Order Relations and a Monotone Convergence Theorem in the Class of Fuzzy Sets on $\mathbb{R}^n$

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Abstract. Concerning with the topics of fuzzy decision processes, a brief survey on ordering of fuzzy numbers on $\mathbb{R}$ is presented and an extension to that of fuzzy sets(numbers) on $\mathbb{R}^n$ are considered. This extension is a pseudo order $\preccurlyeq_K$ defined by a non-empty closed convex cone K and characterized by the projection into its dual cone K^+. Especially a structure of the lattice is presented on the class of rectangle-type fuzzy sets. Moreover, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. the order $\preccurlyeq_K$. Our study is carried out by restricting the class of fuzzy sets into the subclass in which the order $\preccurlyeq_K$ becomes a partial order so that a monotone convergence theorem is proved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class. These results are applied to obtain the limit theorem for a sequence of fuzzy sets defined by the dynamic fuzzy system with a monotone fuzzy relation. Several figures are illustrated to comprehend our results.

1 Introduction

Fuzzy set theory has made applications in many fields of management science, operations research and statistics (cf.[7], [13], [18]), in which order relation of fuzzy sets is a fundamental problem in fuzzy optimization or fuzzy decision making. Many methods of ordering fuzzy numbers have been proposed in the literature. Among them a partial order on the set of fuzzy numbers on $\mathbb{R}$, called the fuzzy max order, introduced by Ramík and Řimánek[16] is very interesting in the concerns of pure mathematics because it is a natural extension of the order over real numbers and includes many theoretical and applicable potentials.

First, the fuzzy max order of fuzzy numbers on $\mathbb{R}$ (cf. [6], [16]) is briefly surveyed and extended to a pseudo order on a class of fuzzy sets defined on $\mathbb{R}^n$. The pseudo order for fuzzy sets on $\mathbb{R}^n$ is defined by a non-empty closed convex cone K in $\mathbb{R}^n$ and characterized by the projection into its dual cone K^+. Also, the structure of a lattice is discussed for the class of rectangle-type fuzzy sets (see [11]). For a lattice-structure of the fuzzy max order, see [1], [25]. So, we can imagine the much wider application to the fuzzy optimization problem. Our idea of the motivation originates from a set-relation in $\mathbb{R}^n$ given by Kuroiwa, Tanaka and Ha[12], in which various types of set-relations in $\mathbb{R}^n$ are used in set-valued optimizations.

A convergence theorem for a sequence of fuzzy sets is also a fundamental tool and mathematically interesting for sequential decision analysis in a fussy environment. In fact, the limiting behavior of fuzzy states of dynamic fuzzy system or sequential fuzzy decision process have been studied by developing a suitable convergence theorem of a sequence of fuzzy sets. (c.f. [8], [9], [10], [20], [21], [22], [23]) Also, the theory of metric space of fuzzy sets has been developed by many authors (c.f. [2], [14], [19]), in which several convergence theorems of fuzzy sets are given. On the other hand, in multiple criteria decision making, the rewards from dynamic system are described in terms of fussy sets and the model is often optimized under some order or pseudo order relation among fuzzy sets. In this case, it is more important to study the convergence theorem related to fuzzy order relation.

From this motivation, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preccurlyeq_K$. Our procedure is done by restricting the class of fuzzy sets into the subclass, in which $\preccurlyeq_K$ becomes a partial order and a monotone convergence theorem is proved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class. These results are applied to obtain the limit theorem for a sequence of fuzzy sets defined by the dynamic fuzzy system with a monotone fuzzy relation.

The outline for this paper is as follows. Section 2 contains some notations and basic concepts of fuzzy set theory referring to the text books (cf. [3], [15]). The fuzzy max order on $\mathbb{R}$ and its related topics are summarized in Section 3. A pseudo order on the class of fuzzy sets on $\mathbb{R}^n$ is introduced as an extension of the fuzzy max order in Section 4. Its characterization and the structure of a lattice are considered for the rectangle-type fuzzy sets in Section 5. In Section 6, we introduce a concept of determining class and give a convergence theorem for a sequence of convex compact subclass of $\mathbb{R}^n$. In Section 7, these results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$. In Section 8, we consider the limit of a sequence of fuzzy sets defined by the monotone dynamic fuzzy system. Several figures are illustrated to comprehend our results.

2 Notations and Basic Concepts

In this section we describe the notation and basic concepts of fuzzy set theory (cf. [3], [7], [15], [24]).

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^n$ an n-dimensional Euclidean space. We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\widetilde{s} : \mathbb{R}^n \to [0,1]$ (see Novák [15] and Zadeh [24]). The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ is defined as

$$\widetilde{s}_\alpha := \{x \in \mathbb{R}^n \mid \widetilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \widetilde{s}_0 := \mathrm{cl}\{x \in \mathbb{R}^n \mid \widetilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\widetilde{s}$ is called convex if

$$\widetilde{s}(\lambda x + (1-\lambda)y) \geq \widetilde{s}(x) \wedge \widetilde{s}(y) \quad x, y \in \mathbb{R}^n,\ \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\widetilde{s}$ is convex if and only if the α-cut $\widetilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\widetilde{s} : \mathbb{R}^n \to [0,1]$ are upper-semicontinuous and normal $(\sup_{x \in \mathbb{R}^n} \widetilde{s}(x) = 1)$ and have a compact support. In the one-dimensional case $n = 1$, $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $\mathcal{C}_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\widetilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)$ $(\alpha \in [0,1])$. We write a rectangle in $\mathcal{C}_r(\mathbb{R}^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ $(i = 1, 2, \cdots, n)$. For the case of $n = 1$, $\mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\widetilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n)$ for all $\alpha \in [0,1]$, $\widetilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

Here we give the extension principle introduced by Zadeh which provides a general method for fuzzification of non-fuzzy mathematical concepts.

The extension principle (cf. Dubois and Prade [3]):

Let f be a map from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^n$ such that $y = f(x_1, \cdots, x_n)$. It allows us to induce a map through f from fuzzy sets $\widetilde{s}_i (i = 1, 2, \ldots, n)$ on $\mathbb{R}^n$ to a fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ such that

$$\widetilde{s}(y) := \sup_{\substack{x_1, \cdots, x_n \in \mathbb{R}^n \\ y = f(x_1, \cdots, x_n)}} \min\{\widetilde{s_1}(x_1), \cdots, \widetilde{s_n}(x_n)\},$$

where $\quad \widetilde{s}(y) := 0$ *if* $f^{-1}(y) = \phi$.

Applying the extension principle, the addition and the scalar multiplication on $\mathbb{R}^n$ are extended to those on $\mathcal{F}(\mathbb{R}^n)$ as follows:

For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$(\widetilde{s} + \widetilde{r})(x) := \sup_{\substack{x_1, x_2 \in \mathbb{R}^n \\ x_1 + x_2 = x}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\}, \tag{2.1}$$

$$(\lambda \widetilde{s})(x) := \begin{cases} \widetilde{s}(x/\lambda) & \text{if } \lambda > 0 \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^n), \tag{2.2}$$

where $\mathbf{1}_{\{\cdot\}}(\cdot)$ is an indicator. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$, the following holds immediately:

$$(\widetilde{s} + \widetilde{r})_\alpha = \widetilde{s}_\alpha + \widetilde{r}_\alpha \quad \text{and} \quad (\lambda \widetilde{s})_\alpha = \lambda \widetilde{s}_\alpha \quad (\alpha \in [0,1]). \tag{2.3}$$

Also, for $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R})$, $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$ and $\widetilde{\min}\{\widetilde{s}, \widetilde{r}\} \in \mathcal{F}(\mathbb{R})$ are defined by

$$\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = \max(x_1, x_2)}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\} \tag{2.4}$$

and

$$\widetilde{\min}\{\widetilde{s}, \widetilde{r}\}(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = \min(x_1, x_2)}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\}. \tag{2.5}$$

The images of $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$ and $\widetilde{\min}\{\widetilde{s}, \widetilde{r}\} \in \mathcal{F}(\mathbb{R})$ are illustrated as follows:

