden value function* $v(x) = \phi^{-1}x^2$.

6.4 Euler equation II(P)

Now we solve

$$\text{PC}_2' \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \left[(x_{n+1} - bx_n)^2 + x_{n+1}^2 \right] & \\ \text{subject to} & \text{(i)} & x_0 = c \\ & \text{(ii)} & x \in R^\infty. \end{array}$$

It suffices to note that

$$\sum_{n=0}^{\infty} \left[(x_{n+1} - bx_n)^2 + x_{n+1}^2 \right] = -c^2 + \sum_{n=0}^{\infty} \left[x_n^2 + (x_{n+1} - bx_n)^2 \right].$$

Both PC_1' and PC_2' have essentially the same objective function with a constant difference $-c^2$. Hence Euler equation of I(P) gives the same optimal path

$$x_n = c\alpha^n, \quad \alpha = \frac{b^2 + 2 - \sqrt{b^4 + 4}}{2b}.$$

7 Dual Process II(D)

This section maximizes a quadratic cost function

$$2bc\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty} \left[(b\lambda_{n+1} - \lambda_n)^2 + \lambda_{n+1}^2 \right],$$

which will be derived from the primal (minimization) problem at the end of section. This maximization problem is also solved as a control process with criterion

$$\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty} (\nu_n^2 + \lambda_{n+1}^2)$$

under the additive dynamics

$$b\lambda_{n+1} = \lambda_n + \nu_n.$$

7.1 Dynamic programming II(D)

Let us solve

$$\text{DC}_2(c) \qquad \begin{array}{lll} \text{minimize} & \displaystyle 2bx_0\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty} (\nu_n^2 + \lambda_{n+1}^2) & \\ \text{subject to} & \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n & \\ & \text{(ii)} \quad -\infty < \nu_n < \infty & n \geq 0 \\ & \text{(iii)} \quad x_0 = c & \end{array}$$

through dynamic programming. The dual process $\mathrm{DC}_1(c)$ generates a family of subprocesses :

$$\mathrm{DC}_2'(\lambda) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} (\lambda_n^2 + \nu_{n+1}^2) \\ \text{subject to} & \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n \qquad n \geq 0 \\ & \text{(ii)} \quad -\infty < \nu_n < \infty \\ & \text{(iii)} \quad \lambda_0 = \lambda. \end{array}$$

Let $v(c)$ be the maximum value of $\mathrm{DC}_2(c)$ and $w(\lambda)$ be the minimum value of $\mathrm{DC}_2'(\lambda)$. Then the two value functions v, w satisfy the Bellman equation (BE):

$$\begin{cases} w(\lambda) = \min\limits_{\nu \in R^1} \left[\nu^2 + \left(\dfrac{\lambda + \nu}{b} \right)^2 + w\left(\dfrac{\lambda + \nu}{b} \right) \right], \quad w(0) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda, c \in R^1 \\ v(c) = \operatorname*{Max}\limits_{\lambda \in R^1} \left[2bc\lambda - \lambda^2 - w(\lambda) \right], \quad v(0) = 0 \end{cases} \tag{19}$$

This equation has a quadratic form $w(\lambda) = w\lambda^2$, $v(c) = vc^2$, where $w, v \in R^1$.

Theorem 7.1 *The control process $\mathrm{DC}_2'(\lambda)$ has a proportional optimal policy $g^\infty, g(\lambda) = q\lambda$, and a quadratic minimum value function $w(\lambda) = w\lambda^2$, where*

$$w = \frac{-b^2 + \sqrt{b^4 + 4}}{2}, \qquad q = -\frac{-b^2 + \sqrt{b^4 + 4}}{2}.$$

The process $\mathrm{DC}_2(c)$ has a proportional maximizer $\lambda^*, \lambda^*(c) = pc$, and a quadratic maximum value function $v(c) = vc^2$, where

$$v = \frac{b^2 - 2 + \sqrt{b^4 + 4}}{2}, \qquad p = \frac{b^2 - 2 + \sqrt{b^4 + 4}}{2b}.$$

Now let us consider case $b = 1$. Then Eq.(19) is reduced to a functional equation on an only v :

$$\begin{cases} v(\lambda) = \min\limits_{\nu \in R^1} \left[\nu^2 + (\lambda + \nu)^2 + v(\lambda + \nu) \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \lambda, c \in R^1, \quad v(0) = 0 \\ v(c) = \operatorname*{Max}\limits_{\lambda \in R^1} \left[2c\lambda - \lambda^2 - v(\lambda) \right] \end{cases} \tag{20}$$

Corollary 7.1 *The equation (20) has a proportional minimizer $\hat{\nu}(\lambda) = -\phi^{-1}\lambda$, a proportional maximizer $\lambda^*(c) = \phi^{-1}c$, and a quadratic value function $v(c) = \phi^{-1}c^2$.*

On the other hand, case $b = -1$ leads the following equation.

$$\begin{cases} v(\lambda) = \min\limits_{\nu \in R^1} \left[\nu^2 + (\lambda + \nu)^2 + v(-\lambda - \nu) \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \lambda, c \in R^1, \quad v(0) = 0 \\ v(c) = \operatorname*{Max}\limits_{\lambda \in R^1} \left[2c\lambda - \lambda^2 - v(\lambda) \right] \end{cases} \tag{21}$$

Corollary 7.2 *The equation (21) has a proportional minimizer $\hat{\nu}(\lambda) = -\phi^{-1}\lambda$, a proportional maximizer $\lambda^*(c) = -\phi^{-1}c$, and a quadratic value function $v(c) = \phi^{-1}c^2$.*

7.2 Euler equation II(D)

Now let us solve

$$\mathrm{DC}_2' \quad \begin{array}{ll} \text{Maximize} & 2bc\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty}\left[(b\lambda_{n+1} - \lambda_n)^2 + \lambda_{n+1}^2\right] \\ \text{subject to} & \text{(i)} \quad \lambda \in R^{\infty} \end{array}$$

through variational approach.

We note that DC_1' has the objective function

$$f(\lambda) = c^2 + 2bc\lambda_0 - \sum_{n=0}^{\infty}\left[\lambda_n^2 + (b\lambda_{n+1} - \lambda_n)^2\right].$$

DC_2' has the objective function

$$g(\lambda) = 2bc\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty}\left[(b\lambda_{n+1} - \lambda_n)^2 + \lambda_{n+1}^2\right].$$

The difference is an only constant $-c^2$:

$$g(\lambda) = -c^2 + f(\lambda).$$

Thus the optimal solution of Euler equation I(D) is an optimal solution of Euler equation II(D) except for the constant $-c^2$ in optimal (maximum) value function.

Hence we have an optimal path

$$\lambda_n = c(b - \alpha)\alpha^n \tag{22}$$

where

$$\alpha = \frac{b^2 + 2 - \sqrt{b^4 + 4}}{2b} = \frac{2b}{b^2 + 2 + \sqrt{b^4 + 4}}.$$

This is a solution of characteristic equation

$$bt^2 - (b^2 + 2)t + b = 0$$

of Euler equation

$$b\lambda_{n+1} - (b^2 + 2)\lambda_n + b\lambda_{n-1} = 0.$$

This optimal path is also obtained by solving the system of variational equations.

$$\begin{array}{ll} \text{(EE)} & \lambda_n - [(b\lambda_{n+1} - \lambda_n) - b\,(b\lambda_n - \lambda_{n-1})] = 0 \qquad n \geq 1 \\ \text{(TC)}_0 & bc - \lambda_0 + (b\lambda_1 - \lambda_0) = 0 \\ \text{(TC)}_\infty & \lim_{n\to\infty}(b\lambda_{n+1} - \lambda_n) = 0. \end{array}$$

7.3 A derivation of dual process II(D)

We show how a dual process is derived from the primal process

$$\mathrm{PC}_2(c) \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty}\left(u_n^2 + x_{n+1}^2\right) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = bx_n + u_n & \\ & \text{(ii)} \quad -\infty < u_n < \infty & n \geq 0 \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

Let $x = \{x_n\}$, $u = \{u_n\}$ satisfy the above conditions and $I(x,u)$ denote the value of objective function :

$$I(x,u) = \sum_{n=0}^{\infty}\left(u_n^2 + x_{n+1}^2\right).$$

Then we have for any *Lagrange multiplier sequence* $\lambda = \{\lambda_n\}$

$$I(x,u) = \sum_{n=0}^{\infty}\left[u_n^2 + x_{n+1}^2 - 2\lambda_n\left(x_{n+1} - bx_n - u_n\right)\right].$$

Here we take $-2\lambda_n$ as a Lagrange multiplier for equality condition (i). By rearranging terms, we have

$$\begin{aligned} I(x,u) &= 2bx_0\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty}\left[\left(b\lambda_{n+1} - \lambda_n\right)^2 + \lambda_{n+1}^2\right] \\ &\quad + \sum_{n=1}^{\infty}\left[x_n - \left(\lambda_{n-1} - b\lambda_n\right)\right]^2 + \sum_{n=0}^{\infty}\left(u_n + \lambda_n\right)^2 \\ &\geq 2bx_0\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty}\left[\left(b\lambda_{n+1} - \lambda_n\right)^2 + \lambda_{n+1}^2\right]. \end{aligned}$$

Letting

$$J(\lambda) = 2bx_0\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty}\left[\left(b\lambda_{n+1} - \lambda_n\right)^2 + \lambda_{n+1}^2\right],$$

we have an inequality

$$I(x,u) \geq J(\lambda)$$

for any feasible (x,u) and any λ. The sign of equality holds iff

$$\begin{aligned} x_n &= \lambda_{n-1} - b\lambda_n \qquad n \geq 1 \\ u_n &= -\lambda_n \qquad n \geq 0. \end{aligned}$$

Thus we have derived a dual problem

$$\begin{aligned} &\text{Maximize} \quad 2bc\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty} \left[(b\lambda_{n+1} - \lambda_n)^2 + \lambda_{n+1}^2 \right] \\ &\text{subject to} \quad \text{(i)} \quad \lambda \in R^{\infty} \end{aligned}$$

Introducing a control variable $\nu_n := b\lambda_{n+1} - \lambda_n$, this problem is formulated as a control process

$$\begin{aligned} &\text{Maximize} \quad 2bx_0\lambda_0 - \lambda_0^2 - \sum_{n=0}^{\infty} (\nu_n^2 + \lambda_{n+1}^2) \\ &\text{subject to} \quad \text{(i)} \quad b\lambda_{n+1} = \lambda_n + \nu_n \qquad n \geq 0 \\ &\qquad\qquad\quad \text{(ii)} \quad -\infty < \nu_n < \infty \\ &\qquad\qquad\quad \text{(iii)} \quad x_0 = c. \end{aligned}$$

This is the desired dual process $\mathrm{DC}_2(c)$.

An optimal path x of primal process $\mathrm{PC}_2(c)$ and an optimal path λ of dual process $\mathrm{DC}_2(c)$ are transformed through

$$\begin{aligned} x_n &= \lambda_{n-1} - b\lambda_n \qquad n \geq 1 \\ \lambda_n &= bx_n - x_{n+1} \qquad n \geq 0. \end{aligned}$$

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A fuzzy CUSUM control chart for LR-fuzzy data under improved Kruse-Meyer approach

Dabuxilatu Wang[1],*Masami Yasuda[2]
[1] Department of probability and statistics,
Guangzhou University, No. 230 Waihuan Xilu ,
Higher Education Mega Center, Guangzhou, 510006, P.R.China
[2] Faculty of Science, Chiba University, Chiba 263, Japan

Abstract

Quality characteristic data is often imperfect (incomplete, censored, vague or partially unknown) in standing for the quality information of the products or services, such imperfectness sometimes may be well complemented by vague, imprecise or linguistic way of expression. In practice the LR-fuzzy number data is frequently recommended to be applied in above cases. LR-fuzzy number itself can be generated with method of Cheng based on expert's evaluations on products or services quality. On the set of LR-fuzzy data used for modelling the subjective human feeling on quality, we propose a fuzzy Cumulative Sum (CUSUM) control chart, in which the possibility distribution determined by the membership function of the fuzzy test statistic is employed, LR-fuzzy data is viewed as a fuzzy random variable with normally distributed center and two χ^2 distributed spreads. Under the distance between two fuzzy numbers proposed by Feng and an improved Kruse-Meyer hypothesis testing methods, a fuzzy decision rule as well as a level-wise average run length (ARL) for the chart are proposed. The simulation results shows that the proposed CUSUM chart has a better performance than fuzzy Shewhart chart under the proposed rule in term of ARL.

keywords: *statistical process control; Cumulative sum chart; fuzzy sets; possibility distribution.*

1 Introduction

Statistical process control is very important in that it is proven to bring processes into control and maintain it, in which the control charts is the principle measure to be designed and applied. Cumulative Sum (CUSUM) control chart proposed by Page [13] is widely used for monitoring and examining modern production processes. The power of CUSUM control chart lies in its ability to detect small shifts in processes as soon as it occurs and to identify abnormal conditions in a production process.

Control chart in many application is used to monitor real life data given as real numbers (real random variables) or real vectors (random vectors) sampling from production line. However, data collected from production lines with evaluation in some situation are considerably difficult to be exactly denoted by real numbers, e.g., the food taste data from the foods production line. Such data are often easily expressed by linguistic way and said to be linguistic data or vague (fuzzy) data, in the same way, data from human perception can be recorded by fuzzy data. Motivated by applying quality control charts to environment involving vague data, there have been some literatures dedicating for the design of control charts with linguistic data or fuzzy data. Wang and

*Corresponding author. E-mail:dbxlt0@yahoo.com

Raz [17] proposed the representative values control charts with both probability rule and membership function rule, for which the linguistic data (fuzzy data) is transformed into scalars referred as representative values of the fuzzy data, four kinds of transformation formula have been proposed, they are fuzzy mode, fuzzy midrange, fuzzy median and fuzzy average. Kanagawa and Tamaki and Ohta[9] proposed another representative values chart by using the barycenter of the fuzzy data, in which the required probability density function needs to be estimated using the Grame-Charlier series method. Höppner [7] proposed a kind of Shewhart chart, EWMA (Exponential Weighted Moving Average) chart with fuzzy data under Kruse and Meyer's hypothesis testing method [11], where the fuzzy data are directly used but mainly using the end-points of the α-cuts. Cen [1] proposed the suitability quality by using fuzzy sets method from an opinion of end-users. Taleb and Limam [15] discussed different precedures of construction control charts for linguistic data, based on fuzzy set and probability theories. A comparison between the fuzzy and probabilistic approaches, based on the average run length and the samples under control, is made by using real data. Cheng [2] proposed a method for generating fuzzy data based on the experts' score from evaluating the products quality, and constructed a control chart using membership method. Yu et al. [21] proposed a sequential probability ratio test (SPRT) control scheme for linguistic data based on Kanagawa et al.'s estimated probability density function, which lays a base for constructing CUSUM chart with linguistic data, however, in which the fuzzy data have to be transformed into its one of the representative value. Wang [18] presented a CUSUM control chart with fuzzy data by using a novel representative values that is a sum of central value of the fuzzy data with its fuzziness value. Hryniewicz [8] presented a general outlook for control charts with fuzzy data. Taleb [16] presented an application of the representative values control charts proposed by Wang and Raze [17] to multivariate attribute process. Gülbay [6] presents a direct fuzzy approach to construct a c-chart with fuzzy data. Faraz [4] presents a Shewhart chart with trapezoidal fuzzy data by using the concept of fuzzy random variables. Ming-Hung Shu and Hsien-Chung Wu [14] presented a fuzzy Shewhart chart and R chart using an expanded fuzzy dominance approach.

