Ordering of convex fuzzy sets
— A brief survey and new results

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Abstract

Concerning with the topics of a fuzzy max order, a brief survey on ordering of fuzzy numbers is presented in this article, and we will consider an extension to that of fuzzy sets. An extension of the fuzzy max order as a pseudo order is investigated and defined on a class of fuzzy sets on $\mathbb{R}^n (n \geq 1)$. This order is developed by using a non-empty closed convex cone and characterized by the projection into its dual cone. Especially a structure of the lattice can be illustrated with the class of rectangle-type fuzzy sets.

Keywords : Fuzzy set, fuzzy max order, pseudo order, n-dimensional fuzzy set, rectangle-type fuzzy set.

1. Introduction

Fuzzy set theory has made applications in many fields of management science, operations research and statistics (cf.[20, 25, 31]), in which ordering or ranking fuzzy sets is a fundamental problem in fuzzy optimization or fuzzy decision making. Many methods of ordering fuzzy numbers have been proposed in the literature (see, for example, the survey paper, Bortolan and Degani[2]). Also, in multiple criteria decision making several procedures for ranking fuzzy multicriteria alternatives are investigated (for example, see [5, 29]). Each method has its own advantages and disadvantages, so that the ordering method should be chosen to be suitable for the particular problem. Among various ordering methods, a partial order on the set of fuzzy numbers, called the fuzzy max order, introduced by Ramík and Římánek[27] is very interesting in the concerns of pure mathematics because it is a natural extension of the order over real numbers and includes many theoretical and applicable potentials.

In this paper, concentrating on the fuzzy max order, we present a brief survey on ordering fuzzy sets in a real line $\mathbb{R}$, which motivates our new attempt of ordering high-dimensional fuzzy sets. The fuzzy max order of fuzzy numbers is extended to a pseudo order on a class of fuzzy sets defined on an n-dimensional Euclidean space $\mathbb{R}^n$.

The pseudo order for fuzzy sets is defined by a non-empty closed convex cone K in $\mathbb{R}^n$ and characterized by the projection into its dual cone K^+. Also, the structure of a lattice is discussed for the class of rectangle-type fuzzy sets. So, we can imagine the much wider application to the fuzzy optimization problem. Our idea of the motivation originates from a set-relation in $\mathbb{R}^n$ given by Kuroiwa[22], Kuroiwa, Tanaka and Ha[23], in which various types of set-relations in $\mathbb{R}^n$ are used in set-valued optimizations.

The outline for this paper is as follows: The next section contains some notations and basic concepts of fuzzy set theory referring to the text books (cf. [10, 26]); several methods of ordering fuzzy sets of a real line are surveyed concentrating on the fuzzy max order and its related topics in the third section; a pseudo order on the class of fuzzy sets on R^n is originally introduced as an extension of the fuzzy max order. Its characterization and the structure of a lattice is considered in the fourth section.

2. Notations and basic concepts

In this section we describe notations and basic concepts of fuzzy set theory (cf. [10, 20, 26, 30, 35]).

Let R be the set of all real numbers and R^n an n-dimensional Euclidean space. We write a fuzzy set on R^n by its membership function $\tilde{s} : R^n \to [0,1]$ (cf. [26, 35]). The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{s}$ on R^n is defined as

$$\tilde{s}_\alpha := \{x \in R^n \mid \tilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in R^n \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{s}$ is called convex if

$$\tilde{s}(\lambda x + (1-\lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad x, y \in R^n, \ \lambda \in [0,1],$$

where $a \wedge b := \min\{a, b\}$. Note that $\tilde{s}$ is convex if and only if the α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(R^n)$ be the set of all convex fuzzy sets whose membership functions $\tilde{s} : R^n \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x \in R^n} \tilde{s}(x) = 1$) and have a compact support. When the one-dimensional case $n = 1$, the fuzzy sets are called fuzzy numbers and $\mathcal{F}(R)$ denotes the set of all fuzzy numbers.

Let $\mathcal{C}(R^n)$ be the set of all compact convex subsets of R^n, and $\mathcal{C}_r(R^n)$ be the set of all rectangles in R^n. For $\tilde{s} \in \mathcal{F}(R^n)$, we have $\tilde{s}_\alpha \in \mathcal{C}(R^n)$ ($\alpha \in [0,1]$). We write a rectangle in $\mathcal{C}_r(R^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in R^n$ with $x_i \leq y_i$ ($i = 1, 2, \cdots, n$). For the case of $n = 1$, $\mathcal{C}(R) = \mathcal{C}_r(R)$ and it denotes the set of all bounded closed intervals. When $\tilde{s} \in \mathcal{F}(R^n)$ satisfies $\tilde{s}_\alpha \in \mathcal{C}_r(R^n)$ for all $\alpha \in [0,1]$, $\tilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(R^n)$ the set of all rectangle-type fuzzy sets on R^n. Obviously $\mathcal{F}_r(R) = \mathcal{F}(R)$.

Here we give the extension principle introduced by Zadeh which provides a general method for fuzzification of non-fuzzy mathematical concepts.

The extension principle (cf. [10]):

Let f be a map from R^n to R such that $y = f(x_1, \cdots, x_n)$. It allows us to induce a map from n fuzzy sets $\tilde{s}_i$ on R to a fuzzy set $\tilde{s}$ on R through f such that

$$\tilde{s}(y) := \sup_{\substack{x_1, \cdots, x_n \in R \\ y = f(x_1, \cdots, x_n)}} \{\min(\tilde{s}_1(x_1), \cdots, \tilde{s}_n(x_n))\}$$

where $\quad \tilde{s}(y) := 0$ if $f^{-1}(y) = \emptyset$.