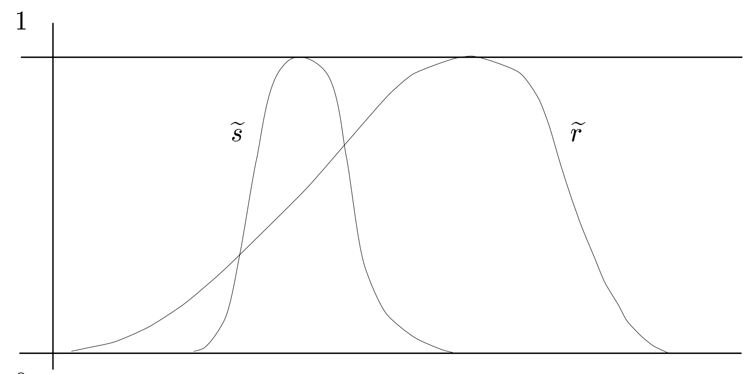

Figure 1: $\widetilde{s}$ and $\widetilde{r} \in \mathcal{F}(\mathbb{R})$

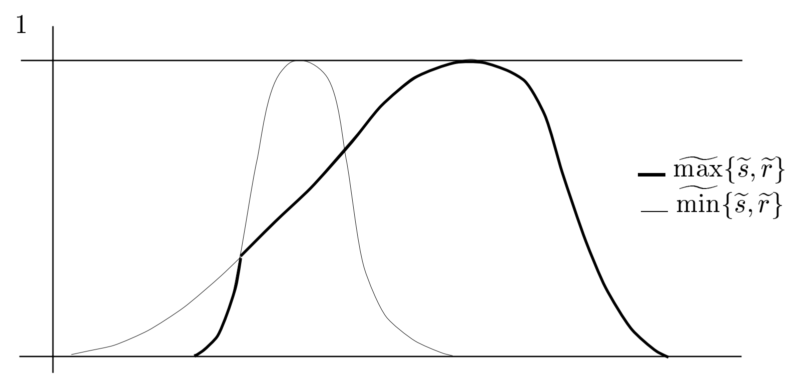

Figure 2: $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$ and $\widetilde{\min}\{\widetilde{s}, \widetilde{r}\} \in \mathcal{F}(\mathbb{R})$

We need a representative theorem (cf. [3], [15]) which is a basic tool for the fuzzy interval analysis.

The representative theorem:

(i) *For any* $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}(x) = \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{\widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}^n.$

(ii) *Conversely, for a family of subsets* $\{D_\alpha \in \mathcal{C}(\mathbb{R}^n) \mid 0 \le \alpha \le 1\}$ *with* $D_\alpha \subset D_{\alpha'}$ *for* $\alpha' \le \alpha$ *and* $\cap_{\alpha'<\alpha} D_{\alpha'} = D_\alpha$, *if we set* $\widetilde{s}(x) := \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{D_\alpha}(x)\},\ x \in \mathbb{R}^n$ *then* $\widetilde{s}$ *belongs to* $\mathcal{F}(\mathbb{R}^n)$ *and satisfies* $\widetilde{s}_\alpha = D_\alpha,\ \alpha \in [0,1]$.

The image of the equation (i) of the representative theorem is illustrated in Figure 3.

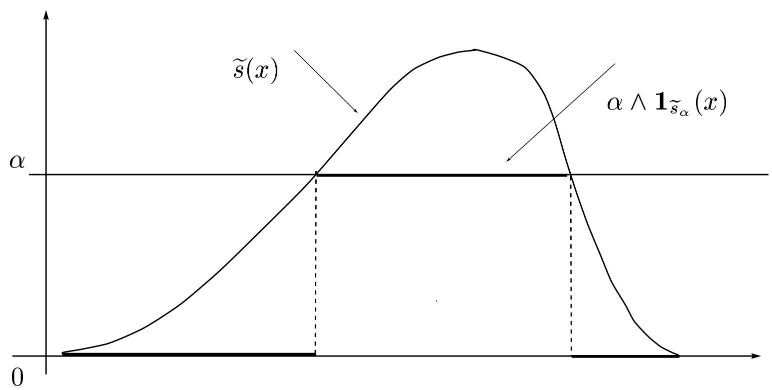

Figure 3: $\widetilde{s}(x) = \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{\widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}^n$

3 Fuzzy Max Order on $\mathcal{F}(\mathbb{R})$

In this section, we review a fuzzy max order in $\mathcal{F}(\mathbb{R})$ which is extended to $\mathcal{F}(\mathbb{R}^n)$ in the next section.

The following binary relation $\preccurlyeq$ has been formulated first by Ramík and Řimánek[16]. Let $\widetilde{s}$ and $\widetilde{r}$ be two fuzzy numbers in $\mathcal{F}(\mathbb{R})$. Then $\widetilde{s} \preccurlyeq \widetilde{r}$ if and only if $\sup \widetilde{s}_\alpha \le \sup \widetilde{r}_\alpha$ and $\inf \widetilde{s}_\alpha \le \inf \widetilde{r}_\alpha$ for each $\alpha \in [0,1]$, where $\widetilde{s}_\alpha$ and $\widetilde{r}_\alpha$ are α-cuts of $\widetilde{s}$ and $\widetilde{r}$ respectively and $\widetilde{s}_\alpha := [\inf \widetilde{s}_\alpha, \sup \widetilde{s}_\alpha]$ and $\widetilde{r}_\alpha := [\inf \widetilde{r}_\alpha, \sup \widetilde{r}_\alpha]$. Obviously the binary relation $\preccurlyeq$ satisfies the axioms of a partial order relation on $\mathcal{F}(\mathbb{R})$ and is called the fuzzy max order. Some properties of the relation $\preccurlyeq$ are investigated in Ramík and Řimánek[16].

Proposition 3.1. *Let* $\widetilde{s}, \widetilde{r}$ *be fuzzy numbers in* $\mathcal{F}(\mathbb{R})$.

(i) *The inequality* $\widetilde{s} \preccurlyeq \widetilde{r}$ *if and only if there exist* $m, n, t^* \in \mathbb{R}$ *with* $m \le t^* \le n$, $\widetilde{s}(m) = \widetilde{r}(n) = 1$ *and* $\widetilde{s}(t) \ge \widetilde{r}(t)$ *for any* $t \le t^*$ *and* $\widetilde{s}(t) \le \widetilde{r}(t)$ *for any* $t > t^*$.

(ii) *The following three conditions* (a) *to* (c) *are equivalent:*
(a) $\widetilde{s} \preccurlyeq \widetilde{r}$, (b) $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\} = \widetilde{r}$, (c) $\widetilde{\min}\{\widetilde{s}, \widetilde{r}\} = \widetilde{s}$,
where $\widetilde{\min}$ *and* $\widetilde{\max}$ *are defined in* (2.4) *and* (2.5).

For the fuzzy max order on $\mathcal{F}(\mathbb{R})$, Congxin and Cong [1] proved that the bounded set of fuzzy numbers on $\mathcal{F}(\mathbb{R})$ must have supremum and infimum. The basic proof is as follows. For any sequence of fuzzy numbers $\{\widetilde{s}_n\}_{n=1}^{\infty}$, let $\underline{D}_\alpha := \lim_{\alpha'\uparrow\alpha} \sup_{n\geq 1} \inf \widetilde{s}_{n\alpha'}$ and $\overline{D}_\alpha := \lim_{\alpha'\uparrow\alpha} \sup_{n\geq 1} \sup \widetilde{s}_{n\alpha'}$, where $\widetilde{s}_{n\alpha}$ is the α-cut of $\widetilde{s}_n$. Then, the family of closed subsets $\{D_\alpha := [\underline{D}_\alpha, \overline{D}_\alpha] \mid \alpha \in [0,1]\}$ satisfies condition (ii) of the representation theorem, so that $\widetilde{s}$ defined by

$$\widetilde{s}(x) := \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{D_\alpha}(x)\} \quad (x \in \mathbb{R}) \tag{3.1}$$

belongs to $\mathcal{F}(\mathbb{R})$ and $\widetilde{s} = \sup_{n\geq 1} \widetilde{s}_n$.

Similarly, the infimum of $\{\widetilde{s}_n\}_{n=1}^{\infty}$ is constructed. These results derived the interesting mathematical fact that the continuous fuzzy-valued function on a closed interval has a maximum and minimum. Also, the structure of lattice for fuzzy numbers is discussed in Zhang and Hirota[25].

To be suitable for computations and treatments, a class of fuzzy numbers, called an L-R-fuzzy number, is introduced in many text books.

Let $L, R : [0,\infty) \to [0,1]$ be two non-increasing and not constant functions with $L(0) = R(0) = 1$ and $L(x_0) = R(x_0) = 0$ for some $x_0 > 0$. A fuzzy number $\widetilde{s}$ is called an L-R-fuzzy number if there exist real numbers $m, n (m \leq n), \alpha, \beta (\alpha, \beta > 0)$ such that

$$\widetilde{s}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{for} \quad x \leq m, \\ 1 & \text{for} \quad m \leq x \leq n, \\ R(\frac{x-n}{\beta}) & \text{for} \quad n \leq x. \end{cases} \tag{3.2}$$

Given functions L, R with the properties in the above definition, the L-R-fuzzy number specified by m, n, α, β is denoted by an ordered tetradic $(m, n, \alpha, \beta)_{L\text{-}R}$, which includes the triangular and trapezoidal fuzzy numbers. Then, the fuzzy max order on the set of L-R-fuzzy numbers is characterized by inequalities of the elements. (cf. [16]).