Most of the works mentioned above considered the Shewhart chart with representative values of fuzzy data, only a few works considered Shewhart chart, c-chart and EWMA chart with fuzzy data without using representative values methods. Since the representative value of a fuzzy data may result in losing important information included in original data, it is desirable to develop a suitable direct fuzzy way in establishing control charts with fuzzy data without using representative values. There are no constructions of CUSUM chart with fuzzy data in some direct fuzzy way reported in literatures. A sort of CUSUM chart with LR-fuzzy data in a direct fuzzy way will be established in this paper.

The rest of the article is organized as follows. In Section 2, some preliminary knowledge on fuzzy number and related concepts such as distance between two fuzzy numbers proposed by Feng, fuzzy max-order, fuzzy statistic, LR-fuzzy random variable are mentioned. In Section 3, we propose a CUSUM control chart with LR-fuzzy data based on fuzzy statistic. In Section 4, a level wise average run length for the proposed chart is considered. Finally, a detail conclusion and some related future research topic are presented.

2 Some statistics based on fuzzy data

Let $\mathbb{R}$ be the set of all real numbers. A fuzzy set on $\mathbb{R}$ is defined to be a mapping $u : \mathbb{R} \to [0,1]$ satisfying following conditions:

(1) $u_\alpha = \{x|u(x) \geq \alpha\}$ is a closed bounded interval for each $\alpha \in (0,1]$, i.e. $u_\alpha = [u_\alpha^-, u_\alpha^+]$.

(2) $u_0 = supp u$ is a closed bounded interval.

(3) $u_1 = \{x|u(x) = 1\}$ is nonempty.

where supp$u = cl\{x|u(x) > 0\}$, cl denotes the *closure* of a set. Such a fuzzy set is also called a *fuzzy number*. By $\mathcal{F}(\mathbb{R})$ we denote the set of all fuzzy numbers, with Zadeh's extension principle [22] the arithmetic operation $*$ on $\mathcal{F}(\mathbb{R})$ can be defined by

$$(u * v)(t) = \sup_{\{t_1 * t_2 = t\}} \{\min(u(t_1), v(t_2))\}, u, v \in \mathcal{F}(\mathbb{R}), t, t_1, t_1 \in \mathbb{R}, * \in \{\oplus, \ominus, \odot\}.$$

Where $\oplus, \ominus, \odot$ denote the addition, subtraction and scalar multiplication among fuzzy numbers, respectively. The fuzzy max-order $\preccurlyeq$ on $\mathcal{F}(\mathbb{R})$ is defined by

$$u \preccurlyeq v \iff \forall \alpha \in [0,1], u_\alpha^+ \leqslant v_\alpha^+, u_\alpha^- \leqslant v_\alpha^-, u, v \in \mathcal{F}(\mathbb{R}).$$

This order can be viewed as an extension of the interval order, in comparison of fuzzy numbers it has some advantages of simplicity in computation. The following parametric class of fuzzy numbers, the so-called LR-fuzzy numbers, are often used in applications:

$$u(x) = \begin{cases} L(\frac{m-x}{l}), & x \leq m \\ R(\frac{x-m}{r}), & x > m \end{cases}$$

Here $L : \mathbb{R}^+ \to [0,1]$ and $R : \mathbb{R}^+ \to [0,1]$ are given left- continuous and non-increasing function with $L(0) = R(0) = 1$. L and R are called left and right shape functions, m the central point of u and $l > 0$, $r > 0$ are the left and right spread of u. An LR-fuzzy number is abbreviated by $u = (m, l, r)_{LR}$, especially $(m, 0, 0)_{LR} := m$. It has been proven that LR-fuzzy numbers possesses some nice properties for operations:

$$\begin{aligned} (m_1, l_1, r_1)_{LR} \oplus (m_2, l_2, r_2)_{LR} &= (m_1 + m_2, l_1 + l_2, r_1 + r_2)_{LR} \\ a \odot (m, l, r)_{LR} &= \begin{cases} (am, al, ar)_{LR}, & a > 0 \\ (am, -ar, -al)_{RL}, & a < 0 \\ 0, & a = 0 \end{cases} \\ (m_1, l_1, r_1)_{LR} \ominus m_2 &= (m_1 - m_2, l_1, r_1)_{LR}. \end{aligned}$$

The last equality can be understood as that the fuzzy number $(m_1, l_1, r_1)_{LR}$ has a shift from m_1 to m_2.

Let $L^{(-1)}(\alpha) := \sup\{x \in R | L(x) \geq \alpha\}, R^{(-1)}(\alpha) := \sup\{x \in R | R(x) \geq \alpha\}$. Then for $u = (m, l, r)_{LR}$, $u_\alpha = [m - lL^{(-1)}(\alpha), m + rR^{(-1)}(\alpha)]$, $\alpha \in [0,1]$.

Körner [10] defined the *LR-fuzzy random variable* on the probability space $(\Omega, \mathcal{A}, P)$ as a measurable mapping $X : \Omega \to \mathcal{F}_{LR}(\mathbb{R})$, $X(\omega) = (m(\omega), l(\omega), r(\omega))_{LR}$, $\omega \in \Omega$, in brief we denote X as $X = (m, l, r)_{LR}$, where m, l, r are three independent real-valued random variables with $P\{l \geq 0\} = P\{r \geq 0\} = 1$. In a fuzzy observation on objects of interest, the outcomes can be viewed as LR-fuzzy data under a proper assumption, i.e., the data are viewed as the realizations of a LR-fuzzy random variable.

There are several metrics defined on fuzzy number space $\mathcal{F}(\mathbb{R})$. Among them a metric d_* proposed by Feng [3] seems to be more simple than other metrics in computation, which is defined as follows,

$$d_*(u, v) := (< u, u > -2 < u, v > + < v, v >)^{1/2},$$

where $< u, v > := \int_0^1 (u_\alpha^- v_\alpha^- + u_\alpha^+ v_\alpha^+) d\alpha$, and $< u, u >, < v, v > < \infty$, $u_\alpha = [u_\alpha^-, u_\alpha^+]$, $v_\alpha = [v_\alpha^-, v_\alpha^+]$. Feng's metric d_* can be employed to calculate the distance between two fuzzy number data of quality characteristics . For LR-fuzzy random variable $X = (m, l, r)_{LR}$, Feng [3] also define the expectation and variance as follows,

$$E(X) = (E(m), E(l), E(r))_{LR},$$

$$Var(X) = \frac{1}{2} \int_0^1 (Var(m - lL^{(-1)}(\alpha)) + Var(m + rR^{(-1)}(\alpha))) d\alpha.$$

Let $X_1, \cdots, X_n$ be a sample of size n from $X = (m, l, r)_{LR}$, then the sample mean $\overline{X}$ and the sample variance S^2 maybe defined by

$$\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i = (\overline{m}, \overline{l}, \overline{r})_{LR}, S^2 := \frac{1}{2(n-1)}\sum_{i=1}^{n} d_*^2(X_i, \overline{X}),$$

i.e.

$$S^2 = \frac{1}{2(n-1)}\sum_{i=1}^{n}\int_0^1 [((m_i - \overline{m}) + (\overline{l} - l_i)L^{(-1)}(\alpha))^2 + ((m_i - \overline{m}) + (r_i - \overline{r})R^{(-1)}(\alpha))^2]d\alpha,$$

and the sample range statistic maybe defined by $\overline{R} := d_*(X_{(1)}, X_{(n)})$, where $X_{(j)}$ denotes the jth order statistic with respect to the fuzzy-max order, and

$$\overline{m} = \frac{1}{n}\sum_{i=1}^{n} m_i, \overline{l} = \frac{1}{n}\sum_{i=1}^{n} l_i, \overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i.$$

3 CUSUM chart with LR-fuzzy data

The conventional CUSUM chart is usually used for monitoring real valued quality characteristics data. For a given sequence of crisp observations $\{X_n, n = 1, 2, \ldots\}$ on normal population, the monitored parameter of interest is typically the process mean, $\mu_n = E(X_n)$, the purpose is to detect a small change in the process mean, one might specifies the levels μ_0 and $\mu_1 > \mu_0$ (or $\mu_1 < \mu_0$) such that under normal conditions the values of μ_i should fall below (or above) μ_0 and the values of μ_n above (or below) μ_1 are considered undesirable and should be detected as soon as possible. The CUSUM chart can be used to monitor above process with the test-statistics $S_n = \max\{0, S_{n-1} + X_n - K\}$ (or $T_n = \min\{0, T_{n-1} + X_n + K\}$) and signal if $S_n > b$ (or $T_n < -b$), where b is the control limit derived from a confidence interval assuming a Gaussian distributed observation, b usually equals four or five times the standard deviation of sample, X_n $(n \geq 1)$ are the sample means at time t_n and $S_0 = T_0 = 0$, and K is the reference value.

We are aware of that the concept of quality has been extended to so called suitability quality by considering comprehensively the opinion of the end-users on products, as a tool the fuzzy sets has been applied to express the suitability quality of products [1], and the suitability quality characteristics of a product sampled from the production line are expressed by some appropriate way such as linguistic or some interval score in order to record the experts perception on the product quality. Fortunately, a sort of methods for generating a LR-fuzzy number from the expert's score (an evaluation) on quality characteristics had been proposed by Cheng [2], where the number of on-line experts is around 5.

We in the sequel assume that the observational process $\{X_i\}$ is consists of a finite sequence of i.i.d. LR-fuzzy random variables generated by method mentioned above, where X_i is the mean of group sample of size e at time t_i, i.e. $X_i = (x_i, l_i, r_i)_{LR}$, where $x_i = \frac{1}{e}\sum_{j=1}^{e} x_{ij}$, $l_i = \frac{1}{e}\sum_{j=1}^{e} l_{ij}$, $r_i = \frac{1}{e}\sum_{i=1}^{e} r_{ij}$, ($e \geq 25$, here we apply the Bootstrape approach for the group sample of size 5), x_i is a Gaussian variables approximately by the central limit theorem, and $l_{ij} > 0, r_{ij} > 0$ and l_i, r_i are approximately modelled by distributions $\chi^2(e_1)$, $\chi^2(e_2)$ $(i = 1, \cdots, n, j = 1, \cdots, e.)$ respectively. Assume that $E(x_{ij}) = \mu, Var(x_{ij}) = \sigma^2$. The test statistics S_n, T_n of a two-sided CUSUM chart then will be expanded to fuzzy statistics (fuzzy quantities) since they totally depend upon the samples X_i, $i = 1, \cdots, n$. By Zadeh's extension principle [22], the fuzzy statistics in this situation can be defined as

$$\tilde{S}_n(s) = \max_{s=max\{a,y\}}\{0(a), (\tilde{S}_{n-1} \oplus X_n \ominus K)(y)\}, s \in R$$

$$\tilde{T}_n(t) = \max_{t=min\{a,z\}}\{0(a), (\tilde{T}_{n-1} \oplus X_n \oplus K)(z)\}, t \in R$$

where 0 denotes a fuzzy number with membership value 1 at 0, and 0 otherwise. K is the reference value defined as follow in three different ways.

In the case (1): using Feng's variance,

$$K := K_1 = \frac{\delta}{2}\sigma_{\overline{X}} = \frac{\delta}{2}\sqrt{\frac{\sigma^2}{ne} + \frac{1}{n}\int_0^1 (e_1(L^{(-1)}(\alpha))^2 + e_2(R^{(-1)}(\alpha))^2)d\alpha},$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n}(x_i, l_i, r_i)_{LR}$, and $X_i - K_1 = (x_i - K_1, l_i, r_i)_{LR}$, δ is a multiple that allows us to measure shift size in term of the deviation $\sigma_{\overline{X}}$ being sampled.

In the case (2): We assume that the reference value is concerned only with the central value x_i of X_i, where some of X_i $(i = 1, \cdots, n.)$ maybe degenerated to crisp value, thus,

$$K := K_2 = \frac{\delta}{2}\sigma_{\overline{X}} = \frac{\delta\sigma}{2\sqrt{ne}},$$

and $X_i - K_2 = (x_i - K_2, l_i, r_i)_{LR}$.

In the case (3): We measure the shift size of the process mean by using the metric d_* based on the state of in-control, i.e.,

$$K := K_3 = \frac{1}{2}d_*[(\mu_0, a_{l0}, a_{r0})_{LR}, (\mu_1, a_l, a_r)_{LR}],$$

and $X_i - K_3 = (x_i - K_3, l_i, r_i)_{LR}$.

Remark: The properties of the operations $\oplus, \ominus$ for LR-fuzzy numbers ensure that the definitions of the fuzzy CUSUM statistic $\tilde{S}_n(s)$, $\tilde{T}_n(t)$ are reasonable if the reference value K is taken to be a real number value. However, in a general case where the sample $X_n \in \mathcal{F}(\mathbb{R})$ and so is the reference value K, then the above fuzzy statistic $\tilde{S}_n(s)$, $\tilde{T}_n(t)$ will lost reasonability since they are not able to measure the shift size of fuzzy data. It maybe an available way to employ the metric d_* to redefine the above fuzzy CUSUM statistic.