Applying the extension principle, the addition and the scalar multiplication on $\mathbb{R}$ are extended to those on $\mathcal{F}(\mathbb{R})$ as follows:

For $\tilde{s}, \tilde{e} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$,

$$(2.1) \qquad (\tilde{s} + \tilde{e})(x) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ x_1 + x_2 = x}} \{\tilde{s}(x_1) \wedge \tilde{e}(x_2)\};$$

$$(2.2) \qquad (\lambda \tilde{s})(x) := \begin{cases} \tilde{s}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R});$$

where $I_{\{\cdot\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}$, the following holds immediately:

$$(2.3) \qquad (\tilde{s} + \tilde{e})_\alpha = \tilde{s}_\alpha + \tilde{e}_\alpha \quad \text{and} \quad (\lambda \tilde{s})_\alpha = \lambda \tilde{s}_\alpha \quad (\alpha \in [0, 1]).$$

Also, for $\tilde{s}, \tilde{e} \in \mathcal{F}(\mathbb{R})$, $\widetilde{\max}(\tilde{s}, \tilde{e})$ and $\widetilde{\min}(\tilde{s}, \tilde{e})$ are defined by

$$(2.4) \qquad \widetilde{\max}(\tilde{s}, \tilde{e})(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = \max(x_1, x_2)}} \{\tilde{s}(x_1) \wedge \tilde{e}(x_2)\}$$

and

$$(2.5) \qquad \widetilde{\min}(\tilde{s}, \tilde{e})(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = \min(x_1, x_2)}} \{\tilde{s}(x_1) \wedge \tilde{e}(x_2)\}$$

respectively. The images of $\widetilde{\max}(\tilde{s}, \tilde{e})$ and $\widetilde{\min}(\tilde{s}, \tilde{e})$ are illustrated as follows:

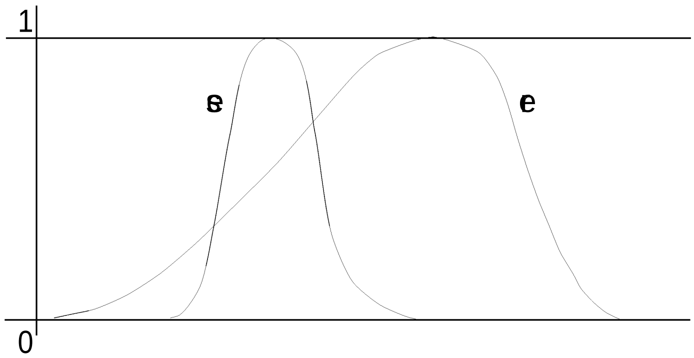

Fig.1: $\tilde{s}$ and $\tilde{e}$

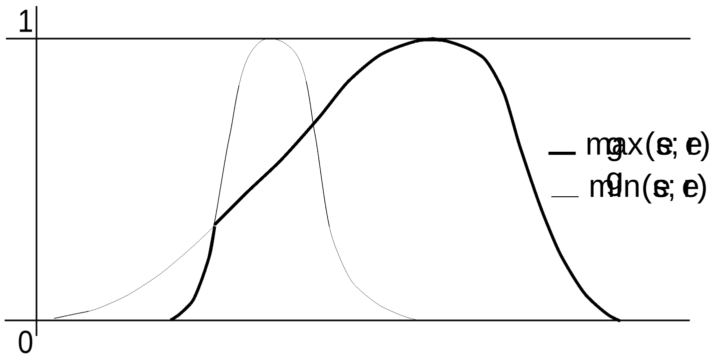

Fig.2: $\widetilde{\max}(\tilde{s}, \tilde{e})$ and $\widetilde{\min}(\tilde{s}, \tilde{e})$

We need a representative theorem (cf. [10, 26]) which is a basic tool for the fuzzy interval analysis.

The representative theorem:

(i) For any $\tilde{a} \in \mathcal{F}(\mathbf{R}^n)$, $\tilde{a}(x) = \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{\tilde{a}_\alpha}(x)\}, \quad x \in \mathbf{R}.$

(ii) Conversely, for a family of subsets $\{D_\alpha \in \mathcal{C}(\mathbf{R}^n) \mid 0 \le \alpha \le 1\}$ with $D_\alpha \subset D_{\alpha'}$ for $\alpha' \le \alpha$ and $\cap_{\alpha' < \alpha} D_{\alpha'} = D_\alpha$, we set $\tilde{a}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(x)\}; \; x \in \mathbf{R}$,

then $\tilde{a}$ belongs to $\mathcal{F}(\mathbf{R}^n)$ and satisfies $\tilde{a}_\alpha = D_\alpha; \; \alpha \in [0,1]$.

3. A brief survey on ordering of fuzzy numbers

In this section, we give a brief survey on method for ordering fuzzy numbers which is mainly concerning to the fuzzy max order.

3.1. Fuzzy max order

The following binary relation $\preceq$ has been formulated first by Ramík and Řimánek[27]. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then $\tilde{a} \preceq \tilde{b}$ if and only if $\sup \tilde{a}_\alpha \le \sup \tilde{b}_\alpha$ and $\inf \tilde{a}_\alpha \le \inf \tilde{b}_\alpha$ for each $\alpha \in [0,1]$, where $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are α-cuts of $\tilde{a}$ and $\tilde{b}$ respectively. Without any confusion we put $\tilde{a}_\alpha := [\inf \tilde{a}_\alpha, \sup \tilde{a}_\alpha]$ and $\tilde{b}_\alpha := [\inf \tilde{b}_\alpha, \sup \tilde{b}_\alpha]$. Obviously the binary relation $\preceq$ satisfies the axioms of a partial order relation on $\mathcal{F}(\mathbf{R})$ and is called the fuzzy max order.

Some properties of the relation $\preceq$ are investigated in Ramík and Řimánek[27].