In particular, the symmetric fuzzy number $\widetilde{s} = (m, m, \alpha, \alpha)_{L\text{-}L}$ is called an L-fuzzy number and denoted by $(m, \alpha)_L$. Furukawa[4] extended the L-fuzzy number $(m, \alpha)_L$ with $\alpha > 0$ to the case of $\alpha \in \mathbb{R}$ and proved that for $\alpha\beta \geq 0$ the fuzzy max order $(m, \alpha)_L \preccurlyeq (n, \beta)_L$ if and only if $x_0|\alpha - \beta| \leq n - m$ where x_0 is the zero point of L. Moreover Furukawa [4] introduced the linear operations on the set of extended L-fuzzy numbers by

$$\begin{aligned} (m, \alpha)_L \oplus (n, \beta)_L &= (m + n, \alpha + \beta)_L, \\ \lambda(m, \alpha)_L &= (\lambda m, \lambda\alpha)_L \quad \text{for any scalar} \quad \lambda \in \mathbb{R}. \end{aligned}$$

The fuzzy max order is proved to be adapted to the above operations. Also Furukawa [5] introduced a parametric order relation on L-fuzzy numbers which is an extension of the fuzzy max order and its total order relation. The fuzzy optimization problems related to the fuzzy max order are dealed with many authors; e.g., Furukawa[5], Kurano et al.[9], Yoshida[22] and others.

4 A Pseudo Order on $\mathcal{F}(\mathbb{R}^n)$

In this section we extend the fuzzy max order on $\mathcal{F}(\mathbb{R})$ to a pseudo order on $\mathcal{F}(\mathbb{R}^n)$ by the argument in [11].

We will review a vector ordering on $\mathbb{R}^n$ by a non-empty convex cone $K \subset \mathbb{R}^n$. Using this K, we can define a pseudo order relation $\preccurlyeq_K$ on $\mathbb{R}$ by $x \preccurlyeq_K y$ if and only if $y - x \in K$. Let $\mathbb{R}^n_+$ be the subset of entrywise non-negative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $\preccurlyeq_K$ will be denoted by $\preccurlyeq_n$ and $x \preccurlyeq_n y$ means that $x_i \le y_i$ for all $i = 1, 2, \cdots, n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

First we introduce a binary relation on $\mathcal{C}(\mathbb{R}^n)$, by which a pseudo order on $\mathcal{F}(\mathbb{R}^n)$ is given. Henceforth we assume that the convex cone $K \subset \mathbb{R}^n$ is given. We define a binary relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ by abuse of notation. For $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \preccurlyeq_K B$ means the following (C.a) and (C.b) (cf. [12]):

(C.a) *For any $x \in A$, there exists $y \in B$ such that $x \preccurlyeq_K y$.*
(C.b) *For any $y \in B$, there exists $x \in A$ such that $x \preccurlyeq_K y$.*

Lemma 4.1. *The binary relation $\preccurlyeq_K$ is a pseudo order on $\mathcal{C}(\mathbb{R}^n)$.*

Proof. It is trivial that $A \preccurlyeq_K A$ for $A \in \mathcal{C}(\mathbb{R}^n)$. Let $A, B, C \in \mathcal{C}(\mathbb{R}^n)$ such that $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$. We will check $A \preccurlyeq_K C$ by two cases (C.a) and (C.b). Case(C.a): Since $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$, for any $x \in A$ there exists $y \in B$ such that $x \preccurlyeq_K y$ and there exists $z \in C$ such that $y \preccurlyeq_K z$. Since $\preccurlyeq_K$ is a pseudo order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $x \in A$ there exists $z \in C$ such that $x \preccurlyeq_K z$. Case(C.b): Since $A \preccurlyeq_K B$ and $B \preccurlyeq_K C$, for any $z \in C$ there exists $y \in B$ such that $y \preccurlyeq_K z$ and there exists $x \in A$ such that $x \preccurlyeq_K y$. Since $\preccurlyeq_K$ is a pseudo order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $z \in C$ there exists $x \in A$ such that $x \preccurlyeq_K z$. From the above (C.a) and (C.b), we obtain $A \preccurlyeq_K C$. Thus the lemma holds. □

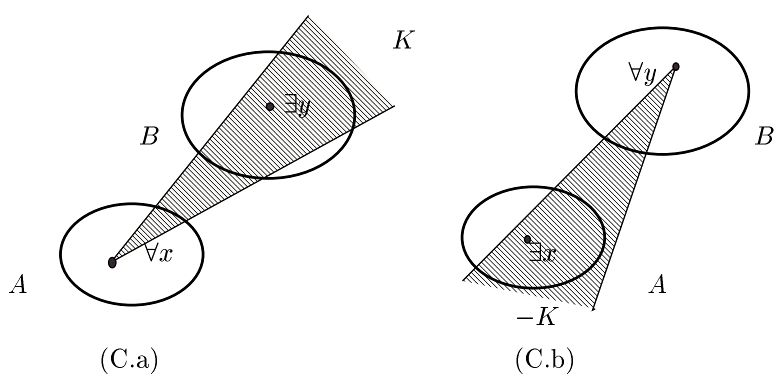

Figure 4: The binary relation $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^2)$

The conditions (C.a) and (C.b) of the binary relation $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^2)$ are illustrated in Figure 4.

When $K = \mathbb{R}^n_+$, the binary relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ will be written simply by $\preccurlyeq_n$ and for $[x,y],[x',y'] \in \mathcal{C}_r(\mathbb{R}^n)$, $[x,y] \preccurlyeq_n [x',y']$ means $x \preccurlyeq_n x'$ and $y \preccurlyeq_n y'$.

Next, we introduce a binary relation $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$: Let $\widetilde{s},\widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$. The relation $\widetilde{s} \preccurlyeq_K \widetilde{r}$ means the following (F.a) and (F.b):

(F.a) *For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \le \widetilde{r}(y)$.*
(F.b) *For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \ge \widetilde{r}(y)$.*

Note that the notation $\preccurlyeq_K$ denotes the binary relation on $\mathbb{R}^n$, $\mathcal{C}(\mathbb{R}^n)$ and $\mathcal{F}(\mathbb{R}^n)$ with some abuse of notation.

Lemma 4.2. *The binary relation $\preccurlyeq_K$ is a pseudo order on $\mathcal{F}(\mathbb{R}^n)$.*

Proof. It is trivial that $\widetilde{s} \preccurlyeq_K \widetilde{s}$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. Let $\widetilde{s},\widetilde{r},\widetilde{p} \in \mathcal{F}(\mathbb{R}^n)$ such that $\widetilde{s} \preccurlyeq_K \widetilde{r}$ and $\widetilde{r} \preccurlyeq_K \widetilde{p}$. We will check $\widetilde{s} \preccurlyeq_K \widetilde{p}$ by following two cases (F.a) and (F.b). Case(F.a): Since $\widetilde{s} \preccurlyeq_K \widetilde{r}$ and $\widetilde{r} \preccurlyeq_K \widetilde{p}$, for any $x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \le \widetilde{r}(y)$, and there exists $z \in \mathbb{R}^n$ such that $y \preccurlyeq_K z$ and $\widetilde{r}(y) \le \widetilde{p}(z)$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$ and $\widetilde{s}(x) \le \widetilde{p}(z)$. Therefore it holds that for any $x \in \mathbb{R}^n$ there exists $z \in \mathbb{R}^n$ such that $x \preccurlyeq_K z$ and $\widetilde{s}(x) \le \widetilde{p}(z)$. Case(F.b): Since $\widetilde{s} \preccurlyeq_K \widetilde{r}$ and $\widetilde{r} \preccurlyeq_K \widetilde{p}$, for any $z \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ such that $y \preccurlyeq_K z$ and $\widetilde{r}(y) \le \widetilde{p}(z)$, and there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \le \widetilde{r}(y)$. Since $\preccurlyeq_K$ is a pseudo-order on $\mathbb{R}^n$, we have $x \preccurlyeq_K z$. Therefore it holds that for any $z \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K z$ and $\widetilde{s}(x) \le \widetilde{p}(z)$. From the above (F.a) and (F.b), we obtain $\widetilde{s} \preccurlyeq_K \widetilde{p}$. Thus the lemma holds. □

The following lemma implies the correspondence between the pseudo order on $\mathcal{F}(\mathbb{R}^n)$ for fuzzy sets and the pseudo order on $\mathcal{C}(\mathbb{R}^n)$ for the α-cuts.