Set

$$s^-(\alpha) := \max\{0, (S_{n-1} + x_n - K)\} - L^{(-1)}(\alpha)\sum_{j=1}^{n} l_j;$$

$$s^+(\alpha) := \max\{0, (S_{n-1} + x_n - K)\} + R^{(-1)}(\alpha)\sum_{j=1}^{n} r_j;$$

$$t^-(\alpha) := \min\{0, (T_{n-1} + x_n + K)\} - L^{(-1)}(\alpha)\sum_{j=1}^{n} l_j;$$

$$t^+(\alpha) := \min\{0, (T_{n-1} + x_n + K)\} + R^{(-1)}(\alpha)\sum_{j=1}^{n} r_j;$$

where S_n, T_n are CUSUM statistic w.r.t. the crisp sample $x_i, i = 1, \cdots, n$ mentioned above.

Using Nguyen's theorem [12], we can obtain

Corollary 2.1 *The α-cut of $\tilde{S}_n, \tilde{T}_n$ are*

$$(\tilde{S}_n)_\alpha = \begin{cases} [s^-(\alpha), s^+(\alpha)], & \text{if } s^-(\alpha) > 0, \\ [0, s^+(\alpha)], & \text{if } s^+(\alpha) > 0, \\ 0, & \text{if } s^+(\alpha) \leqslant 0, \end{cases}$$

$$(\tilde{T}_n)_\alpha = \begin{cases} [t^-(\alpha), t^+(\alpha)], & \text{if } t^+(\alpha) < 0, \\ [t^-(\alpha), 0], & \text{if } t^-(\alpha) < 0, \\ 0, & \text{if } t^-(\alpha) \geqslant 0, \end{cases}$$

for $\alpha \in [0, 1]$.

Theorem 3.1 *For one-sided fuzzy CUSUM chart $\tilde{S}_n$, if the reference value is taken to be K_2 (σ is unknown), then the test statistics mentioned above on each α-level is $s^+(\alpha)$, and the control limit is $5\sqrt{\frac{s_x^2}{n}} + 3R^{(-1)}(\alpha)\sqrt{\frac{2ne_2}{e}}$, where s_x^2 is the sample variance w.r.t. x_i.*

Proof By the definition of fuzzy statistic $\tilde{S}_n$ and the Corollary 1, it is obvious that $s^+(\alpha)$ is the test statistics, and it is a sum of a crisp CUSUM statistics $\max\{0, (S_{n-1}+x_n-K_2)\}$ and a normally distributed statistics $R^{(-1)}(\alpha)\sum_{j=1}^{n} r_j$, then by the control limits of a crisp one-sided CUSUM S_n and the Shewhart chart, the former control limit is obtained as $5\sqrt{\frac{s_x^2}{n}}$ and the later control limit can be carried out as $3\sigma_{r_i} = 3R^{(-1)}(\alpha)\sqrt{\frac{2ne_2}{e}}$, where s_x^2 is the sample variance w.r.t. x_i, thus, the control limit is obtained. □

Corollary 3.2 *The one-sided CUSUM chart $\tilde{T}_n$ on each α-level with reference value K_2 (σ is unknown) possesses the test statistics $t^-(\alpha)$ and control limit $-5\sqrt{\frac{s_x^2}{n}} - 3R^{(-1)}(\alpha)\sqrt{\frac{2ne_2}{e}}$, where s_x^2 is the sample variance w.r.t. x_i.*

Corollary 3.3 *For one-sided fuzzy CUSUM chart $\tilde{S}_n$, if the reference value is taken to be K_1 (σ is unknown), then the test statistics mentioned above on each α-level is $s^+(\alpha)$, and the control limit is $5\sqrt{\frac{S^2}{n}} + 3R^{(-1)}(\alpha)\sqrt{\frac{2ne_2}{e}}$.*

Corollary 3.4 *For one-sided fuzzy CUSUM chart $\tilde{S}_n$, if the reference value is taken to be K_3 (σ is unknown), then the test statistics mentioned above on each α-level is $s^+(\alpha)$, and the control limit is $5d_*[(\mu_0, a_{l0}, a_{r0})_{LR}, (\mu_1, a_l, a_r)_{LR}] + 3R^{(-1)}(\alpha)\sqrt{\frac{2ne_2}{e}}$.*

In the same way we can obtain the corresponding control limits of one-sided CUSUM chart $\tilde{T}_n$ in the case of K_1, K_3.

Based on possibility distribution (membership function) of $\tilde{S}_n, \tilde{T}_n$, we propose a soft control rule for the two-sided CUSUM chart.

Let M be a natural number which stands for the times of checking for different α-levels, and it should be determined reasonably based on a total evaluation from quality experts. There are many approaches to determine M such as the weighted average score of several experts, the Bayesian method using experience or historical data, or belief function generating method, etc. We further choose an arbitrary $z \in \{1, 2, \ldots, M\}$ as a critical value (key number) based on some given possibility levels and choose arbitrarily the levels $\{\alpha_1, \alpha_2, \ldots, \alpha_M\} \subset [0, 1]$. Let

$$\Phi_i(\tilde{S}_n, \tilde{T}_n) = \begin{cases} 1, & \text{if } s^+(\alpha_i) > h_{1\alpha_i} \text{ or } t^-(\alpha_i) < h_{2\alpha_i} \\ 0, & \text{otherwise} \end{cases}$$

where $h_{1\alpha_i} = 5\sqrt{\frac{s_x^2}{n}} + 3R^{(-1)}(\alpha_i)\sqrt{\frac{2ne_2}{e}}, h_{2\alpha_i} = -5\sqrt{\frac{s_x^2}{n}} - 3R^{(-1)}(\alpha_i)\sqrt{\frac{2ne_2}{e}}$ for corresponding $K = K_2$. $h_{1\alpha_i}$ and $h_{2\alpha_i}$ can also be taken the corresponding values when $K = K_1$ or $K = K_3$ as that illustrated in Corollary 3.3 and Corollary 3.4.

Let

$$\Phi(\tilde{S}_n, \tilde{T}_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{M} \Phi_i(\tilde{S}_n, \tilde{T}_n) \geq z \\ 0, & \text{otherwise} \end{cases}$$

Then the decision rule for the fuzzy CUSUM control chart is that: **we stop the process if and only if $\Phi(\tilde{S}_n, \tilde{T}_n) = 1$.**

$$R(\alpha_i) = \begin{cases} \inf\{n | \Phi_i(\tilde{S}_n, \tilde{T}_n) = 1\} \\ \infty, \text{if there is no } n \text{ such that } \Phi_i(\tilde{S}_n, \tilde{T}_n) = 1 \end{cases}$$

is called α_i- level stopping time.

$$R = \begin{cases} \inf\{n | \Phi(\tilde{S}_n, \tilde{T}_n) = 1\} \\ \infty, \text{if there is no } n \text{ such that } \Phi(\tilde{S}_n, \tilde{T}_n) = 1 \end{cases}$$

is called the stopping time.

It is obvious that there exist at least z number $\{i_1, i_2, \ldots, i_z\} \subseteq \{1, 2, \ldots, M\}$ such that $R \geq \max\{R(\alpha_{i_1}), R(\alpha_{i_2}), \ldots, R(\alpha_{i_z})\}$ where $M \in \mathbb{N}$.

4 The computation of a kind of approximate ARL for the fuzzy CUSUM chart

From the definition of the stoping times mentioned in former section, we are aware of that the α-level ($\alpha \neq 1$) stopping time is larger than the 1-level stopping time. This property not only shows a strong flexibility implied in the fuzzy control limit but also exhibit an idea making the control rule depending on the experts's appraisal on the products quality level-wise.

Obviously, on each α_i-level our fuzzy CUSUM control chart can be viewed as an ordinary CUSUM chart, and for which the Average Run Length (ARL) can be carried out based on the basic parameters K and h. Here the ARL of of the fuzzy CUSUM chart on the α_i-level can be approximated by Markov chain method as in the case of [19] (1991), [20] (1994), namely, $ARL(\alpha_i) := \mathbb{E}(R(\alpha_i))$.

Let R and $R(\alpha_i)$ be the stopping time and α_i-level stopping time for the fuzzy CUSUM chart, respectively, where $\alpha_i \in [0,1], \quad i \in \{1,2,\dots,M\}$ and $M \in \mathbb{N}$. J. Höppner [7] (1994) obtained a conclusion on the relation between ARL $\mathbb{E}(R)$ and α_i-level ARL $\mathbb{E}(R(\alpha_i))$ for his EWMA chart with fuzzy data, since it only concerned with the test-statistic Φ, Φ_i, thus an analogue to CUSUM chart can be stated as follows:

Theorem 4.1 *Assume that there exists ARL= $\mathbb{E}(R)$ and α_i-level ARL(α_i)= $\mathbb{E}(R(\alpha_i))$ for the fuzzy CUSUM chart and z is a key number. Then it holds that*

$$\mathbb{E}(R) \leq \frac{1}{M-z+1} \sum_{i=1}^{M} \mathbb{E}(R(\alpha_i)).$$

Corollary 4.1 *If $z = M$ in Theorem 4.1, then it holds that* $\frac{1}{M}\sum_{i=1}^{M}\mathbb{E}(R(\alpha_i)) \leq \mathbb{E}(R) \leq \sum_{i=1}^{M}\mathbb{E}(R(\alpha_i))$.

We now approximate ARL of the fuzzy CUSUM chart, which is $\mathbb{E}(R)$. It is well known that an approximation of ARL for ordinary CUSUM chart usually means to compute the ARL of one-sided S_n-chart or of T_n-chart, since S_n and T_n can not signal simultaneously ([19] (1991),[20] (1994)). However, now the situation becomes so complicated that the $\tilde{S}_n$ and $\tilde{T}_n$ might simultaneous signal at some special α-levels. We only consider the case in which ARL of fuzzy CUSUM chart can be approximated through one-sided $\tilde{S}_n$ chart or $\tilde{T}_n$-chart.

In the following, we employ Markov chain method to approximate the $ARL(\alpha_i)_{\tilde{S}}$, $ARL(\alpha_i)_{\tilde{T}}$.

Consider the one-sided $\tilde{S}_n$-chart on α_i-level, we divide the interval $[0, h_{1\alpha_i}]$ into t subintervals as follows:

Let $t \in \mathbb{N}, \quad \{\alpha_1,\dots,\alpha_M\} \subseteq (0,1)$.

(i) $\delta^i := \frac{h_{1\alpha_i}}{2t-1}$, $2\delta^i$ is the length of subintervals.

(ii) $b^i_j := 2j\delta^i$, $1 \leqslant j \leq t-1$ is the mid-point of (*j+1*)st subinterval.

(iii) $T^i_j := (b^i_j - \delta^i,\, b^i_j + \delta^i]$, $1 \leq j \leq t-1$ is the (*j+1*)st subinterval.

(iv) $I^i_0 := [0,\delta^i]$; $\quad b^i_0 := \frac{\delta^i}{2}$; $\quad I^i_t := (h_{1\alpha_i}, \infty)$.

In Markov chain method, we need to compute the transition probability

$$p^{*i}_{lj} := P(s^+_{(n+1)\alpha_i} \in I^i_j | s^+_{n\alpha_i} \in I^i_l).$$

p^{*i}_{lj} can be approximated by p^i_{lj}, i.e.,

$$p^{*i}_{lj} \approx p^i_{lj},$$

where

$$\begin{aligned} p^i_{lj} : &= P(s^+_{(n+1)\alpha_i} \in I^i_j | s^+_{n\alpha_i} = b^i_l) \\ &= P(K + (j-l)2\delta^i - \delta^i < X^+_{(n+1)\alpha_i} \\ &\quad \le K + (j-l)2\delta^i + \delta^i), \end{aligned}$$

$l = 0, 1, \ldots, t-1; \quad j = 0, 1, \ldots, t-1$ and $K := K_1, K_2, K_3$.

Thus we obtain an approximative transition probability matrix P, a $(t+1) \times (t+1)$ matrix as follows:

$$P = \begin{pmatrix} D & (E-D)d \\ O^T & 1 \end{pmatrix}$$

where $D = \left(p^i_{lj}\right)_{0 \le l, j \le t-1}$ is a $t \times t$ matrix, $d^T = (1, 1, \ldots, 1)$, $O^T = (0, 0, \ldots, 0)$ and E is the $t \times t$ unit matrix.

Theorem 4.2 *Let the observational process $\{X_i\}$ be a sequence of independent and identically distributed (i.i.d.) LR-fuzzy random variables, where each X_i is the mean of group sample of size e at time t_i. Then it holds that*

$$ARL(\alpha_i)_{\tilde{S}} \approx {p_0}^T (E-D)^{-1} d.$$

where ${p_0}^T := (p_0, p_1, \ldots, p_{t-1})$ denotes the start probability vector.

Proof. Since $\{X_i\}$ are i.i.d. fuzzy random variables, then $\tilde{S}_n$ can be viewed as a sum of independent increment in term of fuzzy random variables, for each $\alpha \in (0,1]$ the process $\{s^+_{n\alpha}\}$ can be viewed as a real-valued Markov process. By the approximation method mentioned above, we see that

$$\begin{aligned} &\left(P(s^+_{(n+1)\alpha_i} \in I^i_j | s^+_{1\alpha_i} \in I^i_l)\right)_{0 \le l, j \le t-1} \\ &\approx \quad P^n = \begin{pmatrix} D^n & (E-D^n)d \\ O^T & 1 \end{pmatrix} \end{aligned}$$

note that here

$$\begin{aligned} &P(R(\alpha_i) \le n | s^+_{1\alpha_i} \in I^i_l) \\ &\quad = \quad P(s^+_{(n+1)\alpha_i} \in I^i_t | s^+_{1\alpha_i} \in I^i_l) \\ &\quad \approx \quad ((E-D^n)d)_l, \end{aligned}$$

where $((E-D^n)d)_l$ denotes the lst component of the vector $(E-D^n)d$. Let $p_l := P(s^+_{1\alpha_i} \in I^i_l)$, then $\sum_{l=0}^{t-1} p_l = 1$ and let the vector $p_0 = (p_0, p_1, \ldots, p_{t-1})^T$ be the start probability vector. Then we have

$$\begin{aligned} &P(R(\alpha_i) \leqslant n) \\ &\quad = \quad \sum_{l=0}^{t-1} p_l P(\{R(\alpha_i) \leqslant n | S^+_{1\alpha_i} \in I^i_l\}) \\ &\quad \approx \quad p_0^T (E-D^n)d. \end{aligned}$$

and

$$P(R(\alpha_i) > n) \approx 1 - p_0^T (E-D^n)e = p_0^T (D^n)d,$$

and

$$P(R(\alpha_i) = n) = P(R(\alpha_i) \leqslant n) - P(R(\alpha_i) \le n-1)$$

$$\approx p_0^T (D^{n-1} - D^n)d.$$

by the proof for theorem 3.11 of [20], we obtain that

$$
\begin{aligned}
\mathbb{E}(R(\alpha_i)) &= \sum_{n=1}^{\infty} nP(R(\alpha_i)=n) \\
&\approx p_0^T(\lim_{k\to\infty}\sum_{n=1}^{k} n(D^{n-1}-D^n)d) \\
&= {p_0}^T(E-D)^{-1}d.
\end{aligned}
$$

This completes the proof.