Proposition 1. Let $\tilde{a}, \tilde{b}$ be fuzzy numbers.

(i) The inequality $\tilde{a} \preceq \tilde{b}$ if and only if there are $m, n, t^* \in \mathbf{R}$ with $m \le t^* \le n$, $\tilde{a}(m) = \tilde{b}(n) = 1$ and $\tilde{a}(t) \ge \tilde{b}(t)$ for any $t \le t^*$ and $\tilde{a}(t) \le \tilde{b}(t)$ for any $t > t^*$.

(ii) The following three conditions (a) to (c) are equivalent:
(a) $\tilde{a} \preceq \tilde{b}$, (b) $\widetilde{\max}\{\tilde{a}, \tilde{b}\} = \tilde{b}$, (c) $\widetilde{\min}\{\tilde{a}, \tilde{b}\} = \tilde{a}$,
where $\widetilde{\min}$ and $\widetilde{\max}$ are defined in (2.4) and (2.5).

For the fuzzy max order on fuzzy numbers, Congxin and Cong[6] proved that the bounded set of fuzzy numbers must have supremum and infimum. The basic proof is as follows. For any sequence of fuzzy numbers $\{\tilde{a}_n\}_{n=1}^{\infty}$, let $\underline{D}_\alpha := \lim_{\alpha' \uparrow \alpha} \sup_{n \ge 1} \inf \tilde{a}_{n\alpha'}$ and $\overline{D}_\alpha := \lim_{\alpha' \uparrow \alpha} \sup_{n \ge 1} \sup \tilde{a}_{n\alpha'}$, where $\tilde{a}_{n\alpha}$ is the α-cut of $\tilde{a}_n$. Then, the family of closed subsets $\{D_\alpha := [\underline{D}_\alpha, \overline{D}_\alpha] \mid \alpha \in [0,1]\}$ satisfies condition (ii) of the representation theorem, so that $\tilde{a}$ defined by

$$\tilde{a}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(x)\} \quad (x \in \mathbf{R})$$

belongs to $\mathcal{F}(\mathbf{R})$ and $\tilde{a} = \sup_{n \ge 1} \tilde{a}_n$.

Similarly, the infimum of $\{\tilde{a}_n\}_{n=1}^{\infty}$ is constructed. These results derived the interesting mathematical fact that the continuous fuzzy-valued function on a closed interval has a maximum and minimum. Also, the structure of lattice for fuzzy numbers is discussed in Zhang and Hirota[37].

To be suitable for computations and treatments, a class of fuzzy numbers, called an L-R-fuzzy number, is introduced in many text books.

Let $L, R : [0, \infty) \to [0, 1]$ be two non-increasing and not constant functions with $L(0) = R(0) = 1$ and $L(x_0) = R(x_0) = 0$ for some $x_0 > 0$. A fuzzy number $\tilde{a}$ is called an L-R-fuzzy number if there exist real numbers $m, n (m \le n), \alpha, \beta (\alpha, \beta > 0)$ such that

$$\tilde{a}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{for} \quad x \le m; \\ 1 & \text{for} \quad m \le x \le n; \\ R(\frac{x-n}{\beta}) & \text{for} \quad n \le x. \end{cases}$$

Given functions L, R with the properties in the above definition, the L-R-fuzzy number specified by m, n, α, β is denoted by an ordered tetradic $(m, n, \alpha, \beta)_{L\text{-}R}$, which includes the triangular and trapezoidal fuzzy numbers. Then, the fuzzy max order on the set of L-R-fuzzy numbers is characterized by inequalities of the elements. (cf. [27]).

In particular, the symmetric fuzzy number $\tilde{a} = (m, m, \alpha, \alpha)_{L\text{-}L}$ is called an L-fuzzy number and denoted by $(m, \alpha)_L$. Furukawa[14] extended the L-fuzzy number $(m, \alpha)_L$ with $\alpha > 0$ to the case of $\alpha \in R$ and proved that for $\alpha\beta \ge 0$ the fuzzy max order $(m, \alpha)_L \preccurlyeq (n, \beta)_L$ if and only if $x_0|\alpha - \beta| \le n - m$ where x_0 is the zero point of L. Moreover Furukawa[14] introduced the linear operations on the set of extended L-fuzzy numbers by

$$(m, \alpha)_L \oplus (n, \beta)_L = (m + n, \alpha + \beta)_L,$$

$$\lambda (m, \alpha)_L = (\lambda m, \lambda \alpha)_L \quad \text{for any scalar} \quad \lambda \in R.$$

The fuzzy max order is proved to be adapted to the above operations. Also Furukawa [15] introduced a parametric order relation on L-fuzzy numbers which is an extension of the fuzzy max order and its total order relation. The fuzzy optimization problems related to the fuzzy max order are dealed with many authors; e.g., Furukawa[15], Kurano et al.[21], Yoshida[34] and others.

3.2. Other ordering methods

Besides the fuzzy max order, a large collection of methods for ordering fuzzy sets of linear line has been developed in the literature. A simple method for ordering fuzzy numbers consists of the definition of ordering or ranking function. Let $f : \mathcal{F}(R) \to R$, where a natural ranking relation $\le$ on R is defined. Then, using this ranking function, we can define the order relation $\preccurlyeq$ on $\mathcal{F}(R)$ as follows:

$$\tilde{a} \preccurlyeq \tilde{b} \quad \text{implies} \quad f(\tilde{a}) \le f(\tilde{b}) \quad \text{for} \quad \tilde{a}, \tilde{b} \in \mathcal{F}(R).$$