Lemma 4.3. *Let $\widetilde{s},\widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$. $\widetilde{s} \preccurlyeq_K \widetilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ if and only if $\widetilde{s}_\alpha \preccurlyeq_K \widetilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0,1]$.*

Proof. Let $\widetilde{s},\widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$. Suppose $\widetilde{s} \preccurlyeq_K \widetilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$. Then, two cases (a) and (b) are considered. Case(a): Let $x \in \widetilde{s}_\alpha$. Since $\widetilde{s} \preccurlyeq_K \widetilde{r}$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\alpha \le \widetilde{s}(x) \le \widetilde{r}(y)$. Namely $y \in \widetilde{r}_\alpha$. Case(b): Let $y \in \widetilde{r}_\alpha$. Since $\widetilde{s} \preccurlyeq_K \widetilde{r}$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \ge \widetilde{r}(y) \ge \alpha$. Namely $x \in \widetilde{s}_\alpha$. Therefore we get $\widetilde{s}_\alpha \preccurlyeq_K \widetilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0,1]$ from the above (a) and (b).

On the other hand, suppose $\widetilde{s}_\alpha \preccurlyeq_K \widetilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0,1]$. Then, two cases (a') and (b') are considered. Case(a'): Let $x \in \mathbb{R}^n$. Put $\alpha = \widetilde{s}(x)$. If $\alpha = 0$, then $x \preccurlyeq_K x$ and $\widetilde{s}(x) = 0 \le \widetilde{r}(x)$. While, if $\alpha > 0$, then $x \in \widetilde{s}_\alpha$. Since $\widetilde{s}_\alpha \preccurlyeq_K \widetilde{r}_\alpha$, there exists $y \in \widetilde{r}_\alpha$ such that $x \preccurlyeq_K y$. And we have $\widetilde{s}(x) = \alpha \le \widetilde{r}(y)$. Case(b'): Let $y \in \mathbb{R}^n$. Put $\alpha = \widetilde{r}(y)$. If $\alpha = 0$, then $x \preccurlyeq_K x$ and

$\tilde{s}(x) \geq 0 = \tilde{r}(y)$. While, if $\alpha > 0$, then $y \in \tilde{r}_\alpha$. Since $\tilde{s}_\alpha \preccurlyeq_K \tilde{r}_\alpha$, there exists $x \in \tilde{s}_\alpha$ such that $x \preccurlyeq_K y$. And we have $\tilde{s}(x) \geq \alpha = \tilde{r}(y)$.

Therefore we get $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ from the above Case (a') and (b'). Thus we obtain this lemma. □

The conditions (F.a) and (F.b) of the binary relation $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R})$ and $\mathcal{F}(\mathbb{R}^2)$ are illustrated in Figure 5.

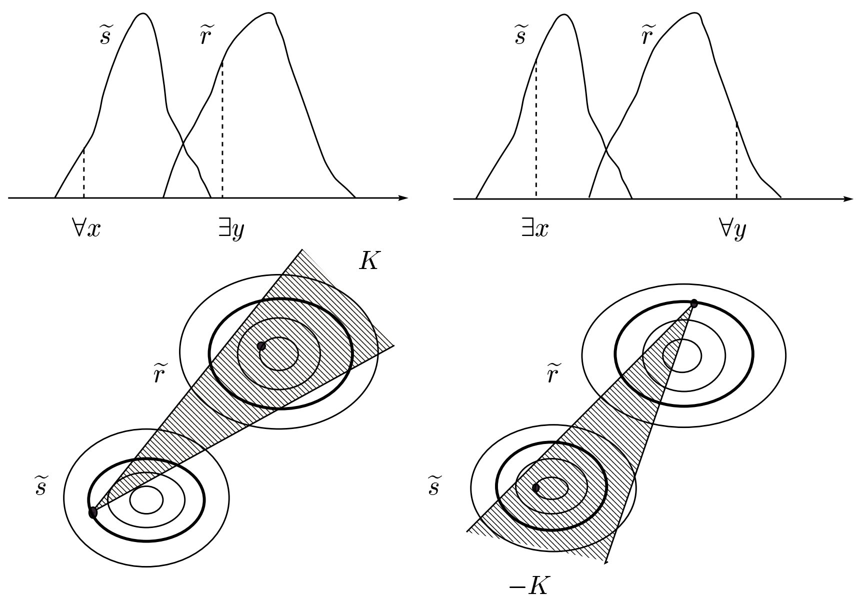

Figure 5: The binary relation $\tilde{s} \preccurlyeq_K \tilde{r}$ on $\mathcal{F}(\mathbb{R})$ and $\mathcal{F}(\mathbb{R}^2)$

For the case of $K = \mathbb{R}_+$, Lemma 4.3 says that the order relation $\preccurlyeq_1$ on $\mathcal{F}(\mathbb{R})$ (that is, $n = 1$) is the fuzzy max order mentioned in Section 3.

Define the dual cone of a cone K by

$$K^+ := \{a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K\},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

$$a \cdot A := \{a \cdot x \mid x \in A\} \ (\subset \mathbb{R}^n). \tag{4.1}$$

The definition (4.1) means that $a \cdot A$ is the projection of A on the parallel line with the vector a if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(\mathbb{R})$ if $A \in \mathcal{C}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

Lemma 4.4. *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$. $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ if and only if $a \cdot A \preccurlyeq_1 a \cdot B$ on $\mathcal{C}(\mathbb{R})$ for all $a \in K^+$.*

Proof. Suppose $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$. Consider the two cases (a) and (b). Case(a): For any $x \in A$, there exists $y \in B$ such that $x \preccurlyeq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. Case(b): For any $y \in B$, there exists $x \in A$ such that $x \preccurlyeq_K y$. Then $y - x \in K$. If $a \in K^+$, then $a \cdot (y - x) \geq 0$ and i.e. $a \cdot x \leq a \cdot y$. From the above cases (a) and (b), we have that $a \cdot A \preccurlyeq_1 a \cdot B$.

On the other hand, to prove the inverse statement, we assume that $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ does not hold. Then we have the following two cases (i) and (ii). Case(i): There exists $x \in A$ such that $y - x \notin K$ for all $y \in B$. Then $B \cap (x + K) = \emptyset$. Since B and $x + K$ are closed convex, by the separation theorem there exists $a \in \mathbb{R}^n$ $(a \neq 0)$ such that $a \cdot y < a \cdot x + a \cdot z$ for all $y \in B$ and all $z \in K$. Now, we suppose that there exists $z \in K$ such that $a \cdot z < 0$. Then $\lambda z \in K$ for all $\lambda \geq 0$ since K is a cone, and so we have $a \cdot x + a \cdot \lambda z = a \cdot x + \lambda a \cdot z \to -\infty$ as $\lambda \to \infty$. This contradicts $a \cdot y < a \cdot x + a \cdot z$. Therefore we obtain $a \cdot z \geq 0$ for all $z \in K$, which implies $a \in K^+$. Especially taking $z = 0 \in K$, we get $a \cdot y < a \cdot x$ for all $y \in B$. This contradicts $a \cdot A \preccurlyeq_1 a \cdot B$. Case(ii): There exists $y \in B$ such that $y - x \notin K$ for all $x \in A$. Then we derive a contradiction in the similar way to the case (i). Therefore the inverse statement holds from the results of the above (i) and (ii). The proof of this lemma is completed. □

The image of Lemma 4.4 is illustrated in Figure 6.

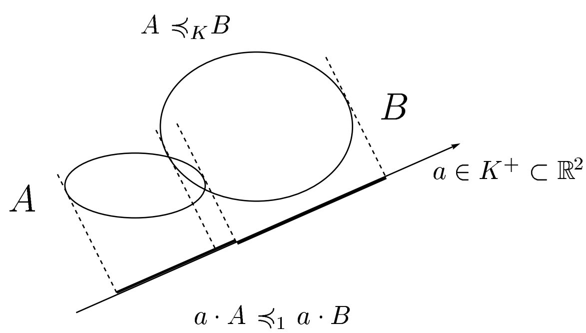

Figure 6: The image of Lemma 4.4

For $a \in \mathbb{R}^n$ and $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, applying the representation theorem we define a fuzzy number $a \cdot \widetilde{s} \in \mathcal{F}(\mathbb{R})$ by

$$a \cdot \widetilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{a \cdot \widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}. \tag{4.2}$$

The following theorem gives the correspondence between the pseudo order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ and the fuzzy max order $\preccurlyeq_1$ on $\mathcal{F}(\mathbb{R})$.