In a similar way, we can compute $ARL(\alpha_i)_{\widetilde{T}}$.

Based on Corollary 4.1 and Corollary 4.2, we can approximate ARL of the fuzzy CUSUM control chart provided that the number M and the key number z are predetermined.

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ON THE MASAMI YASUDA STOPPING GAME

KRZYSZTOF J. SZAJOWSKI

ABSTRACT. The sero-sum stopping game for the stochastic sequences has been formulated in late sixties of the twenty century by Dynkin [5]. The formulation had the assumption about separability of decision moment of the players which simplified the construction of the solution. Further research by Neveu [22] extended the model by admitting more general behaviour of the players and their pay–offs. In new formulation there is the problem with existence of the equilibrium. The proper approach to solution of the problem without restriction of former models was developed by Yasuda [44]. The results was crucial in these research. The author made often reference to the Yasuda's [44] result in his works (see [36, 37, 38]) as well as see results of others stimulated by this paper. Withal, in this note another stopping game model, developed by Yasuda with coauthors (see e.g. [14] and [40]) is recalled. The application of the model to an analysis of system of detectors shows the power of the game theory methods.

In the last part of the paper I would like to express my personal relation to the Masami Yasuda game.

1. INTRODUCTION

The mathematical modelling of economic and engineering systems in stochastic environment leads to various mathematical optimization and game theory problems. If the decision problem relies on choice of intervention moments one can formulate the model of such case as the optimal stopping problem. If it is allowed to react more than once the approach depends on the number of decision makers and their aims. If there is one decision maker and two reactions (or fix number of possible moment of actions) we have the optimal two stopping (multiple stopping) problem. When there are two decision makers with their prescribed aims we usually treat the problem as the stopping game. The related models bring very subtle mathematical questions concerning the correctness of the model, possibility of inference about rational strategies, their realization and the existence of solution in the formulated mathematical model. In this note I would like to focus our attention of two group of models not very precisely

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defined. The first one is related to the existence of solution under mild assumption on the processes defining the payoffs in the zero-sum stopping game related to problem introduced by Dynkin [5] (see the section 1.1). The second group of the problems, which I have applied recently to modelling the sensor networks, developed by Yasuda with co-authors (see e.g. [14] and [40]), is devoted to multivariate stopping problem when there are decision makers having some interactions between each others (see the section 1.2).

1.1. **The randomize strategies in Dynkins game.** *E. B. Dynkin* [5] presented the following problem: two players observe a stochastic sequence X_n, $n = 1, 2, \ldots$. Each of them chooses a stopping time, say λ (resp. μ) be the stopping time chosen by the first (resp. the second) player. It is additionally assumed that the first player can stop at odd and the second at even moments. The pay-off is then: $R(\lambda, \mu) = \mathbf{E}X_{\lambda\wedge\mu}$. The player 1 seeks to maximize the expected pay-off, and the player 2 seeks to minimize it, it means that the solution is a pair $(\lambda^\star, \mu^\star)$ such that

$$R(\lambda, \mu^\star) \leq (\lambda^\star, \mu^\star) \leq (\lambda^\star, \mu). \tag{1.1}$$

J. Neveu [22] modified this problem as follows: there are three random sequences $(X_n, \mathcal{F}_n)$, $(Y_n, \mathcal{F}_n)$ and $(W_n, \mathcal{F}_n)$ with

Assumptions 1.2.

$$X_n \leq W_n \leq Y_n \textit{ for each } n \in \mathbb{N}, \tag{1.3}$$

and the pay-off equals:

$$R(\lambda, \mu) = \mathbf{E}\{X_\lambda \mathbb{I}_{\{\lambda<\mu\}} + W_\lambda \mathbb{I}_{\{\lambda=\mu\}} + Y_\mu \mathbb{I}_{\{\mu<\lambda\}}\}, \tag{1.4}$$

where $\lambda, \mu \in \mathfrak{S}$ are stopping times with respect to $\mathcal{F}_n$.This problem has solution which is presented in [22].

When the assumption 1.2 is not fulfilled then, in general there are no equilibrium in the set of stopping times with respect of observed processes $(X_n, \mathcal{F}_n)$, $(Y_n, \mathcal{F}_n)$ and $(W_n, \mathcal{F}_n)$. It is *M. Yasuda* who shown in [44] that the mixed extension of this game has equilibrium without the assumption 1.2. The mixed extension in this case means that the set of strategies (stopping times) is extended to include randomized stopping time. First, a finite horizon problem is considered. Next, the existence of the value and the equilibrium point in the infinite horizon problem with a discount factor is proved under some natural assumption concerning the integrability of the considered processes.

In that time there were many mathematicians doing research in stopping game (see e.g. the papers by Zabczyk [50], Stettner [34], Ohtsubo [23]). The Yasuda's paper [44] stimulated further research of Rosenberg, Solan and Vieille [28] and Laraki and Solan [17] in the mixed extension of the stopping game for the processes with continuous parameter.

1.2. **The stopping processes by voting procedure.** Let us consider p person stopping game related to the observation of a Markov chain. Let $(X_n, \mathfrak{F}_n, \mathbf{P}_x)$, $n = 0, 1, 2, \ldots$, be a homogeneous Markov chain defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ with state space $(\mathbb{E}, \mathcal{B})$. The players are able to observe the Markov chain sequentially. At each moment n their knowledge is represented by $\mathfrak{F}_n$. Each player has his own utility function $f_i : \mathbb{E} \to \Re$, $i = 1, 2, \ldots, p$, and at each moment n each player declares separately his willingness to stop the observation of the process. The effective ends of the process and realization of the payoffs appears when a suitable subset of players agree to it. The aim of each player is to maximize their expected payoffs. In fact, the problem will be formulated as a p person non-cooperative game with the concept of Nash equilibrium [21] as the solution. On the other hand, one can say that the considered multilateral stopping procedure is based on sequential voting (cf [7], [10], [43] for monotone rule concept and the mathematics of voting).

Such model has been considered in mine and Yasuda paper [40] and Ferguson [6]. Both papers were continuation of Masami Yasuda and his co-workers, Kurano and Nakagami research published in [14], [46], [45], [48]. They have investigated the multilateral version of the optimal stopping problem for independent, identically distributed p dimensional random vectors $\overline{X_n}$. The gain function of the i-th player is X_n^i (i-th coordinate of $\overline{X_n}$). In [14] the following class of strategies is used.

(1) Each player can declare to stop at any stage.
(2) The majority level r $(1 \le r \le p)$ is chosen by the players at the beginning of the game.
(3) During the sequential observation process, if the number of players declaring to stop is greater than or equal to the level r, the process must be stopped.

This class of strategies is generalized in [46] to monotone rules. Roughly speaking, a monotone rule is a p variate, non-decreasing logical function defined on $\{0, 1\}^p$. In both papers the problem is formulated as a p person, non-cooperative game with concept of Nash point as a solution. Paper [14] generalizes the unanimity case, i.e. $p = r$ solved by Sakaguchi [29]. The motivation for the model considered is the secretary problem (see [9] for the formulation of the problem). A solution of some bivariate version of the secretary problem is given in [14]. Presman and Sonin [26] treat this problem with another set of strategies. They considered the model in which each player's decision does not affect the stopping of the process but his reward only. Sakaguchi [30] and Kadane [12] have solved a multilateral sequential decision problem in which decisions whether to stop are made by the players alternately, instead of simultaneous decision under a monotone rule.

The recent paper on the voting stopping problem are [20].

2. Sensors' network and stopping games

In [39] the construction of the mathematical model for a multivariate surveillance system is presented. It is assumed that there is net $\mathfrak{N}$ of p nodes which register (observe) signals modeled by discrete time multivariate stochastic process. At each node the state is the signal at moment $n \in \mathbb{N}$ which is at least one coordinate of the vector $\overrightarrow{x}_n \in \mathbb{E} \subset \Re^m$. The distribution of the signal at each node has two forms and depends on *a pure* or *a dirty* environment of the node. The state of the system change dynamically. We consider the discrete time observed signal as $m \geq p$ dimensional process defined on the fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The observed at each node process is Markovian with two different transition probabilities (see [31] for details). In the signal the visual consequence of the transition distribution changes at moment θ_i, $i \in \mathfrak{N}$ is a change of its character. To avoid false alarm the confirmation from other nodes is needed. The family of subsets (coalitions) of nodes are defined in such a way that the decision of all member of some coalition is equivalent with the claim of the net that the disorder appeared. It is not sure that the disorder has had place. The aim is to define the rules of nodes and a construction of the net decision based on individual nodes claims. Various approaches can be found in the recent research for description or modelling of such systems (see e.g. [42], [27]). The problem is quite similar to a pattern recognition with multiple algorithm when the fusions of individual algorithms results are unified to a final decision. The proposed solution will be based on a simple game and the stopping game defined by a simple game on the observed signals. It gives a centralized, Bayesian version of the multivariate detection with a common fusion center that it has perfect information about observations and *a priori* knowledge of the statistics about the possible distribution changes at each node. Each sensor (player) will declare to stop when it detects disorder at his region. Based on the simple game the sensors' decisions are aggregated to formulate the decision of the fusion center. The sensors' strategies are constructed as an equilibrium strategy in a non-cooperative game with a logical function defined by a simple game (which aggregates their decision).

This approach uses the general description of such multivariate stopping games presented in the section 1.2. The voting aggregation rules are relieved by the simple game (see Ferguson [6]) and the underlining processes form Markov sequences (see [40]).

The model of disorder detection at each sensor are presented in the next section. It allows to define the individual pay-off of the players (sensors). It is assumed that the sensors are distributed in homogeneous way in the guarded area and the intruders behaviour are well modelled by symmetric random walk. By these assumptions in the section 3 the *a priori* distribution of the disorder moment at each node can be chosen in such a way that it gives the best model of the structure of sensors and the behaviour of intruder . The section 4 introduces the aggregation method based on a simple game of the sensors. The section 5 contains derivation of the non-cooperative game and existence theorem for equilibrium strategy. The final decision based on the state of the sensors is given by the fusion center and it is described in the section 6.1. The natural direction of further research is formulated also in the same section. A conclusion and resume of an algorithm for rational construction of the surveillance system is included in the section 6.2.

The extension of non-cooperative games to the case when the communication between player is allowed leads to various solutions concepts. The voting stopping game is interesting approach also in this direction of research.

3. Detection of disorder at sensors

Following the consideration of Section 1, let us suppose that the process $\{\overrightarrow{X}_n, n \in \mathbb{N}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, is observed sequentially in such a way that each sensor, *e.g.* r (gets its coordinates in the vector $\overrightarrow{X}_n$ at moment n). By assumption, it is a stochastic sequence that has the Markovian structure given random moment θ_r, in such a way that the process after θ_r starts from state $\overrightarrow{X}_{n\,\theta_r-1}$. The objective is to detect these moments based on the observation of $\overrightarrow{X}_n$. at each sensor separately. There are some results on the discrete time case of such disorder detection which generalize the basic problem stated by Shiryaev in [32] (see e.g. Brodsky and Darkhovsky [2], Bojdecki [1], Poor and Hadjiliadis [25], Yoshida [49], Szajowski [35]) in various directions.

Application of the model for the detection of traffic anomalies in networks has been discussed by Tartakovsky et al. [41]. The version of the problem when the moment of disorder is detected with given precision will be used here (see [31]).

3.1. Formulation of the problem.

The observable random variables $\{\overrightarrow{X}_n\}_{n\in\mathbb{N}}$ are consistent with the filtration $\mathcal{F}_n$ (or $\mathcal{F}_n = \sigma(\overrightarrow{X}_0, \overrightarrow{X}_1, \ldots, \overrightarrow{X}_n)$). The random vectors $\overrightarrow{X}_n$ take values in $(\mathbb{E}, \mathcal{B})$, where $\mathbb{E} \subset \Re^m$. On the same probability space there are defined unobservable (hence not measurable with respect to $\mathcal{F}_n$) random variables $\{\theta_r\}_{r=1}^m$ which have the geometric distributions:

$$\mathbf{P}(\theta_r = j) = p_r^{j-1} q_r,\ q_r = 1 - p_r \in (0,1),\ j = 1, 2, \ldots. \tag{3.1}$$

The sensor r follows the process which is based on switching between two, time homogeneous and independent, Markov processes $\{X_{rn}^i\}_{n\in\mathbb{N}}$, $i = 0, 1$, $r \in \mathfrak{N}$ with the state space $(\mathbb{E}, \mathcal{B})$, both independent of $\{\theta_r\}_{r=1}^m$. Moreover, it is assumed that the processes $\{X_{rn}^i\}_{n\in\mathbb{N}}$ have transition densities with respect to the σ-finite measure μ, i.e., for any $B \in \mathcal{B}$ we have

$$\mathbf{P}^i_x(X^i_{r1} \in B) = \mathbf{P}(X^i_{r1} \in B | X^i_{r0} = x) = \int_B f^{ri}_x(y)\mu(dy). \tag{3.2}$$

The random processes $\{X_{rn}\}$, $\{X^0_{rn}\}$, $\{X^1_{rn}\}$ and the random variables θ_r are connected via the rule: conditionally on $\theta_r = k$

$$X_{rn} = \begin{cases} X^0_{rn}, & \text{if } k > n, \\ X^1_{r\,n+1-k}, & \text{if } k \leq n, \end{cases}$$

where $\{X^1_{rn}\}$ is started from $X^0_{r\,k-1}$ (but is otherwise independent of $X^0_{r\cdot}$).