If, for all $\tilde{a}, \tilde{b} \in \mathcal{F}(R)$, $f(\tilde{a} + \tilde{b}) = f(\tilde{a}) + f(\tilde{b})$ and $f(\lambda \tilde{a}) = \lambda f(\tilde{a})$ for $\lambda \ge 0$ are satisfied, it is called a linear ranking function. As a simple ranking function, we can construct a family of ranking procedures as follows:

$$f_p(\tilde{a}) := \int_0^1 (p_\alpha \inf \tilde{a}_\alpha + q_\alpha \sup \tilde{a}_\alpha)\, d\alpha; \quad \text{where} \quad \int_0^1 (p_\alpha + q_\alpha)\, d\alpha = 1.$$

Several variants of this family are discussed by the authors; see, e.g., Adamo[1], Campos, Bonzalez and Vila[4], Fortemps and Roubens[12] and Yager[33]. Gonzalez and Vila[17] defined the dominance relations by means of a ranking function evaluated into R^n. For a R^n-valued ranking function $f : \mathcal{F}(R) \to R^n$, the order relation $\preccurlyeq$ on $\mathcal{F}(R)$ is defined using an order relation $\preccurlyeq_n$ on R^n(defined in the next section) as follows:

$$\tilde{a} \preccurlyeq \tilde{b} \quad \text{implies} \quad f(\tilde{a}) \preccurlyeq_n f(\tilde{b}).$$

From flexibility of order relations on R^n, this seems to be useful in the suitable optimization problem of fuzzy decision making.

Another approach to ordering fuzzy numbers is discussed by using the fuzzy ordering (cf. [36]). From the point of view of possibility theory, Dubois and Prade[8] defined four fuzzy relations on $\mathcal{F}(R)$: $\mathrm{Pos}(\tilde{a} \preccurlyeq \tilde{b}), \mathrm{Pos}(\tilde{a} \prec \tilde{b}), \mathrm{Nes}(\tilde{a} \preccurlyeq \tilde{b})$ and $\mathrm{Nes}(\tilde{a} \prec \tilde{b})$.

4. On an extension to a pseudo order on $\mathcal{F}(R^n)$

In this section, we extend the fuzzy max order on $\mathcal{F}(R)$ to a pseudo order on $\mathcal{F}(R^n)$.

4.1. A pseudo order on $\mathcal{F}(R^n)$

We will review a vector ordering on R^n by a non-empty convex cone $K \subset R^n$. Using this K, we can define a pseudo order relation $\preccurlyeq_K$ on R by $x \preccurlyeq_K y$ if and only if $y - x \in K$. Let R^n_+ be the subset of entrywise non-negative elements in R^n. When $K = R^n_+$, the order $\preccurlyeq_K$ will be denoted by $\preccurlyeq_n$ and $x \preccurlyeq_n y$ means that $x_i \le y_i$ for all $i = 1, 2, \cdots, n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in R^n$.

First we introduce a binary relation on $\mathcal{C}(R^n)$, by which a pseudo order on $\mathcal{F}(R^n)$ is given. Henceforth we assume that the convex cone $K \subset R^n$ is given. We define a binary relation $\preccurlyeq_K$ on $\mathcal{C}(R^n)$ by abuse of notation. For $A, B \in \mathcal{C}(R^n)$, $A \preccurlyeq_K B$ means the following (C.a) and (C.b) (cf. [22, 23]):

(C.a) For any $x \in A$, there exists $y \in B$ such that $x \preccurlyeq_K y$.

(C.b) For any $y \in B$, there exists $x \in A$ such that $x \preccurlyeq_K y$.

Lemma 4.1. The binary relation $\preccurlyeq_K$ is a pseudo order on $\mathcal{C}(R^n)$.

Proof. It is trivial that $A \preccurlyeq_K A$ for $A \in \mathcal{C}(R^n)$. Let $A, B, C \in \mathcal{C}(R^n)$ such that $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$. We will check $A \preccurlyeq_K C$ by two cases (C.a) and (C.b). Case(C.a): Since $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$, for any $x \in A$ there exists $y \in B$ such that $x \preccurlyeq_K y$ and there exists $z \in C$ such that $y \preccurlyeq_K z$. Since $\preccurlyeq_K$ is a pseudo order on R^n, we have $x \preccurlyeq_K z$.

Therefore it holds that for any $x \in A$ there exists $z \in C$ such that $x \ 4_K \ z$. Case(C.b): Since $A \ 4_K \ B$ and $B \ 4_K \ C$, for any $z \in C$ there exists $y \in B$ such that $y \ 4_K \ z$ and there exists $x \in A$ such that $x \ 4_K \ y$. Since 4_K is a pseudo order on R^n, we have $x \ 4_K \ z$. Therefore it holds that for any $z \in C$ there exists $x \in A$ such that $x \ 4_K \ z$. From the above (C.a) and (C.b), we obtain $A \ 4_K \ C$. Thus the lemma holds. ¤

When $K = R^n_+$, the binary relation 4_K on $\mathcal{C}(R^n)$ will be written simply by 4_n and for $[x; y]; [x^0; y^0] \in \mathcal{C}_r(R^n)$, $[x; y] \ 4_n \ [x^0; y^0]$ means $x \ 4_n \ x^0$ and $y \ 4_n \ y^0$.

Next, we introduce a binary relation 4_K on $\mathcal{F}(R^n)$: Let $\mathbf{s}; \mathbf{e} \in \mathcal{F}(R^n)$. The relation $\mathbf{s} \ 4_K \ \mathbf{e}$ means the following (F.a) and (F.b):

(F.a) For any $x \in R^n$, there exists $y \in R^n$ such that $x \ 4_K \ y$ and $\mathbf{s}(x) \leq \mathbf{e}(y)$.

(F.b) For any $y \in R^n$, there exists $x \in R^n$ such that $x \ 4_K \ y$ and $\mathbf{s}(x) \geq \mathbf{e}(y)$.

Note that the notation 4_K denotes the binary relation on $R^n; \mathcal{C}(R^n); \mathcal{F}(R^n)$ with some abuse of notation.