Theorem 4.1. *For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ if and only if $a \cdot \widetilde{s} \preccurlyeq_1 a \cdot \widetilde{r}$ for all $a \in K^+$.*

Proof. By (4.2) and the representative theorem, we have $(a \cdot \widetilde{s})_\alpha = a \cdot \widetilde{s}_\alpha$ for all $\alpha \in [0, 1]$. On the other hand, from Lemmas 4.3 and 4.4, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ if and only if $a \cdot \widetilde{s}_\alpha \preccurlyeq_1 a \cdot \widetilde{r}_\alpha$ for all $a \in K^+$. Thus, noting the definition of the max order $\preccurlyeq_1$ on $\mathcal{F}(\mathbb{R})$, Theorem 4.1 follows. □

The image of Theorem 4.1 is illustrated in Figure 7.

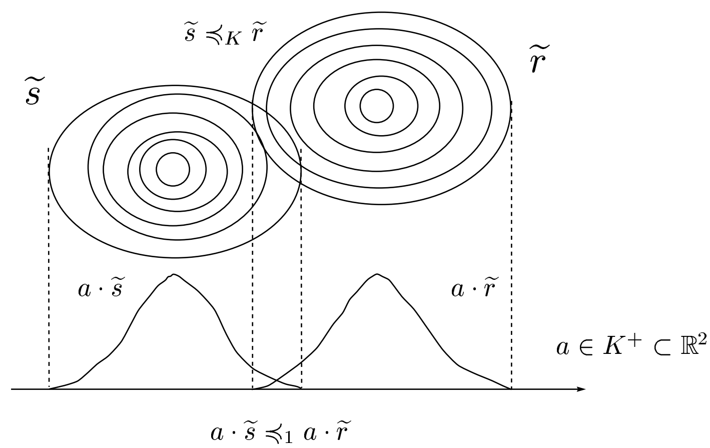

Figure 7: The image of Theorem 4.1

5 A Pseudo Order on $\mathcal{F}_r(\mathbb{R}^n)$

In this section as a special case of the previous section, we shall investigate the pseudo order $\preccurlyeq_K$ on $\mathcal{F}_r(\mathbb{R}^n)$ for a polyhedral cone K with $K^+ \subset \mathbb{R}^n$. To this end, we need the following lemma.

Lemma 5.1. Let $a, b \in \mathbb{R}^n_+$ and $A \in \mathcal{C}_r(\mathbb{R}^n)$. Then for any scalars $\lambda_1, \lambda_2 \geq 0$, it holds

$$(\lambda_1 a + \lambda_2 b) \cdot A = \lambda_1 (a \cdot A) + \lambda_2 (b \cdot A), \tag{5.1}$$

where the arithmetic in (5.1) is defined in (4.1).

Proof. Let $\lambda_1 a \cdot x + \lambda_2 b \cdot y \in \lambda_1 (a \cdot A) + \lambda_2 (b \cdot B)$ with $x, y \in A$. It suffices to show that $\lambda_1 a \cdot x + \lambda_2 b \cdot y \in (\lambda_1 a + \lambda_2 b) \cdot A$. Define $z = (z_1, z_2, \cdots, z_n)$ by

$$z_i := \begin{cases} (\lambda_1 a_i x_i + \lambda_2 b_i y_i)/(\lambda_1 a_i + \lambda_2 b_i) & \text{if } (\lambda_1 a_i + \lambda_2 b_i) > 0 \\ x_i & \text{if } (\lambda_1 a_i + \lambda_2 b_i) = 0 \end{cases} \quad (i = 1, \cdots, n).$$

Then, clearly $(\lambda_1 a + \lambda_2 b) \cdot z = \lambda_1 a \cdot x + \lambda_2 b \cdot y$. Since $A \in \mathcal{C}_r(\mathbb{R}^n), z \in A$, so that $\lambda_1 a \cdot x + \lambda_2 b \cdot y \in (\lambda_1 a + \lambda_2 b) \cdot A$. □

Henceforth, we assume that K is a polyhedral convex cone with $K^+ \subset \mathbb{R}^n_+$, i.e., there exist vectors $b^i \in \mathbb{R}^n_+ (i = 1, 2, \cdots, m)$ such that

$$K = \{x \in \mathbb{R}^n \mid b^i \cdot x \geq 0 \text{ for all } \ i = 1, 2, \cdots, m\}.$$

Then, it is well-known (cf. [17]) that K^+ can be written as

$$K^+ = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^m \lambda_i b^i, \ \lambda_i \geq 0, \ i = 1, 2, \cdots, m\}.$$

The above dual cone K^+ is denoted simply by

$$K^+ = \text{cone}\{b^1, b^2, \cdots, b^m\},$$

where cone S denotes the conical hull of set S. The pseudo order $\preccurlyeq_K$ on $\mathcal{C}_r(\mathbb{R}^n)$ is characterized by the pseudo order $\preccurlyeq_1$ on $\mathcal{C}_r(\mathbb{R})$.

Corollary 5.1. Let $K^+ = \text{cone}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbb{R}^n_+$. Then, for $A, B \in \mathcal{C}_r(\mathbb{R}^n)$, $A \preccurlyeq_K B$ if and only if $b^i \cdot A \preccurlyeq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$.

Proof. We assume that $b^i \cdot A \preccurlyeq_1 b^i \cdot B$ for all $i = 1, 2, \cdots, m$. For any $a \in K^+$, there exist $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. From Lemma 4.1, we have:

$$a \cdot A = \sum_{i=1}^m \lambda_i (b^i \cdot A) \ \preccurlyeq_1 \ \sum_{i=1}^m \lambda_i (b^i \cdot B) = a \cdot B.$$

Thus, by Lemma 4.4, $A \ \preccurlyeq_K \ B$ follows. By applying Lemma 4.4 again, the 'only if' part of Corollary holds. □

Lemma 5.2. Let $a, b \in \mathbb{R}^n_+$ and $\widetilde{s} \in \mathcal{F}_r(\mathbb{R}^n)$. Then, for any $\lambda_1, \lambda_2 \geq 0$,

$$(\lambda_1 a + \lambda_2 b) \cdot \widetilde{s} = \lambda_1 (a \cdot \widetilde{s}) + \lambda_2 (b \cdot \widetilde{s}), \tag{5.2}$$

where the arithmetic in (5.2) is given in (2.1), (2.2) and (4.2).

Proof. For any $\alpha \in [0, 1]$, it follows from the definition and Lemma 5.1 that

$$\begin{aligned} ((\lambda_1 a + \lambda_2 b) \cdot \widetilde{s})_\alpha &= (\lambda_1 a + \lambda_2 b) \cdot \widetilde{s}_\alpha = \lambda_1 (a \cdot \widetilde{s}_\alpha) + \lambda_2 (b \cdot \widetilde{s}_\alpha) \\ &= \lambda_1 (a \cdot \widetilde{s})_\alpha + \lambda_2 (b \cdot \widetilde{s})_\alpha = (\lambda_1 (a \cdot \widetilde{s}) + \lambda_2 (b \cdot \widetilde{s}))_\alpha. \end{aligned}$$

The last equality follows from (2.3). The above shows that (5.2) holds. □

The main result in this section is given in the following.

Theorem 5.1. Let $K^+ = \text{cone}\{b^1, b^2, \cdots, b^m\}$ with $b^i \in \mathbb{R}^n$. Then, for $\widetilde{s}, \widetilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$,

$$\widetilde{s} \preccurlyeq_K \widetilde{r} \quad \text{if and only if} \quad b^i \cdot \widetilde{s} \preccurlyeq_1 b^i \cdot \widetilde{r} \text{ for } i = 1, 2, \cdots, m.$$

Proof. It suffices to prove the 'if' part of Theorem 5.1. For any $a \in K^+$, there exist $\lambda_i \geq 0$ with $a = \sum_{i=1}^m \lambda_i b^i$. Applying Lemma 5.2, we have

$$a \cdot \widetilde{s} = \sum_{i=1}^m \lambda_i (b^i \cdot \widetilde{s}) \preccurlyeq_1 \sum_{i=1}^m \lambda_i (b^i \cdot \widetilde{r}) = a \cdot \widetilde{r},$$

From Theorem 4.1, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ follows. □

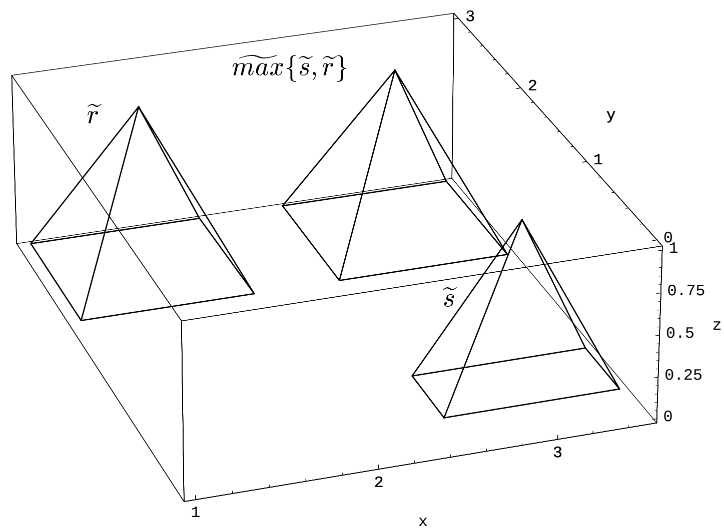

Figure 8: $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$ on $\mathcal{F}_r(\mathbb{R}^2)$