For any fixed $d_r \in \{0, 1, 2, \ldots\}$ we are looking for the stopping time $\tau^*_r \in \mathcal{T}$ such that

$$\mathbf{P}_x(|\theta_r - \tau^*_r| \leq d_r) = \sup_{\tau \in \mathfrak{S}^X} \mathbf{P}_x(|\theta_r - \tau| \leq d_r) \tag{3.3}$$

where $\mathfrak{S}^X$ denotes the set of all stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. The parameters d_r determines the precision level of detection and it can be different for too early and too late detection. These payoff functions measure the chance of detection of intruder.

3.2. **Construction of the optimal detection strategy.** In [31] the construction of τ^* by transformation of the problem to the optimal stopping problem for the Markov process $\overrightarrow{\xi}$ has been made, such that $\overrightarrow{\xi}_{rn} = (\underline{\overrightarrow{X}}_{r\,n-1-d_r,n}, \Pi_n)$, where $\underline{\overrightarrow{X}}_{r\,n-1-d_r,n} = (\overrightarrow{X}_{r\,n-1-d_r}, \ldots, \overrightarrow{X}_{r\,n})$ and Π_{rn} is the posterior process:

$$\begin{aligned} \Pi_{r0} &= 0, \\ \Pi_{rn} &= \mathbf{P}_x(\theta_r \leq n \mid \mathcal{F}_n), \ n = 1, 2, \ldots \end{aligned}$$

which is designed as information about the distribution of the disorder instant θ_r. In this equivalent the problem of the payoff function for sensor r is $h_r(\overrightarrow{x}_{r\,d_r+2}, \alpha)$.

4. The aggregated decision via the cooperative game

There are various methods combining the decisions of several classifiers or sensors. Each ensemble member contributes to some degree to the decision at any point of the sequentially delivered states. The fusion algorithm takes into account all the decision outputs from each ensemble member and comes up with an ensemble decision. When classifier outputs are binary, the fusion algorithms include the majority voting [15], [16], naïve Bayes combination [3], behavior knowledge space [11], probability approximation [13] and singular value decomposition [18].

The majority vote is the simplest. The extension of this method is a simple game.

4.1. **A simple game.** Let us assume that there are many nodes absorbing information and make decision if the disorder has appeared or not. The final decision is made in the fusion center which aggregates information from all sensors. The nature of the system and their role is to detect intrusion in the system as soon as possible but without false alarm.

The voting decision is made according to the rules of *a simple game.* Let us recall that a coalition is a subset of the players. Let $\mathcal{C} = \{C : C \subset \mathfrak{N}\}$ denote the class of all coalitions.

Definition 4.1. (see [24], [6]) *A simple game* is coalition game having the characteristic function, $\phi(\cdot) : \mathcal{C} \to \{0,1\}$.

Let us denote $\mathcal{W} = \{C \subset \mathfrak{N} : \phi(C) = 1\}$ and $\mathcal{L} = \{C \subset \mathfrak{N} : \phi(C) = 0\}$. The coalitions in $\mathcal{W}$ are called the winning coalitions, and those from $\mathcal{L}$ are called the losing coalitions.

Assumptions 4.2. *By assumption the characteristic function satisfies the properties:*

(1) $\mathfrak{N} \in \mathcal{W}$;
(2) $\emptyset \in \mathcal{L}$;
(3) *(the monotonicity):* $T \subset S \in \mathcal{L}$ *implies* $T \in \mathcal{L}$.

4.2. **The aggregated decision rule.** When the simple game is defined and the players can vote presence or absence, $x_i = 1$ or $x_i = 0$, $i \in \mathfrak{N}$, of the intruder then the aggregated decision is given by the logical function

$$\pi(x_1, x_2, \ldots, x_p) = \sum_{C \in \mathcal{W}} \prod_{i \in C} x_i \prod_{i \notin C} (1 - x_i). \tag{4.3}$$

For the logical function π we have (cf [46])

$$\pi(x^1, \ldots, x^p) = x^i \cdot \pi(x^1, \ldots, \overset{i}{\breve{1}}, \ldots, x^p) + \overline{x}^i \cdot \pi(x^1, \ldots, \overset{i}{\breve{0}}, \ldots, x^p).$$

5. A non-cooperative stopping game

Following the results of the author and Yasuda [40] the multilateral stopping of a Markov chain problem can be described in the terms of the notation used in the non-cooperative game theory (see [21], [4], [19], [24]). Let $(\overrightarrow{X}_n, \mathfrak{F}_n, \mathbf{P}_x)$, $n = 0, 1, 2, \ldots, N$, be a homogeneous Markov chain with state space $(\mathbb{E}, \mathcal{B})$. The horizon can be finite or infinite. The players are able to observe the Markov chain sequentially. Each player has their utility function $f_i : \mathbb{E} \to \Re$, $i = 1, 2, \ldots, p$, such that $\mathbf{E}_x|f_i(\overrightarrow{X}_1)| < \infty$. If process is not stopped at moment n, then each player, based on $\mathfrak{F}_n$, can declare independently their willingness to stop the observation of the process.

Definition 5.1. (see [46]) An individual stopping strategy of the player i (ISS) is the sequence of random variables $\{\sigma_n^i\}_{n=1}^N$, where $\sigma_n^i : \Omega \to \{0,1\}$, such that σ_n^i is $\mathfrak{F}_n$-measurable.

The interpretation of the strategy is following. If $\sigma_n^i = 1$ then player i declares that they would like to stop the process and accept the realization of X_n. Denote $\sigma^i = (\sigma_1^i, \sigma_2^i, \dots, \sigma_N^i)$ and let $\mathfrak{S}^i$ be the set of ISSs of player i, $i = 1, 2, \dots, p$. Define

$$\mathfrak{S} = \mathfrak{S}^1 \times \mathfrak{S}^2 \times \dots \times \mathfrak{S}^p.$$

The element $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^p)^T \in \mathfrak{S}$ will be called the stopping strategy (SS). The stopping strategy $\sigma \in \mathfrak{S}$ is a random matrix. The rows of the matrix are the ISSs. The columns are the decisions of the players at successive moments. The factual stopping of the observation process, and the players realization of the payoffs is defined by the stopping strategy exploiting p-variate logical function. Let $\pi : \{0,1\}^p \to \{0,1\}$. In this stopping game model the stopping strategy is the list of declarations of the individual players. The aggregate function π converts the declarations to an effective stopping time.

Definition 5.2. A stopping time $\mathfrak{t}_\pi(\sigma)$ generated by the SS $\sigma \in \mathfrak{S}$ and the aggregate function π is defined by

$$\mathfrak{t}_\pi(\sigma) = \inf\{1 \le n \le N : \pi(\sigma_n^1, \sigma_n^2, \dots, \sigma_n^p) = 1\}$$

($\inf(\emptyset) = \infty$). Since π is fixed during the analysis we skip index π and write $\mathfrak{t}(\sigma) = \mathfrak{t}_\pi(\sigma)$.

We have $\{\omega \in \Omega : \mathfrak{t}_\pi(\sigma) = n\} = \bigcap_{k=1}^{n-1}\{\omega \in \Omega : \pi(\sigma_k^1, \sigma_k^2, \dots, \sigma_k^p) = 0\} \cap \{\omega \in \Omega : \pi(\sigma_n^1, \sigma_n^2, \dots, \sigma_n^p) = 1\} \in \mathfrak{F}_n$, then the random variable $\mathfrak{t}_\pi(\sigma)$ is stopping time with respect to $\{\mathfrak{F}_n\}_{n=1}^N$. For any stopping time $\mathfrak{t}_\pi(\sigma)$ and $i \in \{1, 2, \dots, p\}$, let

$$f_i(X_{\mathfrak{t}_\pi(\sigma)}) = \begin{cases} f_i(X_n) & \text{if } \mathfrak{t}_\pi(\sigma) = n, \\ \limsup_{n\to\infty} f_i(X_n) & \text{if } \mathfrak{t}_\pi(\sigma) = \infty \end{cases}$$

(cf [33], [40]). If players use SS $\sigma \in \mathfrak{S}$ and the individual preferences are converted to the effective stopping time by the aggregate rule π, then player i gets $f_i(X_{\mathfrak{t}_\pi(\sigma)})$.

Let ${}^*\sigma = ({}^*\sigma^1, {}^*\sigma^2, \dots, {}^*\sigma^p)^T$ be fixed SS. Denote

$${}^*\sigma(i) = ({}^*\sigma^1, \dots, {}^*\sigma^{i-1}, \sigma^i, {}^*\sigma^{i+1}, \dots, {}^*\sigma^p)^T.$$

Definition 5.3. (cf. [40]) Let the aggregate rule π be fixed. The strategy ${}^*\sigma = ({}^*\sigma^1, {}^*\sigma^2, \dots, {}^*\sigma^p)^T \in \mathfrak{S}$ is an equilibrium strategy with respect to π if for each $i \in \{1, 2, \dots, p\}$ and any $\sigma^i \in \mathfrak{S}^i$ we have

$$\mathbf{E}_x f_i(\overrightarrow{X}_{\mathfrak{t}_\pi({}^*\sigma)}) \ge \mathbf{E}_x f_i(\overrightarrow{X}_{\mathfrak{t}_\pi({}^*\sigma(i))}). \tag{5.4}$$

The set of SS $\mathfrak{S}$, the vector of the utility functions $f = (f_1, f_2, \dots, f_p)$ and the monotone rule π define the non-cooperative game $\mathcal{G} = (\mathfrak{S}, f, \pi)$. The construction of the equilibrium strategy $^*\sigma \in \mathfrak{S}$ in $\mathcal{G}$ is provided in [40]. For completeness this construction will be recalled here. Let us define an individual stopping set on the state space. This set describes the ISS of the player. With each ISS of player i the sequence of stopping events $D_n^i = \{\omega : \sigma_n^i = 1\}$ combines. For each aggregate rule π there exists the corresponding set value function $\Pi : \mathfrak{F} \to \mathfrak{F}$ such that $\pi(\sigma_n^1, \sigma_n^2, \dots, \sigma_n^p) = \pi\{\mathbb{I}_{D_n^1}, \mathbb{I}_{D_n^2}, \dots, \mathbb{I}_{D_n^p}\} = \mathbb{I}_{\Pi(D_n^1, D_n^2, \dots, D_n^p)}$. For solution of the considered game the important class of ISS and the stopping events can be defined by subsets $C^i \in \mathcal{B}$ of the state space $\mathbb{E}$. A given set $C^i \in \mathcal{B}$ will be called the stopping set for player i at moment n if $D_n^i = \{\omega : X_n \in C^i\}$ is the stopping event.

For the logical function π we have

$$\pi(x^1, \dots, x^p) = x^i \cdot \pi(x^1, \dots, \overset{i}{\breve{1}}, \dots, x^p) + \overline{x}^i \cdot \pi(x^1, \dots, \overset{i}{\breve{0}}, \dots, x^p).$$

It implies that for $D^i \in \mathfrak{F}$

$$\begin{aligned} \Pi(D^1, \dots, D^p) = \ & \{D^i \cap \Pi(D^1, \dots, \overset{i}{\breve{\Omega}}, \dots, D^p)\} \\ & \cup \{\overline{D}^i \cap \Pi(D^1, \dots, \overset{i}{\breve{\emptyset}}, \dots, D^p)\}. \end{aligned} \tag{5.5}$$

Let f_i, g_i be the real valued, integrable (i.e. $\mathbf{E}_x|f_i(X_1)| < \infty$) function defined on $\mathbb{E}$. For fixed D_n^j, $j = 1, 2, \dots, p$, $j \neq i$, and $C^i \in \mathcal{B}$ define

$$\psi(C^i) = \mathbf{E}_x \left[f_i(X_1)\mathbb{I}_{{}^iD_1(D_1^i)} + g_i(X_1)\mathbb{I}_{\overline{{}^iD_1(D_1^i)}} \right]$$

where ${}^iD_1(A) = \Pi(D_1^1, \dots, D_1^{i-1}, A, D_1^{i+1}, \dots, D_1^p)$ and $D_1^i = \{\omega : X_n \in C^i\}$. Let $a^+ = \max\{0, a\}$ and $a^- = \min\{0, -a\}$.

Lemma 5.6. *Let f_i, g_i, be integrable and let $C^j \in \mathcal{B}$, $j = 1, 2, \dots, p$, $j \neq i$, be fixed. Then the set $^*C^i = \{x \in \mathbb{E} : f_i(x) - g_i(x) \geq 0\} \in \mathcal{B}$ is such that*

$$\psi(^*C^i) = \sup_{C^i \in \mathcal{B}} \psi(C^i)$$

and

$$\begin{aligned} \psi(^*C^i) \ = \ & \mathbf{E}_x(f_i(X_1) - g_i(X_1))^+ \mathbb{I}_{{}^iD_1(\Omega)} \\ & - \mathbf{E}_x(f_i(X_1) - g_i(X_1))^- \mathbb{I}_{{}^iD_1(\Omega)} + \mathbf{E}_x g_i(X_1). \end{aligned} \tag{5.7}$$

Based on Lemma 5.6 we derive the recursive formulae defining the equilibrium point and the equilibrium payoff for the finite horizon game.

5.1. **The finite horizon game.** Let horizon N be finite. If the equilibrium strategy ${}^*\sigma$ exists, then we denote $v_{i,N}(x) = \mathbf{E}_x f_i(X_{t({}^*\sigma)})$ the equilibrium payoff of i-th player when $X_0 = x$. For the backward induction we introduce a useful notation. Let $\mathfrak{S}_n^i = \{\{\sigma_k^i\}, k = n, \ldots, N\}$ be the set of ISS for moments $n \leq k \leq N$ and $\mathfrak{S}_n = \mathfrak{S}_n^1 \times \mathfrak{S}_n^2 \times \ldots \times \mathfrak{S}_n^p$. The SS for moments not earlier than n is ${}^n\sigma = ({}^n\sigma^1, {}^n\sigma^2, \ldots, {}^n\sigma^p) \in \mathfrak{S}_n$, where ${}^n\sigma^i = (\sigma_n^i, \sigma_{n+1}^i, \ldots, \sigma_N^i)$. Denote

$$t_n = t_n(\sigma) = t({}^n\sigma) = \inf\{n \leq k \leq N : \pi(\sigma_k^1, \sigma_k^2, \ldots, \sigma_k^p) = 1\}$$

to be the stopping time not earlier than n.