Lemma 4.2. The binary relation 4_K is a pseudo order on $\mathcal{F}(R^n)$.

Proof. It is trivial that $\mathbf{s} \ 4_K \ \mathbf{s}$ for $\mathbf{s} \in \mathcal{F}(R^n)$. Let $\mathbf{s}; \mathbf{e}; \mathbf{p} \in \mathcal{F}(R^n)$ such that $\mathbf{s} \ 4_K \ \mathbf{e}$ and $\mathbf{e} \ 4_K \ \mathbf{p}$. We will check $\mathbf{s} \ 4_K \ \mathbf{p}$ by following two cases (F.a) and (F.b). Case(F.a): Since $\mathbf{s} \ 4_K \ \mathbf{e}$ and $\mathbf{e} \ 4_K \ \mathbf{p}$, for any $x \in R^n$ there exists $y \in R^n$ such that $x \ 4_K \ y$ and $\mathbf{s}(x) \leq \mathbf{e}(y)$, and there exists $z \in R^n$ such that $y \ 4_K \ z$ and $\mathbf{e}(y) \leq \mathbf{p}(z)$. Since 4_K is a pseudo-order on R^n, we have $x \ 4_K \ z$ and $\mathbf{s}(x) \leq \mathbf{p}(z)$. Therefore it holds that for any $x \in R^n$ there exists $z \in R^n$ such that $x \ 4_K \ z$ and $\mathbf{s}(x) \leq \mathbf{p}(z)$. Case(F.b): Since $\mathbf{s} \ 4_K \ \mathbf{e}$ and $\mathbf{e} \ 4_K \ \mathbf{p}$, for any $z \in R^n$ there exists $y \in R^n$ such that $y \ 4_K \ z$ and $\mathbf{e}(y) \leq \mathbf{p}(z)$, and there exists $x \in R^n$ such that $x \ 4_K \ y$ and $\mathbf{s}(x) \leq \mathbf{e}(y)$. Since 4_K is a pseudo-order on R^n, we have $x \ 4_K \ z$. Therefore it holds that for any $z \in R^n$ there exists $x \in R^n$ such that $x \ 4_K \ z$ and $\mathbf{s}(x) \leq \mathbf{p}(z)$. From the above (F.a) and (F.b), we obtain $\mathbf{s} \ 4_K \ \mathbf{p}$. Thus the lemma holds. ¤

The following lemma implies the correspondence between the pseudo order on $\mathcal{F}(R^n)$ for fuzzy sets and the pseudo order on $\mathcal{C}(R^n)$ for the ®-cuts.

Lemma 4.3. Let $\mathbf{s}; \mathbf{e} \in \mathcal{F}(R^n)$. $\mathbf{s} \ 4_K \ \mathbf{e}$ on $\mathcal{F}(R^n)$ if and only if $\mathbf{s}_® \ 4_K \ \mathbf{e}_®$ on $\mathcal{C}(R^n)$ for all $® \in (0; 1]$.

Proof. Let $\mathbf{s}; \mathbf{e} \in \mathcal{F}(R^n)$ and $® \in (0; 1]$. Suppose $\mathbf{s} \ 4_K \ \mathbf{e}$ on $\mathcal{F}(R^n)$. Then, two cases (a) and (b) are considered. Case(a): Let $x \in \mathbf{s}_®$. Since $\mathbf{s} \ 4_K \ \mathbf{e}$, there exists $y \in R^n$ such that $x \ 4_K \ y$ and $® \leq \mathbf{s}(x) \leq \mathbf{e}(y)$. Namely $y \in \mathbf{e}_®$. Case(b): Let $y \in \mathbf{e}_®$. Since $\mathbf{s} \ 4_K \ \mathbf{e}$, there exists $x \in R^n$ such that $x \ 4_K \ y$ and $\mathbf{s}(x) \geq \mathbf{e}(y) \geq ®$. Namely $x \in \mathbf{s}_®$. Therefore we get $\mathbf{s}_® \ 4_K \ \mathbf{e}_®$ on $\mathcal{C}(R^n)$ for all $® \in (0; 1]$ from the above (a) and (b).

On the other hand, suppose $\mathbf{s}_® \ 4_K \ \mathbf{e}_®$ on $\mathcal{C}(R^n)$ for all $® \in (0; 1]$. Then, two cases (a') and (b') are considered. Case(a'): Let $x \in R^n$. Put $® = \mathbf{s}(x)$. If $® = 0$, then $x \ 4_K \ x$ and $\mathbf{s}(x) = 0 \leq \mathbf{e}(x)$. While, if $® > 0$, then $x \in \mathbf{s}_®$. Since $\mathbf{s}_® \ 4_K \ \mathbf{e}_®$, there exists $y \in \mathbf{e}_®$ such that $x \preceq_K y$. And we have $\tilde{s}(x) = \alpha \leq \tilde{e}(y)$. Case(b'): Let $y \in R^n$. Put $\alpha = \tilde{e}(y)$. If $\alpha = 0$, then $x \preceq_K x$ and $\tilde{s}(x) \geq 0 = \tilde{e}(y)$. While, if $\alpha > 0$, then $y \in \tilde{e}_\alpha$. Since $\tilde{s}_\alpha \preceq_K \tilde{e}_\alpha$, there exists $x \in \tilde{s}_\alpha$ such that $x \preceq_K y$. And we have $\tilde{s}(x) \geq \alpha = \tilde{e}(y)$.

Therefore we get $\tilde{s} \preceq_K \tilde{e}$ on $\mathcal{F}(R^n)$ from the above Case (a') and (b'). Thus we obtain this lemma. $\square$

For the case of $K = R_+$, Lemma 4.3 says that the order relation $\preceq_1$ on $\mathcal{F}(R)$ (That is, $n = 1$) is the fuzzy max order mentioned in Section 3.