Zhang and Hirota[25] described the structure of the fuzzy number lattice $(\mathcal{F}_r(\mathbb{R}), \preccurlyeq_1)$. When $K = \mathbb{R}^n$, $K^+ = \mathbb{R}^n$ and $K^+ = \text{cone}\{e^1, e^2, \cdots, e^n\}$. So that, by Theorem 5.1, we see that for $\widetilde{s}, \widetilde{r} \in \mathcal{F}_r(\mathbb{R}^n)$, $\widetilde{s} \preccurlyeq_n \widetilde{r}$ means $e^i \cdot \widetilde{s} \preccurlyeq_1 e^i \cdot \widetilde{r}$ for all $i = 1, 2, \cdots, n$. Therefore, by applying the same method as Zhang and Hirota[25], we can describe the structure of the fuzzy set lattice $(\mathcal{F}_r(\mathbb{R}^n), \preccurlyeq_n)$. Figure 8 illustrates $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$ for $\widetilde{s}, \widetilde{r} \in \mathcal{F}_r(\mathbb{R}^2)$.

6 Sequences in $\mathcal{C}(\mathbb{R}^n)$

In this section, we introduce the concept of a determining class in which the monotone convergence theorem for the sequences in $\mathcal{C}(\mathbb{R}^n)$ is proved.

A closed cone $K \subset \mathcal{C}(\mathbb{R}^n)$ is said to be acute (c.f. [17]) if there exists an $a \in \mathbb{R}^n$ such that $a \cdot x > 0$ for all $x \in K$ with $x \neq 0$.

We have the following lemma.

Lemma 6.1. *Let K be a closed, acute convex cone and $x_0, y_0 \in \mathbb{R}^n$ with $x_0 \preccurlyeq_K y_0$. Then, $(x_0 + K) \cap (y_0 - K)$ is nonempty and bounded.*

Proof. By $x_0 \preccurlyeq_K y_0$, it follows that $y_0 \in (x_0 + K) \cap (y_0 - K)$. Suppose that $(x_0 + K) \cap (y_0 - K)$ is not bounded. Then, there exists a sequence $\{z_k'\} \subset (x_0 + K) \cap (y_0 - K)$ with $\| z_k' \| \to \infty$ as $k \to \infty$, where $\| \cdot \|$ is a norm in $\mathbb{R}^n$. Since $z_k' \in (x_0 + K)$, for each $k \geq 1$, there exists $z_k \in K$ with $z_k' = x_0 + z_k$. By acuteness of K, there exists $a \in \mathbb{R}^n$ such that

$$a \cdot z_k > 0 \quad \text{for all} \quad k \geq 1. \tag{6.1}$$

Also, from $z_k' \in y_0 - K$, $y_0 - x_0 - z_k \in K$, which implies, together with (6.1), that

$$a \cdot (y_0 - x_0) > a \cdot z_k > 0 \quad \text{for all} \quad k \geq 1. \tag{6.2}$$

It clearly holds that

$$\inf_{\|z\|=1, z \in K} a \cdot z = a \cdot z_0 > 0 \quad \text{for some} \quad z_0 \in \mathbb{R}^n.$$

From (6.2), we have

$$\frac{a \cdot (y_0 - x_0)}{\| z_k \|} > a \cdot \left(\frac{z_k}{\| z_k \|} \right) \geq a \cdot z_0 > 0 \quad \text{for all} \quad k \geq 1.$$

As $\| z_k \| \to \infty$ $(k \to \infty)$, the above inequality leads a contradiction. □

Let ρ_n be the Hausdorff metric on $\mathcal{C}(\mathbb{R}^n)$, that is, for $A, B \in \mathcal{C}(\mathbb{R}^n)$, $\rho_n(A, B) = \max_{a \in A} d(a, B) \vee \max_{b \in B} d(b, A)$, where d is a metric in $\mathbb{R}^n$ and $d(x, Y) = \min_{y \in Y} d(x, y)$ for $x \in \mathbb{R}^n$ and $Y \in \mathcal{C}(\mathbb{R}^n)$. It is well-known that $(\mathcal{C}(\mathbb{R}^n), \rho_n)$ is a complete metric space. A sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^n)$ converges to $D \in \mathcal{C}(\mathbb{R}^n)$ w.r.t. ρ_n if $\rho_n(D_\ell, D) \to 0$ as $\ell \to \infty$.

Definition (Convergence of fuzzy set, [23]).
For $\{\widetilde{s}_\ell\}_{\ell=1}^{\infty} \subset \mathcal{F}(\mathbb{R}^n)$ and $\widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}_\ell$ converges to $\widetilde{r}$ w.r.t. ρ_n if $\rho_n(\widetilde{s}_{\ell,\alpha}, \widetilde{r}_\alpha) \to 0$ as $\ell \to \infty$ except at most countable $\alpha \in [0, 1]$.

Lemma 6.2. *Let $\{\widetilde{s}_k\}_{k=1}^{\infty} \subset \mathcal{F}(\mathbb{R})$ and $\widetilde{s} \in \mathcal{F}(\mathbb{R})$ such that $\widetilde{s}_k \preccurlyeq_1 \widetilde{s}_{k+1}$ $(k \geq 1)$ and $\lim_{k \to \infty} \widetilde{s}_k = \widetilde{s}$. Then we have $\widetilde{s}_1 \preccurlyeq_1 \widetilde{s}$.*

Proof. Trivial. □

Lemma 6.3. *Let $\{\widetilde{s}_k\}_{k=1}^{\infty} \subset \mathcal{F}(\mathbb{R}^n)$ and $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ such that $\widetilde{s}_k \preccurlyeq_K \widetilde{s}_{k+1}$ $(k \geq 1)$ and $\lim_{k \to \infty} \widetilde{s}_k = \widetilde{s}$. Then we have $\widetilde{s}_1 \preccurlyeq_K \widetilde{s}$.*

Proof. From Theorem 4.1, for all $a \in K^+$ it holds that $a \cdot \widetilde{s}_k \preccurlyeq_1 a \cdot \widetilde{s}_{k+1}$ $(k \geq 1)$. Also, since $(a \cdot \widetilde{s}_k)_\alpha = a \cdot \widetilde{s}_{k\alpha}$ from (4.2) and $\rho_1(a \cdot \widetilde{s}_{k\alpha}, a \cdot \widetilde{s}_\alpha) \leq \|a\|\rho_n(\widetilde{s}_{k\alpha}, \widetilde{s}_\alpha)$ for all $k \geq 1$, we get $\lim_{k\to\infty} a \cdot \widetilde{s}_k = a \cdot \widetilde{s}$ where $\|a\|$ is a norm of a. By Lemma 6.2, it holds that $a \cdot \widetilde{s}_1 \preccurlyeq_1 a \cdot \widetilde{s}$ for all $a \in K^+$. From Theorem 4.1, we have $\widetilde{s}_1 \preccurlyeq_K \widetilde{s}$. □

Let K be a convex cone. The sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preccurlyeq_K$ if there exists $F, D \in \mathcal{C}(\mathbb{R}^n)$ such that $F \preccurlyeq_K D_\ell \preccurlyeq_K D$ for all $\ell \geq 1$ and said to be monotone w.r.t. $\preccurlyeq_K$ if $D_1 \preccurlyeq_K D_2 \preccurlyeq_K \cdots$.

Let $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that A is a determining class for $\mathfrak{L}$ if $a \cdot D = a \cdot F$ for all $a \in A$ and $D, F \in \mathfrak{L}$ implies $D = F$. For example, the set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ in $\mathbb{R}^n$ is a determining class for $\mathcal{C}_r(\mathbb{R}^n)$, which is the result of Theorem 5.1. Also, by the separation theorem, $\mathbb{R}^n$ is a determining class for $\mathcal{C}(\mathbb{R}^n)$.

Two example are illustrated in Figure 9.