Definition 5.8. The stopping strategy ${}^{n*}\sigma = ({}^{n*}\sigma^1, {}^{n*}\sigma^2, \ldots, {}^{n*}\sigma^p)$ is an equilibrium in $\mathfrak{S}_n$ if

$$\mathbf{E}_x f_i(X_{t_n({}^*\sigma)}) \geq \mathbf{E}_x f_i(X_{t_n({}^*\sigma(i))}) \quad \mathbf{P}_x - \text{a.e.}$$

for every $i \in \{1, 2, \ldots, p\}$, where

$${}^{n*}\sigma(i) = ({}^{n*}\sigma^1, \ldots, {}^{n*}\sigma^{i-1}, {}^n\sigma^i, {}^{n*}\sigma^{i+1}, \ldots, {}^{n*}\sigma^p).$$

Denote

$$v_{i,N-n+1}(X_{n-1}) = \mathbf{E}_x[f_i(X_{t_n({}^*\sigma)})|\mathfrak{F}_{n-1}] = \mathbf{E}_{X_{n-1}} f_i(X_{t_n({}^*\sigma)}).$$

At moment $n = N$ the players have to declare to stop and $v_{i,0}(x) = f_i(x)$. Let us assume that the process is not stopped up to moment n, the players are using the equilibrium strategies ${}^*\sigma_k^i$, $i = 1, 2, \ldots, p$, at moments $k = n+1, \ldots, N$. Choose player i and assume that other players are using the equilibrium strategies ${}^*\sigma_n^j$, $j \neq i$, and player i is using strategy σ_n^i defined by stopping set C^i. Then the expected payoff $\varphi_{N-n}(X_{n-1}, C^i)$ of player i in the game starting at moment n, when the state of the Markov chain at moment $n-1$ is X_{n-1}, is equal to

$$\varphi_{N-n}(X_{n-1}, C^i) = \mathbf{E}_{X_{n-1}}\left[f_i(X_n)\mathbb{I}_{{}^{i*}D_n(D_n^i)} + v_{i,N-n}(X_n)\mathbb{I}_{\overline{{}^{i*}D_n(D_n^i)}}\right],$$

where ${}^{i*}D_n(A) = \Pi({}^*D_n^1, \ldots, {}^*D_n^{i-1}, A, {}^*D_n^{i+1}, \ldots, {}^*D_n^p)$.

By Lemma 5.6 the conditional expected gain $\varphi_{N-n}(X_{N-n}, C^i)$ attains the maximum on the stopping set ${}^*C_n^i = \{x \in \mathbb{E} : f_i(x) - v_{i,N-n}(x) \geq 0\}$ and

$$\begin{aligned} v_{i,N-n+1}(X_{n-1}) &= \mathbf{E}_x[(f_i(X_n) - v_{i,N-n}(X_n))^+\mathbb{I}_{{}^{i*}D_n(\Omega)}|\mathfrak{F}_{n-1}] \\ &\quad -\mathbf{E}_x[(f_i(X_n) - v_{i,N-n}(X_n))^-\mathbb{I}_{{}^{i*}D_n(\emptyset)}|\mathfrak{F}_{n-1}] \\ &\quad +\mathbf{E}_x[v_{i,N-n}(X_n)|\mathfrak{F}_{n-1}] \end{aligned} \tag{5.1}$$

$\mathbf{P}_x$–a.e.. It allows to formulate the following construction of the equilibrium strategy and the equilibrium value for the game $\mathcal{G}$.

Theorem 5.2. *In the game $\mathcal{G}$with finite horizon N we have the following solution.*

(i): *The equilibrium value $v_i(x)$, $i = 1, 2, \dots, p$, of the game $\mathcal{G}$ can be calculated recursively as follows:*

(1) $v_{i,0}(x) = f_i(x)$;

(2) *For* $n = 1, 2, \dots, N$ *we have* $\mathbf{P}_x$*–a.e.*

$$\begin{aligned} v_{i,n}(x) = \mathbf{E}_x&[(f_i(X_{N-n+1}) - v_{i,n-1}(X_{N-n+1}))^+ \mathbb{I}_{{}^{i*}D_{N-n+1}(\Omega)} | \mathfrak{F}_{N-n}] \\ -\mathbf{E}_x&[(f_i(X_{N-n+1}) - v_{i,n-1}(X_{N-n+1}))^- \mathbb{I}_{{}^{i*}D_{N-n+1}(\emptyset)} | \mathfrak{F}_{N-n}] \\ &+\mathbf{E}_x[v_{i,n-1}(X_{N-n+1}) | \mathfrak{F}_{N-n}], \end{aligned}$$

for $i = 1, 2, \dots, p$.

(ii): *The equilibrium strategy ${}^*\sigma \in \mathfrak{S}$ is defined by the SS of the players ${}^*\sigma_n^i$, where ${}^*\sigma_n^i = 1$ if $X_n \in {}^*C_n^i$, and ${}^*C_n^i = \{x \in \mathbb{E} : f_i(x) - v_{i,N-n}(x) \geq 0\}$, $n = 0, 1, \dots, N$.*

We have $v_i(x) = v_{i,N}(x)$, and $\mathbf{E}_x f_i(X_{t({}^*\sigma)}) = v_{i,N}(x)$, $i = 1, 2, \dots, p$.

6. Infinite horizon game

In this class of games the equilibrium strategy is presented in Definition 5.3 but in class of SS

$$\mathfrak{S}_f^* = \{\sigma \in \mathfrak{S}^* : \mathbf{E}_x f_i^-(X_{t(\sigma)}) < \infty \quad \text{for every} \quad x \in \mathbb{E},\ i = 1, 2, \dots, p\}.$$

Let ${}^*\sigma \in \mathfrak{S}_f^*$ be an equilibrium strategy. Denote

$$v_i(x) = \mathbf{E}_x f_i(X_{t({}^*\sigma)}).$$

Let us assume that ${}^{(n+1)*}\sigma \in \mathfrak{S}_{f,n+1}^*$ is constructed and it is an equilibrium strategy. If players $j = 1, 2, \dots, p$, $j \neq i$, apply at moment n the equilibrium strategies ${}^*\sigma_n^j$, player i the strategy σ_n^i defined by stopping set C^i and ${}^{(n+1)*}\sigma$ at moments $n+1, n+2, \dots$, then the expected payoff of the player i, when history of the process up to moment $n-1$ is known, is given by

$$\varphi_n(X_{n-1}, C^i) = \mathbf{E}_{X_{n-1}} \left[f_i(X_n) \mathbb{I}_{{}^{i*}D_n(D_n^i)} + v_i(X_n) \mathbb{I}_{\overline{{}^{i*}D_n(D_n^i)}} \right],$$

where ${}^{i*}D_n(A) = \Pi({}^*D_n^1, \dots, {}^*D_n^{i-1}, A, {}^*D_n^{i+1}, \dots, {}^*D_n^p)$, ${}^*D_n^j = \{\omega \in \Omega : {}^*\sigma_n^j = 1\}$, $j = 1, 2, \dots, p$, $j \neq i$, and $D_n^i = \{\omega \in \Omega : \sigma_n^i = 1\} = 1\} = \{\omega \in \Omega : X_n \in C^i\}$. By Lemma 5.6 the conditional expected gain $\varphi_n(X_{n-1}, C^i)$ attains the maximum on the stopping set ${}^*C_n^i = \{x \in \mathbb{E} : f_i(x) \geq v_i(x)\}$ and

$$\begin{aligned} \varphi_n(X_{n-1}, {}^*C^i) \quad = \quad & \mathbf{E}_x[(f_i(X_n) - v_i(X_n))^+ \mathbb{I}_{{}^{i*}D_n(\Omega)} | \mathfrak{F}_{n-1}] \\ & -\mathbf{E}_x[(f_i(X_n) - v_i(X_n))^- \mathbb{I}_{{}^{i*}D_n(\emptyset)} | \mathfrak{F}_{n-1}] \\ & +\mathbf{E}_x[v_i(X_n) | \mathfrak{F}_{n-1}]. \end{aligned}$$

Let us assume that there exists solution $(w_1(x), w_2(x), \ldots, w_p(x))$ of the equations

$$(6.1) \qquad \begin{aligned} w_i(x) &= \mathbf{E}_x(f_i(X_1) - w_i(X_1))^+ \mathbb{I}_{i*D_1(\Omega)} \\ &\quad - \mathbf{E}_x(f_i(X_1) - w_i(X_1))^- \mathbb{I}_{i*D_1(\emptyset)} + \mathbf{E}_x w_i(X_1), \end{aligned}$$

$i = 1, 2, \ldots, p$. Consider the stopping game with the following payoff function for $i = 1, 2, \ldots, p$.

$$\phi_{i,N}(x) = \begin{cases} f_i(x) & \text{if } n < N, \\ v_i(x) & \text{if } n \geq N. \end{cases}$$

Lemma 6.2. *Let* $^*\sigma \in \mathfrak{S}_f^*$ *be an equilibrium strategy in the infinite horizon game* $\mathcal{G}$. *For every* N *we have*

$$\mathbf{E}_x \phi_{i,N}(X_{t^*}) = v_i(x).$$

Let us assume that for $i = 1, 2, \ldots, p$ and every $x \in \mathbb{E}$ we have

$$(6.3) \qquad \mathbf{E}_x[\sup_{n \in \mathbb{N}} f_i^+(X_n)] < \infty.$$

Theorem 6.4. *Let* $(X_n, \mathfrak{F}_n, \mathbf{P}_x)_{n=0}^{\infty}$ *be a homogeneous Markov chain and the payoff functions of the players fulfill (6.3). If* $t^* = t(^*\sigma)$, $^*\sigma \in \mathfrak{S}_f^*$ *then* $\mathbf{E}_x f_i(X_{t^*}) = v_i(x)$.

Theorem 6.5. *Let the stopping strategy* $^*\sigma \in \mathfrak{S}_f^*$ *be defined by the stopping sets* $^*C_n^i = \{x \in \mathbb{E} : f_i(x) \geq v_i(x)\}$, $i = 1, 2, \ldots, p$, *then* $^*\sigma$ *is the equilibrium strategy in the infinite stopping game* $\mathcal{G}$.

6.1. **Determining the strategies of sensors.** Based on the model constructed in Sections 3–5 for the net of sensors with the fusion center determined by a simple game, one can determine the rational decisions of each nodes. The rationality of such a construction refers to the individual aspiration for the highest sensitivity to detect the disorder without false alarm. The Nash equilibrium fulfills requirement that nobody deviates from the equilibrium strategy because its probability of detection will be smaller. The role of the simple game is to define wining coalitions in such a way that the detection of intrusion to the guarded area is maximal and the probability of false alarm is minimal. The method of constructing the optimum winning coalitions family is not the subject of the research in this article. However, there are some natural methods of solving this problem.

The research here is focused on constructing the solution of the non-cooperative stopping game as to determine the detection strategy of the sensors. To this end, the game analyzed in Section 5 with the payoff function of the players defined by the individual disorder problem formulated in Section 3 should be derived.

The proposed model disregards correlation of the signals. It is also assumed that the fusion center has perfect information about signals and the information is available at each node. The further research should help to qualify these real needs of such models and to extend the model to more general cases. In some type of distribution of sensors, e.g. when the distribution of the pollution in the given direction is observed, the multiple disorder model should work better than the game approach. In this case the *a priori* distribution of disorder moment has the form of sequentially dependent random moments and the fusion decision can be formulated as the threshold one: stop when k^* disorder is detected. The method of a cooperative game was used in [8] to find the best coalition of sensors in the problem of the target localization. The approach which is proposed here shows possibility of modelling the detection problem by multiple agents at a general level.

6.2. **Final conclusion concerning the disorder detection system.** In a general case the consideration of the paper [39] leads to the algorithm of constructing the disorder detection system.

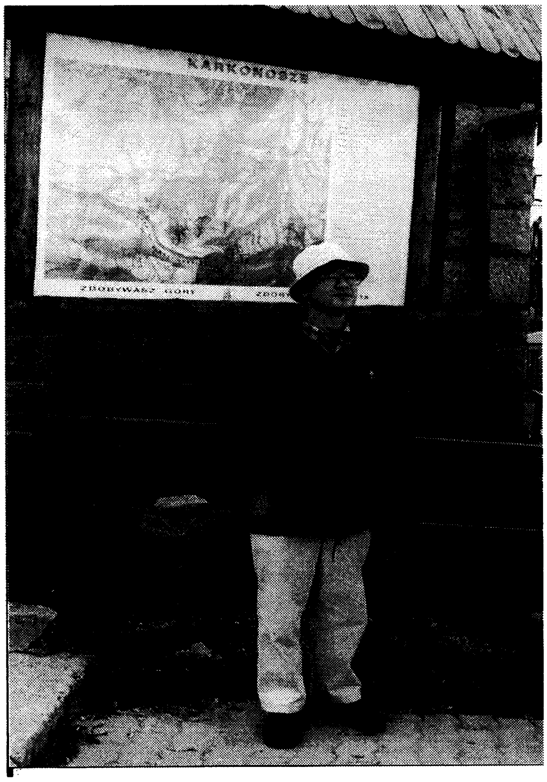

FIGURE 1. Sudety Mountains. International Conference on Mathematical Statistics STAT'2000. Szklarska Poręba, Poland, August, 2000

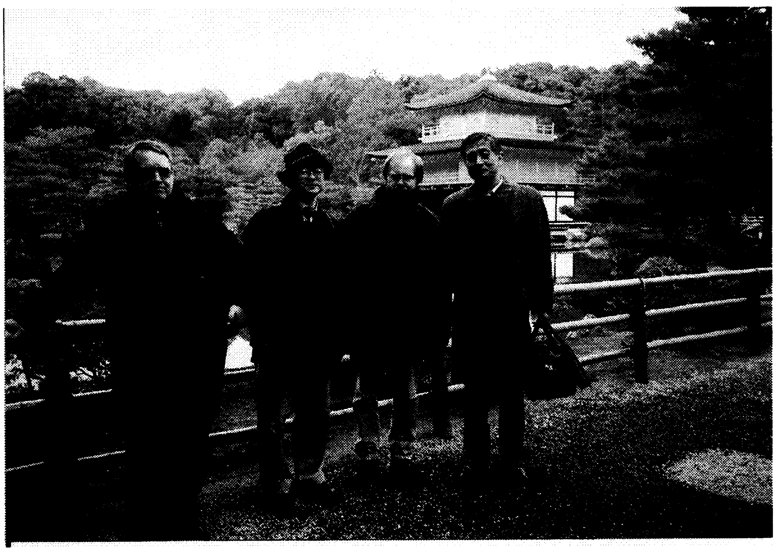

FIGURE 2. RIMS Conference 2002, Kyoto, Japan. From the left: the author of this note, professors Masami Yasuda, Vladimir V. Mazalov and Mitsushi Tamaki

6.2.1. *Algorithm.*

(1) Define a simple game on the sensors.
(2) Describe signal processes and *a priori* distribution of the disorder moments at all sensors. Establish the *a posteriori* processes: $\overrightarrow{\Pi}_n = (\Pi_{1n}, \ldots, \Pi_{mn})$, where $\Pi_{kn} = \mathbf{P}(\theta \leq n|\mathcal{F}_n)$.
(3) Solve the multivariate stopping game on the simple game to get the individual strategies of the sensors.