Define the dual cone of a cone K by

$$K^+ := \{a \in R^n \mid a \cdot x \geq 0 \text{ for all } x \in K\},$$

where $x \cdot y$ denotes the inner product on R^n for $x, y \in R^n$. For a subset $A \subset R^n$ and $a \in R^n$, we define

$$a \cdot A := \{a \cdot x \mid x \in A\} \ (\subset R). \tag{4.1}$$

The equation (4.1) means that $a \cdot A$ is the projection of A on the parallel line with the vector a if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(R)$ if $A \in \mathcal{C}(R^n)$ and $a \in R^n$.

Lemma 4.4. Let $A, B \in \mathcal{C}(R^n)$. $A \preceq_K B$ on $\mathcal{C}(R^n)$ if and only if $a \cdot A \preceq_1 a \cdot B$ on $\mathcal{C}(R)$ for all $a \in K^+$.

Proof. Suppose $A \preceq_K B$ on $\mathcal{C}(R^n)$. Consider the two cases (a) and (b). Case(a): For any $x \in A$, there exists $y \in B$ such that $x \preceq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. Case(b): For any $y \in B$, there exists $x \in A$ such that $x \preceq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. From the above cases (a) and (b), we have that $a \cdot A \preceq_1 a \cdot B$.

On the other hand, to prove the inverse statement, we assume that $A \preceq_K B$ on $\mathcal{C}(R^n)$ does not hold. Then we have the following two cases (i) and (ii). Case(i): There exists $x \in A$ such that $y - x \notin K$ for all $y \in B$. Then $B \cap (x + K) = \emptyset$. Since B and $x + K$ are closed convex, by the separation theorem there exists $a \in R^n$ ($a \neq 0$) such that $a \cdot y < a \cdot x + a \cdot z$ for all $y \in B$ and all $z \in K$. Now, we suppose that there exists $z \in K$ such that $a \cdot z < 0$. Then $\lambda z \in K$ for all $\lambda \geq 0$ since K is a cone, and so we have $a \cdot x + a \cdot \lambda z = a \cdot x + \lambda a \cdot z \to -\infty$ as $\lambda \to \infty$. This contradicts $a \cdot y < a \cdot x + a \cdot z$. Therefore we obtain $a \cdot z \geq 0$ for all $z \in K$, which implies $a \in K^+$. Especially taking $z = 0 \in K$, we get $a \cdot y < a \cdot x$ for all $y \in B$. This contradicts $a \cdot A \preceq_1 a \cdot B$. Case(ii): There exists $y \in B$ such that $y - x \notin K$ for all $x \in A$. Then we derive a contradiction in the similar way to the case (i). Therefore the inverse statement holds from the results of the above (i) and (ii). The proof of this lemma is completed. $\square$

For $a \in R^n$ and $\tilde{s} \in \mathcal{F}(R^n)$, applying the representation theorem we define a fuzzy number $a \cdot \tilde{s} \in \mathcal{F}(R)$ by

$$a \cdot \tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{a \cdot \tilde{s}_\alpha}(x)\}, \quad x \in R. \tag{4.2}$$

The following theorem gives the correspondence between the pseudo order $\preccurlyeq_K$ on $\mathcal{F}(R^n)$ and the fuzzy max order $\preccurlyeq_1$ on $\mathcal{F}(R)$.

Theorem 4.1. For $\tilde{a}, \tilde{b} \in \mathcal{F}(R^n)$, $\tilde{a} \preccurlyeq_K \tilde{b}$ if and only if $a \cdot \tilde{a} \preccurlyeq_1 a \cdot \tilde{b}$ for all $a \in K^+$.

Proof. By (4.2) and the representative theorem, we have $(a \cdot \tilde{a})_\alpha = a \cdot \tilde{a}_\alpha$ for all $\alpha \in [0, 1]$. On the other hand, from Lemmas 4.3 and 4.4, $\tilde{a} \preccurlyeq_K \tilde{b}$ if and only if $a \cdot \tilde{a}_\alpha \preccurlyeq_1 a \cdot \tilde{b}_\alpha$ for all $a \in K^+$. Thus, noting the definition of the max order $\preccurlyeq_1$ on $\mathcal{F}(R)$, Theorem 4.1 follows. □

For $\{\tilde{a}_k\}_{k=1}^{\infty} \subset \mathcal{F}(R^n)$ and $\tilde{a} \in \mathcal{F}(R^n)$, $\lim_{k\to\infty} \tilde{a}_k = \tilde{a}$ means that $\sup_{\alpha \in [0,1]} \rho_n(\tilde{a}_{k\alpha}, \tilde{a}_\alpha) \to 0$ $(k \to \infty)$, where $\tilde{a}_{k\alpha}$ is the α-cut of $\tilde{a}_k$ and ρ_n is the Hausdorff metric on $\mathcal{C}(R^n)$.

Lemma 4.5. Let $\{\tilde{a}_k\}_{k=1}^{\infty} \subset \mathcal{F}(R)$ and $\tilde{a} \in \mathcal{F}(R)$ such that $\tilde{a}_k \preccurlyeq_1 \tilde{a}_{k+1}$ $(k \geq 1)$ and $\lim_{k\to\infty} \tilde{a}_k = \tilde{a}$. Then we have $\tilde{a}_1 \preccurlyeq_1 \tilde{a}$.

Proof. Trivial. □

Theorem 4.2. Let $\{\tilde{a}_k\}_{k=1}^{\infty} \subset \mathcal{F}(R^n)$ and $\tilde{a} \in \mathcal{F}(R^n)$ such that $\tilde{a}_k \preccurlyeq_K \tilde{a}_{k+1}$ $(k \geq 1)$ and $\lim_{k\to\infty} \tilde{a}_k = \tilde{a}$. Then we have $\tilde{a}_1 \preccurlyeq_K \tilde{a}$.