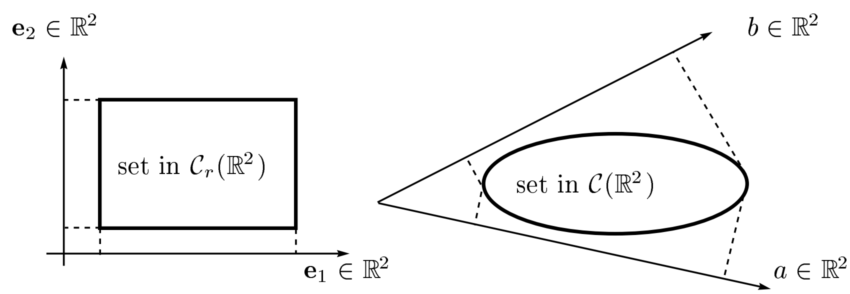

Figure 9: The example of determining class

Theorem 6.1. *Let K be a closed convex cone of $\mathbb{R}^n$. Suppose that K^+ is a determining class for $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$. Then, the pseudo order $\preccurlyeq_K$ becomes a partial order in the restricted class $\mathfrak{L}$.*

Proof. It suffices to show that $\preccurlyeq_K$ is antisymmetric in $\mathfrak{L}$. Let $D, F \in \mathfrak{L}$ satisfy that $D \preccurlyeq_K F$ and $F \preccurlyeq_K D$. By Lemma 4.4, $a \cdot D \preccurlyeq_1 a \cdot F$ and $a \cdot F \preccurlyeq_1 a \cdot D$ for all $a \in K^+$. Since $\preccurlyeq_1$ is a partial order, $a \cdot F = a \cdot D$ for all $a \in K^+$, which implies $F = D$ from the determining property of K^+. □

As a simple application of Theorem 6.1, we have the following.

Corollary 6.1. *Let K be a closed convex cone of $\mathbb{R}^n$. Suppose that K^+ is a determining class for $\mathfrak{L}$. Then, any sequence $\{D_l\} \subset \mathfrak{L}$ which is monotone w.r.t. $\preccurlyeq_K$ and satisfies $D_l \subset X$ $(l \geq 1)$ for some compact subset X of $\mathbb{R}^n$ converges w.r.t. ρ_n.*

Proof. Let $\mathcal{C}(X) = \{X \cap D \mid D \in \mathcal{C}(\mathbb{R}^n)\}$. Then, $\mathcal{C}(X)$ is compact w.r.t. ρ_n. So, the sequence $\{D_l\}$ has at most one limiting point. Since $\preccurlyeq_K$ is a partial order from Theorem 6.1, all the limiting points are equal, which completes the proof. □

In order to continue a further discussion, we need the acuteness of the ordering cone K. Then we have the following.

Lemma 6.4. *Let K be a closed, acute convex cone and $D, F, G \in \mathcal{C}(\mathbb{R}^n)$ with $D \preccurlyeq_K F \preccurlyeq_K G$. Let*

$$X := \bigcup_{\substack{x \preccurlyeq_K y \\ x \in D, y \in G}} (x+K) \cap (y-K). \tag{6.3}$$

Then, it holds that $F \subset X$ and X is bounded.

Proof. From $D \preccurlyeq_K F \preccurlyeq_K G$, for any $z \in F$, there exists $x \in D, y \in G$ such that $x \preccurlyeq_K z \preccurlyeq_K y$, which implies $z \in (x+K) \cap (y-K)$.

Now, suppose that X is unbounded. Then, there exists a sequence $\{z_t\} \subset X$ with $\| z_t \| \to \infty$ as $t \to \infty$. By $z_t \in X$, there exists $x_t \in D, y_t \in F$ with $x_t \preccurlyeq_K y_t$ and $z_t \in (x_t + K) \cap (y_t - K)$. Noting that both D and F are compact, there is no loss of generality in assuming that $x_t \to x \in D$ and $y_t \to y \in F$ as $t \to \infty$.

Since $(x_t + K) \cap (y_t - K) \to (x+K) \cap (y-K)$ as $t \to \infty$, $(x+K) \cap (y-K)$ is unbounded. However, from Lemma 6.3, $x \preccurlyeq_K y$, so that $(x+K) \cap (y-K)$ is bounded by Lemma 6.1, which leads to a contradiction. □

Theorem 6.2. *Let K be a closed, acute convex cone of $\mathbb{R}^n$ and $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$. Suppose that K^+ is a determining class for $\mathfrak{L}$. Then, any sequence $\{D_l\}_{l=1}^{\infty} \subset \mathfrak{L}$ which is bounded and monotone w.r.t. $\preccurlyeq_K$ converges w.r.t. ρ_n.*

Proof. By boundedness of the sequence $\{D_l\}$, there exists $D, G \in \mathcal{C}(\mathbb{R}^n)$ with $D \preccurlyeq_K D_l \preccurlyeq_K G$ for all $l \geq 1$. By Lemma 6.4, there exists a compact subset X of $\mathbb{R}^n$ such that $D_l \subset X$ $(l \geq 1)$. Thus, applying Corollary 6.1, the proof is completed. □

As applications of Theorem 6.2, we have the following Corollaries.

Corollary 6.2. *For any $a \in \mathbb{R}^n$ $(a \neq 0)$, let $K_a := \{\lambda a \mid \lambda \geq 0\}$. Then, any sequence of solid spheres in K_a^+ with monotonicity and boundedness w.r.t. $\preccurlyeq_{K_a}$ converges w.r.t. ρ_n.*

Corollary 6.3. *Any sequence in $\mathcal{C}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preccurlyeq_n$ converges w.r.t. ρ_n.*

For any $D \in \mathcal{C}(\mathbb{R}^n)$ and $\varepsilon > 0$, the ε-closed neighborhood of D will be denoted by

$$S_\varepsilon(D) := \{x \in \mathbb{R}^n \mid d(x, D) \leq \varepsilon\}, \tag{6.4}$$

which is a compact convex subset of $\mathbb{R}^n$. Note that

$$S_\varepsilon(D) = D + \varepsilon U_0, \tag{6.5}$$

where U_0 is the closed unit ball (cf. [2]).

The following lemma is useful in the sequel.

Lemma 6.5. *The following* (i) *to* (iii) *hold.*

(i) *For any* $D, F \in \mathcal{C}(\mathbb{R}^n)$, *if* $S_{\delta_1}(D) \subset S_{\delta_2}(F)$ *for some* $\delta_1, \delta_2 \geq 0$, *then* $S_{\delta_1+\varepsilon}(D) \subset S_{\delta_2+\varepsilon}(F)$ *for any* $\varepsilon \geq 0$.
(ii) *For any* $D \in \mathcal{C}(\mathbb{R}^n)$ *and* $\lambda > 0$, $S_\varepsilon(\lambda D) = \lambda S_{\varepsilon/\lambda}(D)$.
(iii) *For any sequence* $\{D_l\} \subset \mathcal{C}(\mathbb{R}^n)$ *and* $D \in \mathcal{C}(\mathbb{R}^n)$, *if* $D_l \to D$ *as* $l \to \infty$, *then* $S_\delta(D_l) \to S_\delta(D)$ *as* $l \to \infty$ $(\delta \geq 0)$.

Proof. For any $D, F \in \mathcal{C}(\mathbb{R}^n)$ and $\delta_1, \delta_2 \geq 0$, $S_{\delta_1}(D) \subset S_{\delta_2}(F)$ means from (6.5) that $D + \delta_1 U_0 \subset F + \delta_2 U_0$, so that $D + \delta_1 U_0 + \varepsilon U_0 \subset F + \delta_2 U_0 + \varepsilon U_0$. Since U_0 is convex, $\delta_1 U_0 + \varepsilon U_0 = (\delta_1 + \varepsilon) U_0$ and $\delta_2 U_0 + \varepsilon U_0 = (\delta_2 + \varepsilon) U_0$, which leads to $S_{\delta_1+\varepsilon}(D) \subset S_{\delta_2+\varepsilon}(F)$. Also, $S_\varepsilon(\lambda D) = \lambda D + \varepsilon U_0 = \lambda(D + (\varepsilon/\lambda) U_0)$, so that (ii) follows. For (iii), by the properties of the Hausdorff metric ρ_n (cf. [2]) $\rho_n(S_\delta(D_l), S_\delta(D)) = \rho(D_l, D)$, as required. □

For any closed convex cone $K \subset \mathbb{R}^n$, let $\mathfrak{L}(K^+)$ be the set of all $D \in \mathcal{C}(\mathbb{R}^n)$ satisfying that for any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ with $x_0 \notin S_\varepsilon(D)$ there exists $a \in K^+$ $(a \neq 0)$ such that

$$a \cdot y \geq a \cdot x_0 \quad \text{for all} \quad y \in S_\varepsilon(D).$$

The properties of $\mathfrak{L}(K^+)$ are stated in the following lemma.