7. The contribution to the mathematical education, the scientific cooperation and the friendship

It was 1994 when I came to Japan for the first time based on Prof. Minoru Sakaguchi and Prof. Katsunori Ano invitation to take part in the International Conference on Stochastic Models and Optimal Stopping, Nanzan University, Nagoya. Since this event Professor Masami Yasuda is my guide in the mathematics and the Japanese culture. When the Internet connected the people we discussed the game model, which I call myself the Masami Yasuda game described in this note in the sections 1.2 and 2. Based on the discussion we have written the papers [40] and [47]. Our meeting and discussion were in Poland (see Figure 1), Japan (see Figure 2) and Russia (see Figure 3). Last year, during the one day workshop, the possible further research and academic cooperation was the topic of our discussion. Based on European Union integration processes the Polish

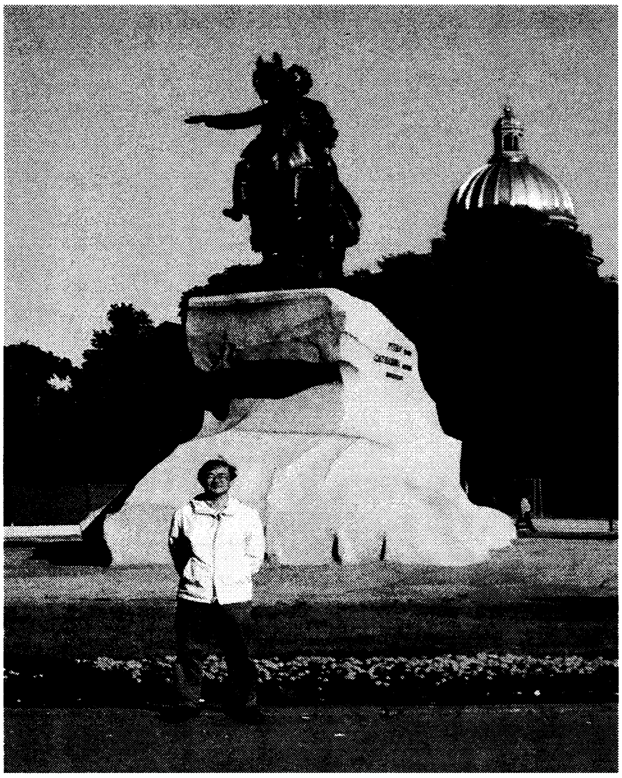

FIGURE 3. 13^{th} Symposium of ISDG 2008, St.Petersburg, Russia

educational system is under very intensive reconstruction process. The mathematical education of engineering faculties students is very fragile task. Professor Yasuda provides us his extensive academic experience by his contribution to our local conferences devoted to teaching mathematics for non-mathematical major students. Such organized events were in Ibaraki National College of Technology, Hitachinaka (2006) and Wrocław Univeristy of Technology (2008).

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Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław, Poland

E-mail address: `Krzysztof.Szajowski@pwr.wroc.pl`

Why is Uncertainty Theory Useful?

Baoding Liu
Uncertainty Theory Laboratory
Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China
liu@tsinghua.edu.cn *http://orsc.edu.cn/liu*

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1 What is uncertainty theory?

When the sample size is too small (even no-sample) to estimate a probability distribution, we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events, the belief degree may have much larger variance than the real frequency. Perhaps some people think that the belief degree is subjective probability. However, it is inappropriate because probability theory may lead to counterintuitive results in this case. In order to distinguish from randomness, this phenomenon was named "uncertainty".

How do we understand uncertainty? How do we model uncertainty? In order to answer those questions, an uncertainty theory was founded by Liu [6] in 2007 and refined by Liu [9] in 2010. Nowadays uncertainty theory has become a branch of mathematics for modeling human uncertainty.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to $[0,1]$ is called an uncertain measure if it satisfies the following axioms (Liu [6]):

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}. \tag{1}$$

The triplet $(\Gamma,\mathcal{L},\mathcal{M})$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [8], thus producing the fourth axiom of uncertainty theory:

Axiom 4. (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\} \tag{2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$, respectively.

An uncertain variable is defined as a measurable function ξ from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi\in B\}=\{\gamma\in\Gamma \mid \xi(\gamma)\in B\} \tag{3}$$

is an event. In order to describe an uncertain variable in practice, the concept of uncertainty distribution is defined by

$$\Phi(x)=\mathcal{M}\{\xi\le x\},\quad \forall x\in\Re. \tag{4}$$

Peng and Iwamura [14] proved that a function $\Phi : \Re \to [0,1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. The expected value of an uncertain variable ξ is defined by Liu [6] as an average value of the uncertain variable in the sense of uncertain measure, i.e.,

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}\mathrm{d}r \tag{5}$$

provided that at least one of the two integrals is finite. If ξ has an uncertainty distribution Φ, then the expected value may be calculated by

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \tag{6}$$

Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{7}$$

This is the operational law of uncertain variables. For example, let ξ_1 and ξ_2 be independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Since $x_1 + x_2$ is a strictly increasing function with respect to (x_1, x_2), the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha). \tag{8}$$

In addition, since $x_1 - x_2$ is strictly increasing with respect to x_1 and strictly decreasing with respect to x_2, the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha). \tag{9}$$

Furthermore, Liu and Ha [12] proved that the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{10}$$

For exploring the recent developments of uncertainty theory, the readers may consult my book *Uncertainty Theory* at http://orsc.edu.cn/liu/ut.pdf.

2 What is uncertainty?

Uncertainty theory is a branch of mathematics for modeling human uncertainty. Perhaps some readers may complain that I never clarify what uncertainty is. In fact, I really have no idea how to use natural language to define the concept of uncertainty clearly, and I think all existing definitions by natural language are specious because they are just like riddles. A very personal and ultra viewpoint is that the words like randomness, fuzziness, roughness, vagueness, greyness, and uncertainty are nothing but ambiguity of human language! Perhaps those concepts cannot be defined directly by natural language.

However, fortunately, some "mathematical scales" have been invented to measure the truth values of propositions, for example, probability measure, capacity, fuzzy measure, possibility measure as well as uncertain measure. All of those measures have been defined clearly and precisely by axiomatic methods.

Let us go back to the first question "what is uncertainty". Perhaps we can answer it this way. If it happens that some phenomenon can be quantified by uncertain measure, then we call the phenomenon uncertainty.

How do we verify that a phenomenon can be quantified by uncertain measure? In order to answer it, let us consider the question "how many students are there in Uncertainty Theory Laboratory". Assume the "number of students" is not exactly known but between 7 and 9. In this case, we may derive 8 events (i.e., a σ-algebra) from the concept of "number of students". The 8 events are listed as follows,

$$\emptyset,\ \{7\},\ \{8\},\ \{9\},\ \{7,8\},\ \{7,9\},\ \{8,9\},\ \{7,8,9\}.$$

In order to indicate the belief degree that each event will occur, we must assign to each event a number between 0 and 1, for example,

$$\mathcal{M}\{7\} = 0.6, \quad \mathcal{M}\{8\} = 0.3, \quad \mathcal{M}\{9\} = 0.2,$$
$$\mathcal{M}\{7,8\} = 0.8, \quad \mathcal{M}\{7,9\} = 0.7, \quad \mathcal{M}\{8,9\} = 0.4,$$
$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{7,8,9\} = 1.$$

This assignment implies that a set function $\mathcal{M}$ is defined from those 8 events to $[0,1]$. If it happens that such a set function satisfies the axioms of uncertainty theory (i.e., it is an uncertain measure), then the "number of students" is an uncertain variable.

Perhaps the above approach is the unique scientific way to judge if a phenomenon is uncertainty. Liu [9] suggested that *uncertainty is anything that satisfies the axioms of uncertainty theory.* In other words, uncertainty is anything that can be quantified by the uncertain measure.

Please note that the word "uncertainty" has been widely used or abused. In a wide sense, Knight [4] and Keynes [3] used uncertainty to represent any non-probabilistic phenomena. This type of uncertainty is also known as Knightian uncertainty, Keynesian uncertainty, or true uncertainty. Unfortunately, it seems impossible for us to develop a unified mathematical theory to deal with such a broad class of uncertainty because the concept of non-probability represents too many things. In a narrow sense, Liu [9] defined uncertainty as anything that satisfies the axioms of uncertainty theory. It is emphasized that uncertainty in the narrow sense is a scientific terminology, but uncertainty in the wide sense is not.

Some people think that *uncertainty and probability are synonymous.* This is a wrong viewpoint either in the wide sense or in the narrow sense. Uncertainty and probability are undoubtedly two different concepts. Otherwise, the terminology "uncertainty" becomes superfluous and we should use "probability" only.

Some people believe that *everything is probability or subjective probability.* When the sample size is large enough, the estimated probability may be close enough to the real one. Meanwhile, perhaps the viewpoint is somewhat true. However, we are frequently lack of observed data, and then the estimated probability may be far from the real one. Liu [11] asserted that probability theory may lead to counterintuitive results in this case. More extensively, Hicks [2] concluded that "we should always ask ourselves, before we apply [stochastic methods], whether they are appropriate to the problem at hand. Very often they are not."

Some people affirm that *probability theory is the only legitimate approach.* Perhaps this misconception is rooted in Cox's theorem [1] that any measure of belief is "isomorphic" to a probability measure. However, uncertain measure is considered coherent but not isomorphic to any probability measure. What goes wrong with Cox's theorem? Personally I think that Cox's theorem presumes the truth value of conjunction of two propositions is a twice differentiable function of the truth values of individual propositions, and then excludes uncertain measure from its start. In fact, there does not exist any evidence that the truth value of conjunction is completely determined by the truth values of individual propositions, let alone a twice differentiable function.

3 In what situations does uncertainty arise?

Frequency is the percentage of all the occurrences of an event in the experiment. An event's frequency is a factual property, and does not change with our state of knowledge. In other words, the frequency in the long run exists and is relatively invariant, no matter if it is observed by us.

A fundamental premise of applying probability theory is that the estimated probability is close enough to the real frequency, no matter whether the probability is interpreted as subjective or objective. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable.

However, very often we are lack of observed data about the unknown state of nature, not only for economic reasons, but also for technical difficulties. How do we deal with this case? It seems that we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events (Tversky and Kahneman [15]), the belief degree may have much larger variance than the real frequency, and we should deal with it by uncertainty theory.

Could we deal with the belief degree by probability theory when the belief degree deviates from the frequency? Some people do think so and call it subjective probability. However, it is inappropriate because probability theory may lead to counterintuitive results in this case.

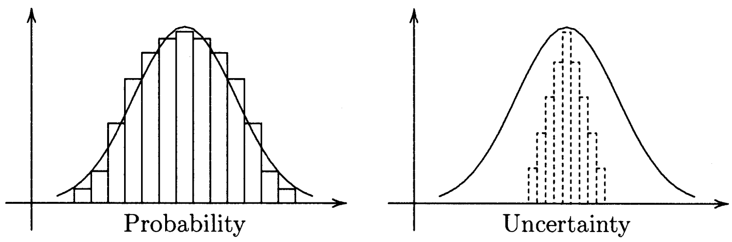

Figure 1: When the estimated probability (curve) is close enough to the real frequency (solid histogram), we should use probability theory. When the belief degree (curve) has much larger variance than the real frequency (dashed histogram), we should deal with it by uncertainty theory.

Consider a counterexample presented by Liu [11]. Assume the weight of a truck is 90 tons and the strengths of 50 bridges are iid normal random variables $\mathcal{N}(100, 1)$ in tons (I am afraid this fact cannot be verified without the help of God). For simplicity, it is admitted that there is only one truck on the bridge at every time, and a bridge collapses whenever its real strength is less than the weight of truck. Now let us have the truck cross over the 50 bridges one by one. It is easy to verify that

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} \approx 1. \tag{11}$$

That is to say, the truck may cross over the 50 bridges successfully.

However, when there do not exist any observed data for the strength of each bridge at the moment, we have to invite some domain experts to evaluate the belief degree about it. As we stated before, usually the belief degree has much larger variance than the real strength of bridges. Assume the belief degree looks like a normal probability distribution $\mathcal{N}(100, 100)$. Let us imagine what will happen if the belief degree is treated as probability. At first, we have no choice but to regard the strengths of the 50 bridges as iid normal random variables with expected value 100 and variance 100 in tons. If we have the truck cross over the 50 bridges one by one, then we immediately have

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} \approx 0. \tag{12}$$

Thus it is almost impossible that the truck crosses over the 50 bridges successfully. Unfortunately, the results (11) and (12) are at opposite poles. This conclusion seems unacceptable and then the belief degree cannot be treated as probability.

Within the research area of information science, Liu [11] suggested a basic principle that *a possible proposition cannot be judged impossible.* In other words, if a proposition is possibly true, then its truth value should not be zero. Equivalently, if a proposition is possibly false, then its truth value should not be unity. Thus probability theory is not appropriate to human uncertainty on the basis of such a principle.

In summary, Liu [11] wrote that *"when the sample size is too small (even no-sample) to estimate a probability distribution, we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events, the belief degree may have much larger variance than the real frequency and then probability theory is no longer valid. In this situation, we should deal with it by uncertainty theory."* When the sample size becomes large, the uncertainty disappears. Meanwhile, the problem at hand becomes probabilistic and we should use probability theory instead of uncertainty theory.

4 What is the difference between uncertain variable and uncertain set?

Uncertain variable (Liu [6]) and uncertain set (Liu [10]) are two basic tools in uncertainty theory. What is the difference between them? As their names suggest, both of them belong to the same broad category of uncertain concepts. However, they are differentiated by their mathematical definitions: an uncertain set is a set-valued function while an uncertain variable is a real-valued function. In other words, the former refers to a collection of values, while the latter to one value.