Proof. From Theorem 4.1, for all $a \in K^+$ it holds that $a \cdot \tilde{a}_k \preccurlyeq_1 a \cdot \tilde{a}_{k+1}$ $(k \geq 1)$. Also, since $(a \cdot \tilde{a}_k)_\alpha = a \cdot \tilde{a}_{k\alpha}$ from (4.2) and $\rho_1(a \cdot \tilde{a}_{k\alpha}, a \cdot \tilde{a}_\alpha) \leq \|a\| \rho_n(\tilde{a}_{k\alpha}, \tilde{a}_\alpha)$ for all $k \geq 1$, we get $\lim_{k\to\infty} a \cdot \tilde{a}_k = a \cdot \tilde{a}$ where $\|a\|$ is a norm of a. By Lemma 4.5, it holds that $a \cdot \tilde{a}_1 \preccurlyeq_1 a \cdot \tilde{a}$ for all $a \in K^+$. From Theorem 4.1, we have $\tilde{a}_1 \preccurlyeq_K \tilde{a}$. □

Remark. Let consider a continuous map $g : [0, 1] \to \mathcal{F}(R^n)$. A point x_0 is said to be efficient if $x_0 \in [0, 1]$ and $g(x_0) \preccurlyeq_K g(x)$ for some $x \in [0, 1]$ implies $g(x) = g(x_0)$. Then, by applying the same idea as in Lemma 3.2 of Furukawa[13], we observe that there exists at least one efficient point in $[0, 1]$. In fact, considering, if necessary, a partial order $\preccurlyeq_K$ on the class of the quotient sets with respect to the equivalence relation $\sim_K$ defined by $\tilde{a} \sim_K \tilde{b}$ if and only if $\tilde{a} \preccurlyeq_K \tilde{b}$ and $\tilde{b} \preccurlyeq_K \tilde{a}$, we can assume that $\preccurlyeq_K$ is a partial order on $\mathcal{F}(R^n)$. By Theorem 4.2 and the continuity of g, $\{g(x) \mid x \in [0, 1]\}$ can be proved to be an inductively ordered set. So, by Zorn's lemma $\{g(x) \mid x \in [0, 1]\}$ has an efficient element.

4.2. Further results

In this section, we shall investigate the above pseudo order $\preccurlyeq_K$ on $\mathcal{F}_r(R^n)$ for a polyhedral cone K with $K^+ \subset R^n$. To this end, we need the following lemma.

Lemma 4.6. Let $a, b \in R^n_+$ and $A \in \mathcal{C}_r(R^n)$. Then for any scalars $\lambda_1, \lambda_2 \geq 0$, it holds

$$(4.3) \qquad (\lambda_1 a + \lambda_2 b) \cdot A = \lambda_1 (a \cdot A) + \lambda_2 (b \cdot A),$$

where the arithmetic in (4.3) is defined in (4.1).

Proof. Let $\lambda_1 a\cdot x + \lambda_2 b\cdot y \in \lambda_1(a\cdot A) + \lambda_2(b\cdot B)$ with $x, y \in A$. It suffices to show that $\lambda_1 a\cdot x + \lambda_2 b\cdot y \in (\lambda_1 a + \lambda_2 b)\cdot A$. Define $z = (z_1, z_2, \cdots, z_n)$ by

$$z_i := \begin{cases} (\lambda_1 a_i x_i + \lambda_2 b_i y_i)/(\lambda_1 a_i + \lambda_2 b_i) & \text{if} \quad (\lambda_1 a_i + \lambda_2 b_i) > 0 \\ x_i & \text{if} \quad (\lambda_1 a_i + \lambda_2 b_i) = 0 \end{cases} \quad (i = 1, 2, \cdots, n).$$

Then, clearly $(\lambda_1 a + \lambda_2 b)\cdot z = \lambda_1 a\cdot x + \lambda_2 b\cdot y$. Since $A \in \mathcal{C}_r(\mathbf{R}^n)$, $z \in A$, so that $\lambda_1 a\cdot x + \lambda_2 b\cdot y \in (\lambda_1 a + \lambda_2 b)\cdot A$. ∎

Henceforth, we assume that K is a polyhedral convex cone with $K^+ \subset \mathbf{R}^n_+$, i.e., there exist vectors $b^i \in \mathbf{R}^n_+ (i = 1, 2, \cdots, m)$ such that

$$K = \{x \in \mathbf{R}^n \mid b^i\cdot x \geq 0 \text{ for all } \ i = 1, 2, \cdots, m\}.$$

Then, it is well-known (cf. [30]) that K^+ is expressed as

$$K^+ = \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{m} \lambda_i b^i, \ \lambda_i \geq 0, \ i = 1, 2, \cdots, m\}.$$

The above dual cone K^+ is denoted simply by

$$K^+ = \text{cone convex}\{b^1, b^2, \cdots, b^m\}.$$

The pseudo order $\preceq_K$ on $\mathcal{C}_r(\mathbf{R}^n)$ is characterized by the pseudo order $\preceq_1$ on $\mathcal{C}_r(\mathbf{R})$.

Corollary 4.1. Let $K^+ = \text{cone convex}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbf{R}^n_+$. Then, for $A, B \in \mathcal{C}_r(\mathbf{R}^n)$, $A \preceq_K B$ if and only if $b^i\cdot A \preceq_1 b^i\cdot B$ for all $i = 1, 2, \cdots, m$.