Essentially, the difference between uncertain variable and uncertain set focuses on the property of *exclusivity*. In fact, the same word can be either uncertain variable or uncertain set. They will be distinguished from the context. If the concept has exclusivity, then it is an uncertain variable. Otherwise, it is an uncertain set. A few examples will illustrate the difference between the two concepts.

Example 1: Consider the statement "Tom came into the classroom at approximately three o'clock". Is "approximately three o'clock" an uncertain variable or an uncertain set? If we are interested in when Tom came into the classroom, then the phrase "approximately three o'clock" is an uncertain variable rather than an uncertain set because it is an exclusive concept (Tom's arrival time cannot be more than one value). For example, if Tom came into the classroom at 2:59, then it is impossible that Tom came into the classroom at 3:01. In other words, "Tom came into the classroom at 2:59" does exclude the possibility that "Tom came into the classroom at 3:01". By contrast, if we are interested in what time can be considered "approximately three o'clock", then "approximately three o'clock" is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 2:59 and 3:01 can be considered "approximately three o'clock". In other words, "2:59 is approximately three o'clock" does not exclude the possibility that "3:01 is approximately three o'clock".

Example 2: Consider the statement "John is a young man". Is "young" an uncertain variable or an uncertain set? If we are interested in John's real age, then "young" is an uncertain variable rather than an uncertain set because it is an exclusive concept (John's age cannot be more than one value). For example, if John is 20 years old, then it is impossible that John is 25 years old. In other words, "John is 20 years old" does exclude the possibility that "John is 25 years old". By contrast, if we are interested in what ages can be regarded "young", then "young" is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 20-year-old and 25-year-old men can be considered "young". In other words, "a 20-year-old man is young" does not exclude the possibility that "a 25-year-old man is young".

Example 3: Consider the statement "James is a tall man". Is "tall" an uncertain variable or an uncertain set? If we are interested in James' real height, then "tall" is an uncertain variable rather than an uncertain set because the concept now is exclusive (James' height cannot be more than one value). For example, if James is 180cm in height, then it is impossible that James is 185cm in height. In other words, "James is 180cm in height" does exclude the possibility that "James is 185cm in height". By contrast, if we are interested in what heights can be considered "tall", then "tall" is an uncertain set rather than an uncertain variable because the concept in this case has no exclusivity. For example, both 180cm and 185cm can be considered "tall". In other words, "a 180cm-tall man is tall" does not exclude the possibility that "a 185cm-tall man is tall".

Example 4: Consider the statement "today is a warm day". Is "warm" an uncertain variable or an uncertain set? If we are interested in today's real temperature, then "warm" is an uncertain variable rather than an uncertain set because the concept now is exclusive. For example, if today is 20°C, then it is impossible that today is 25°C. In other words, "today is 20°C" does exclude the possibility that "today is 25°C". By contrast, if we are interested in what temperatures can be regarded "warm", then "warm" is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 20°C and 25°C can be considered "warm". In other words, "20°C is a warm day" does not exclude the possibility that "25°C is a warm day".

Example 5: Consider the statement "most students are boys". Is "most" an uncertain variable or an uncertain set? If we are interested in what percentage of students are boys, then "most" is an uncertain variable rather than an uncertain set because the concept in this case is exclusive. For example, if 80% of students are boys, then it is impossible that 85% of students are boys. By contrast, if we are interested in what percentages can be considered "most", then "most" is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 80% and 85% can be considered "most".

5 Why is fuzzy variable not suitable for modeling uncertain quantities?

A fuzzy variable is a function from a possibility space to the set of real numbers (Nahmias [13]). Some people think that fuzzy variable is a suitable tool for modeling uncertain quantities. Is it really true? Unfortunately, the answer is negative.

Let us reconsider the counterexample presented by Liu [11]. If the strength of bridge, "about 100 tons", is regarded as a fuzzy concept, then we may assign it a membership function, say

$$\mu(x) = \begin{cases} (x-80)/20, & \text{if } 80 \le x \le 100 \\ (120-x)/20, & \text{if } 100 \le x \le 120 \end{cases} \tag{13}$$

that is just the triangular fuzzy variable $(80, 100, 120)$.

Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on the membership function μ and the definition of possibility measure

$$\mathrm{Pos}\{B\} = \sup_{x \in B} \mu(x), \tag{14}$$

the possibility theory will immediately conclude the following three propositions:

(a) the strength is "exactly 100 tons" with possibility measure 1,

(b) the strength is "not 100 tons" with possibility measure 1,

(c) "exactly 100 tons" and "not 100 tons" are equally likely.

However, it is doubtless that the belief degree of "exactly 100 tons" is almost zero. Nobody is so naive to expect that "exactly 100 tons" is the true strength of the bridge. On the other hand, "exactly 100 tons" and "not 100 tons" have the same belief degree in possibility measure. Thus we have to regard them "equally likely". It seems that no human being can accept this conclusion. This paradox shows that those imprecise quantities like "about 100 tons" cannot be quantified by possibility measure and then they are not fuzzy concepts.

6 Why is fuzzy set not suitable for modeling unsharp concepts?

A fuzzy set is defined by its membership function μ which assigns to each element x a real number $\mu(x)$ in the interval $[0, 1]$, where the value of $\mu(x)$ represents the grade of membership of x in the fuzzy set. This definition was given by Zadeh [16] in 1965. However, there are too many fuzzy sets that share the same membership functions. Perhaps this is the root of debate about fuzzy set theory. A more precise definition states that a fuzzy set is a function from a possibility space to a collection of sets.

Some people believe that fuzzy set is a suitable tool to model unsharp concepts. Unfortunately, it is not true. In order to convince the reader, let us examine the concept of "young". Without loss of generality, assume "young" has a trapezoidal membership function $(15, 20, 30, 40)$, i.e.,

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 30 \\ (40-x)/10, & \text{if } 30 \le x \le 40 \\ 0, & \text{if } x \ge 40. \end{cases}$$

It follows from the fuzzy set theory that "young" may take any values of α-cut of μ. Thus we immediately conclude two propositions:

(a) "young" includes [20yr, 30yr] with possibility measure 1,

(b) "young" is just [20yr, 30yr] with possibility measure 1.

The first conclusion sounds good. However, the second conclusion seems unacceptable because it is almost impossible that "young" does mean the ages just from 20 to 30. This fact says that "young" cannot be regarded as a fuzzy set.

Liu [11] claimed that "we should always ask ourselves, before we apply fuzzy set theory, whether it is appropriate to the problem at hand. Almost it is not."

7 What is the difference between uncertainty theory and fuzzy mathematics?

Uncertainty theory (Liu [6][9]) is a branch of mathematics for modeling human uncertainty, while fuzzy mathematics (Zadeh [16]) is a branch of mathematics for studying the behavior of fuzzy phenomena.

What is the difference between uncertainty theory and fuzzy mathematics? The essential difference is that fuzzy mathematics assumes

$$\text{Pos}\{A \cup B\} = \text{Pos}\{A\} \vee \text{Pos}\{B\} \tag{15}$$

for any events A and B no matter if they are independent or not, and uncertainty theory assumes

$$\mathcal{M}\{A \cup B\} = \mathcal{M}\{A\} \vee \mathcal{M}\{B\} \tag{16}$$

only for independent events A and B. However, a lot of surveys showed that the measure of the union of events is usually greater than the maximum when the events are not independent. This fact states that human brains do not behave fuzziness.

Both uncertainty theory and fuzzy mathematics attempt to model human belief degree, where the former uses the tool of uncertain measure and the latter uses the tool of possibility measure. Thus they are complete competitors.

8 What is the difference between uncertainty theory and probability theory?

Uncertainty theory (Liu [6][9]) is a branch of mathematics for modeling human uncertainty, while probability theory (Kolmogorov [5]) is a branch of mathematics for studying the behavior of random phenomena.

What is the difference between uncertainty theory and probability theory? The main difference is that the product uncertain measure is the minimum of uncertain measures of uncertain events, i.e.,

$$\mathcal{M}\{A \times B\} = \mathcal{M}\{A\} \wedge \mathcal{M}\{B\}, \tag{17}$$

and the product probability measure is the product of probability measures of random events, i.e.,

$$\Pr\{A \times B\} = \Pr\{A\} \times \Pr\{B\}. \tag{18}$$

This difference implies that uncertain variables and random variables obey different operational laws.

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with the phenomena whose outcomes cannot be exactly predicted in advance. Personally I think probability measure should be interpreted as frequency while uncertain measure should be interpreted as belief degree. Probability theory is not an appropriate tool to model belief degree, just like that uncertainty theory is not an appropriate tool to model frequency.

9 Conclusion

When the sample size is too small (even no-sample) to estimate a probability distribution, we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events, the belief degree may have much larger variance than the real frequency and then probability theory is no longer valid. In this situation, we should deal with it by uncertainty theory.

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安田先生へ

江連馨

私が大学三年生で、次年度の研究テーマを悩んでいる時でした。他のゼミのテーマ紹介では、非常に小難しい単語が並び、あまり魅力を感じなかったのですが、安田先生の文章には、「最も恐いジェットコースターとはどんなものか、などを研究します」と書かれており、「これだ！」と即決しました。最大最小問題というのは、実生活でも非常に役に立ちます。例えば、民間企業で働く際には、様々な条件下で利益を最大にするという考え方は、当たり前のようで実践するのは難しいものです。私は安田先生の元でその概念を基礎から徹底的に学べたため、日々働く中で、周囲よりも正解に辿り着くのが早いと実感する場面が多々あります。また、数学科というのは、定理の証明の繰り返しで、非常に単調な授業が多かったのですが、安田先生はそういう難しい話を、具体例を交え、学生に分かりやすく説明してくれました。授業に出席するのが、楽しかった記憶があります。また、社会人になった今でも、卒業研究、そして大学院で学んだ知識を、周囲に説明する機会が存在します。あの頃、安田先生の元で研究した内容というのは、実生活にも応用できるものが多く、スピーチや飲み会でのネタで話すと、皆が感心します。安田先生の元で学べて、本当に感謝しています。あと、安田先生といえば、忘れてはいけないのが焼肉です。学業の節目には、安田先生のご自宅近くの焼肉屋さんに、よく連れていっていただきました。ユッケを初めて食べたのもあの時ですし、焼肉の美味しい焼き方を学んだのも、あの時でした。以上のように、私は恵まれた環境で、公私ともに沢山のことを学ばせていただきました。安田先生は退任されますが、ご指導いただいた弟子達は社会に飛び出し、活躍しています。我々は安田ゼミ卒業生として恥ずかしくないよう、これからも社会に貢献していきます。あたたかく見守っていただければ、幸いです。しばらくごゆっくり、お過ごしください。そしてまたいつか、あの焼肉屋さんで美味しい食事ができることを夢見ています。

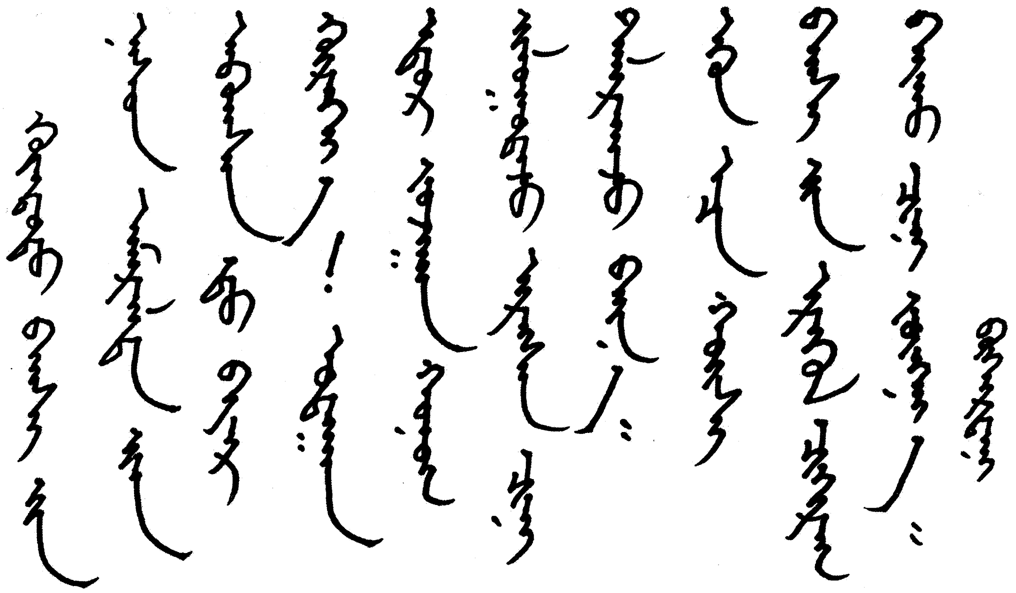
師の影みえず

影山正幸 (統計数理研究所、4 月より名古屋市立大学赴任予定)

この 10 年もっとも多くの時間を共有したのが安田正實教授です。しかし、初めて会ったときから全く距離が近くならいな、というのが正直なところです。物静かで穏やかな語り口で数学を教示していただき、Teacher と Professor の違いを語らずとも教えてくれた大切な恩師です。学位論文作成のための夜遅くまでのセミナー、初めての論文投稿での試行錯誤、北京、ストックホルムで過ごした夜、今となってはどれも大切な財産です。昨今の理論研究に対する風あたりの厳しい環境下で、本当の意味で次世代を見据えたその研究態度に、僕は北川敏男先生の DNA を感じずにはいられません。学生時代読んだ、北川先生と湯川秀樹先生との対談本「物理の世界 数理の世界」（中公新書）のなかで北川先生は「数学者は楽屋を見せてはいけない」といった趣旨のことを述べられていますが、今日の安田先生、蔵野正美先生に間違いなくその北川イズムは受け継がれているんだなと思うと同時に、僕もこの DNA を後世に残していかなくては、と身の引き締まる思いでいます。北川敏男先生のご子息であられる北川源四郎統計数理研究所所長の下で 3 年間勉強させていただいたことも何かの縁かな、とその幸運さに今は感謝の気持ちでいっぱいです。いつの日か師の影くらいは確認したいものです。安田先生、いつまでもお元気で。

春蚕蝋燭　良師益友

布和額尔敦、侯　平軍 (千葉大理学研究科博士課程)



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安田老师，長年的学校教育生活，您辛苦了，恭贺您完成了神圣而又伟大的工作，同时，也从心底里感谢您多年来对我的谆谆教诲，祝愿您身体健康，万事如意。

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