Proof. We assume that $b^i\cdot A \preceq_1 b^i\cdot B$ for all $i = 1, 2, \cdots, m$. For any $a \in K^+$, there exist $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. From Lemma 4.1, we have:

$$a\cdot A = \sum_{i=1}^{m} \lambda_i(b^i\cdot A) \ \preceq_1 \sum_{i=1}^{m} \lambda_i(b^i\cdot B) = a\cdot B.$$

Thus, by Lemma 4.4, $A \preceq_K B$ follows. By applying Lemma 4.4 again, the 'only if' part of Corollary holds. ∎

Lemma 4.7. Let $a, b \in \mathbf{R}^n_+$ and $\tilde{u} \in \mathcal{F}_r(\mathbf{R}^n)$. Then, for any $\lambda_1, \lambda_2 \geq 0$,

$$(\lambda_1 a + \lambda_2 b)\cdot \tilde{u} = \lambda_1(a\cdot \tilde{u}) + \lambda_2(b\cdot \tilde{u}), \tag{4.4}$$

where the arithmetic in (4.4) is given in (2.1), (2.2) and (4.2).

Proof. For any $\alpha \in [0, 1]$, it follows from the definition and Lemma 4.6 that

$$\begin{aligned}((\lambda_1 a + \lambda_2 b)\cdot \tilde{u})_\alpha &= (\lambda_1 a + \lambda_2 b)\cdot \tilde{u}_\alpha = \lambda_1(a\cdot \tilde{u}_\alpha) + \lambda_2(b\cdot \tilde{u}_\alpha) \\ &= \lambda_1(a\cdot \tilde{u})_\alpha + \lambda_2(b\cdot \tilde{u})_\alpha = (\lambda_1(a\cdot \tilde{u}) + \lambda_2(b\cdot \tilde{u}))_\alpha.\end{aligned}$$

The last equality follows from (2.3). The above shows that (4.4) holds. ∎

The main result in this section is given in the following.

Theorem 4.3. Let $K^+ = \text{cone convex}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in R^n$. Then, for $\mu, \nu \in \mathcal{F}_r(R^n)$,

$$\mu \preceq_K \nu \quad \text{if and only if} \quad b^i \cdot \mu \preceq_1 b^i \cdot \nu \text{ for } i = 1, 2, \cdots, m.$$

Proof. It suffices to prove the 'if' part of Theorem 4.3. For any $a \in K^+$, there exist $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. Applying Lemma 4.7, we have

$$a \cdot \mu = \sum_{i=1}^m \lambda_i (b^i \cdot \mu) \preceq_1 \sum_{i=1}^m \lambda_i (b^i \cdot \nu) = a \cdot \nu,$$

From Theorem 4.1, $\mu \preceq_K \nu$ follows. □

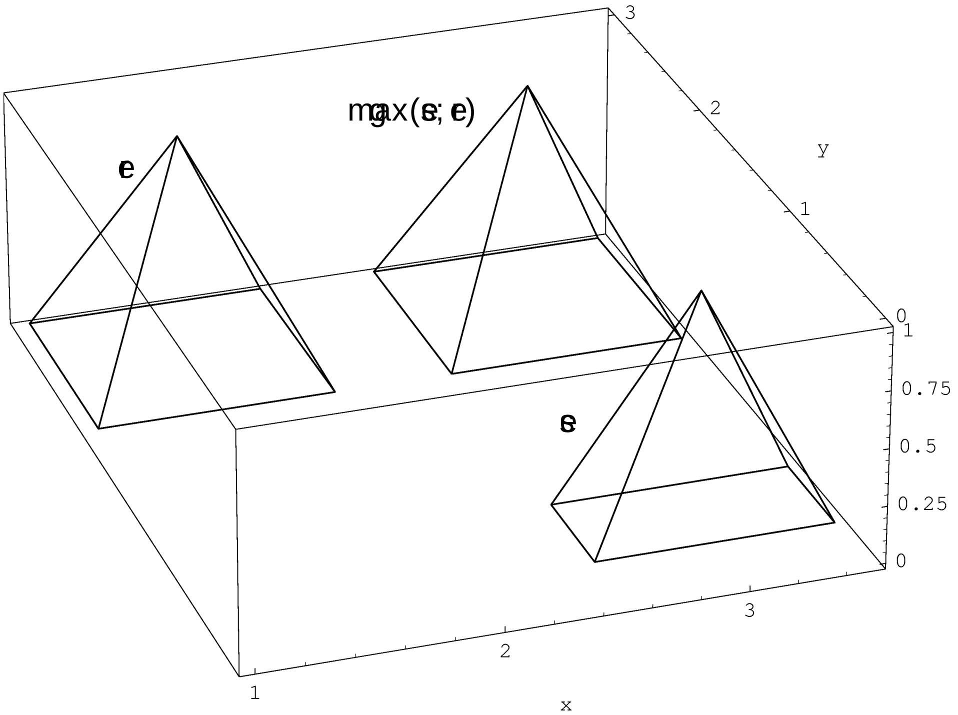

Fig 3: $\widetilde{\max}(\mu, \nu)$

Zhang and Hirota[37] described the structure of the fuzzy number lattice $(\mathcal{F}_r(R), \preceq_1)$. When $K = R^n$, $K^+ = R^n$ and $K^+ = \text{cone convex}\{e^1, e^2, \cdots, e^n\}$. So that, by Theorem 4.3, we see that for $\mu, \nu \in \mathcal{F}_r(R^n)$, $\mu \preceq_n \nu$ means $e^i \cdot \mu \preceq_1 e^i \cdot \nu$ for all $i = 1, 2, \cdots, n$. Therefore, by applying the same method as Zhang and Hirota[37], we can describe the structure of the fuzzy set lattice $(\mathcal{F}_r(R^n), \preceq_n)$. Figure 3 illustrates $\widetilde{\max}(\mu, \nu)$ for $\mu, \nu \in \mathcal{F}_r(R^2)$.

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