Lemma 6.6. *The following* (i) *to* (iii) *hold.*

(i) K^+ *is a determining class for* $\mathfrak{L}(K^+)$.
(ii) $\mathfrak{L}(K^+)$ *is closed w.r.t.* ρ_n.
(iii) *For any* $D \in \mathfrak{L}(K^+)$, $\lambda D + \mu D \in \mathfrak{L}(K^+)$ $(\lambda, \mu \geq 0)$.

Proof. For (i), suppose that there exist $D, F \in \mathfrak{L}(K^+)$ such that $a \cdot D = a \cdot F$ for all $a \in K^+$ but there exists x_0 with $x_0 \notin D$ and $x_0 \in F$. By $x_0 \notin D$, $x_0 \notin S_\delta(D)$ for some $\delta > 0$. Then, there exists $a \in K^+$ such that $a \cdot x \geq a \cdot x_0$ for all $x \in S_\delta(D)$, so that $a \cdot x > a \cdot x_0$ for all $x \in D$, which implies $a \cdot D \neq a \cdot F$, in a contradiction.

Let $\overline{\mathfrak{L}}(K^+)$ be the closure of $\mathfrak{L}(K^+)$ w.r.t. ρ_n. For any $D \in \overline{\mathfrak{L}}(K^+)$, there exists a sequence $\{D_l\} \subset \mathfrak{L}(K^+)$ such that $\rho_n(D_l, D) \to 0$ as $l \to \infty$.

Now, for each $\varepsilon > 0$, let $x_0 \notin S_\varepsilon(D)$. Obviously there exists $\delta > 0$ with $x_0 \notin S_{\varepsilon+\delta}(D)$. By the definition of the Housdorff metric ρ_n, there exists L (depending on δ) for which $D_l \subset S_\delta(D)$ for all $l \geq L$. Applying Lemma 6.5(i), $S_\varepsilon(D_l) \subset S_{\varepsilon+\delta}(D)$, which implies $x_0 \notin S_\varepsilon(D_l)$ for all $l \geq L$.

Since $D_l \in \mathfrak{L}(K^+)$, by the definition there exists $a^l \in K^+$ such that

$$a^l \cdot x \geq a^l \cdot x_0 \quad \text{for any} \quad x \in S_\varepsilon(D_l). \tag{6.6}$$

Without loss of generality, we can assume that $\| a^l \| = 1$ and $a^l \to a \in K^+$ as $l \to \infty$. As $l \to \infty$ in (6.6), from Lemma 6.5(iii), we get that

$$a \cdot x \geq a \cdot x_0 \quad \text{for any} \quad x \in S_\varepsilon(D). \tag{6.7}$$

The above shows $D \in \mathfrak{L}(K^+)$.

For (iii), observing that $(\lambda + \mu)D = \lambda D + \mu D \quad (\lambda, \mu \geq 0)$, we only need to show $\lambda D \in \mathfrak{L}(K^+)$ for any $\lambda > 0$ and $D \in \mathfrak{L}(K^+)$. For any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ with $x_0 \notin S_\varepsilon(\lambda D)$, from Lemma 6.5(ii) it holds that $\lambda^{-1} x_0 \notin S_{\varepsilon/\lambda}(D)$. That $D \in \mathfrak{L}(K^+)$ implies that there exists $a \in K^+ \quad (a \neq 0)$ such that

$$a \cdot x \geq a \cdot (\lambda^{-1} x_0) \quad \text{for all} \quad x \in S_{\varepsilon/\lambda}(D). \tag{6.8}$$

Obviously, (6.8) leads to $a \cdot x \geq a \cdot x_0$ for any $x \in S_\varepsilon(\lambda D)$, as required. □

Noting that $K^+ = \mathbb{R}^2_+$ when $K = \mathbb{R}^2_+$ in $\mathbb{R}^2$, the sets included in $\mathfrak{L}(\mathbb{R}^2_+)$ are illustrated in Figure 10.

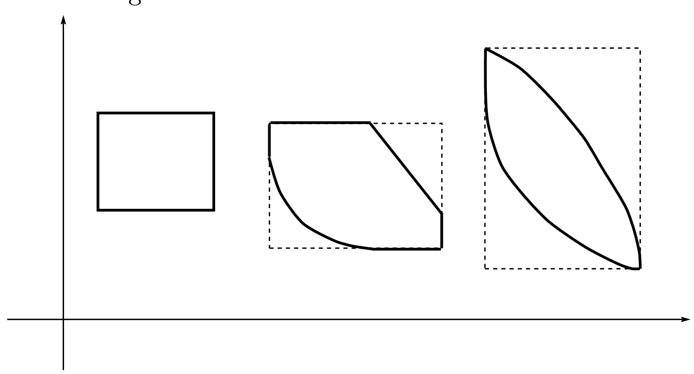

Figure 10: The example of sets in $\mathfrak{L}(\mathbb{R}^2_+)$

We have the following.

Theorem 6.3. *Let K be a closed, acute convex cone of $\mathbb{R}^n$. Then, any sequence $\{D_l\}_{l=1}^{\infty} \subset \mathfrak{L}(K^+)$ which is bounded and monotone w.r.t. $\preccurlyeq_K$ converges w.r.t. ρ_n.*

Proof. From Lemma 6.3, K^+ is a determining class for $\mathfrak{L}(K^+)$ and $\mathfrak{L}(K^+)$ is closed. Thus, by applying Theorem 6.3, the result follows. □

7 Sequences in $\mathcal{F}(\mathbb{R}^n)$

In this section, the monotone convergence theorem for a sequence in $\mathcal{F}(\mathbb{R}^n)$ is given.

Let $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we call A a determining class for $\widetilde{\mathfrak{L}}$ if $a \cdot \widetilde{s} = a \cdot \widetilde{r}$ for all $a \in A$ and $\widetilde{s}, \widetilde{r} \in \widetilde{\mathfrak{L}}$ implies $\widetilde{s} = \widetilde{r}$.

A natural extension of Theorem 6.1 to fuzzy sets will be given in the following theorem.

Theorem 7.1. *Let K be a closed convex cone of $\mathbb{R}^n$ and $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$. Suppose that K^+ is a determining class for $\widetilde{\mathfrak{L}}$. Then, a pseudo order $\preccurlyeq_K$ is a partial order in $\widetilde{\mathfrak{L}}$.*

Proof. Let us show that $\preccurlyeq_K$ is antisymmetric in $\widetilde{\mathfrak{L}}$. For any $\widetilde{s}, \widetilde{r} \in \widetilde{\mathfrak{L}}$ with $\widetilde{s} \preccurlyeq_K \widetilde{r}$ and $\widetilde{r} \preccurlyeq_K \widetilde{s}$, by Theorem 4.4 it holds that $a \cdot \widetilde{s} \preccurlyeq_1 a \cdot \widetilde{r}$ and $a \cdot \widetilde{r} \preccurlyeq_1 a \cdot \widetilde{s}$ for all $a \in K^+$.

Noting $(a \cdot \widetilde{s})_\alpha = a \cdot \widetilde{s}_\alpha$, we have $a \cdot \widetilde{s}_\alpha \preccurlyeq_1 a \cdot \widetilde{r}_\alpha$ and $a \cdot \widetilde{r}_\alpha \preccurlyeq_1 a \cdot \widetilde{s}_\alpha$ for all $a \in K^+$. Since $\preccurlyeq_1$ is a partial order on $\mathcal{C}(\mathbb{R})$, $a \cdot \widetilde{s}_\alpha = a \cdot \widetilde{r}_\alpha$ for all $a \in K^+$, which means $a \cdot \widetilde{s} = a \cdot \widetilde{r}$ for all $a \in K^+$. Thus, that K^+ is a determining class leads to $\widetilde{s} = \widetilde{r}$. $\square$

Let K be a convex cone. The sequence $\{\widetilde{s}_l\} \subset \mathcal{F}(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preccurlyeq_K$ if there exists $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R}^n)$ such that $\widetilde{u} \preccurlyeq_K \widetilde{s}_l \preccurlyeq_K \widetilde{v}$ for all $l \geq 1$ and said to be monotone w.r.t. $\preccurlyeq_K$ if $\widetilde{s}_1 \preccurlyeq_K \widetilde{s}_2 \preccurlyeq_K \cdots$.

In order to obtain the convergence theorem, we need the concept of directionality given in [23]. Denote the surface of the unit ball by $U := \{x \in \mathbb{R}^n \mid \| x \| = 1\}$. Let $V \subset U$. Then, for $D, D' \in \mathcal{C}(\mathbb{R}^n)$ with $D \subset D'$, we call D' V-directional to D (written by $D' \supset_V D$) if there exists a real $\lambda > 0, y \in D$ and $z \in D'$ such that
(i) $d(z, y) = \rho_n(D', D)$ and (ii) $z - y = \lambda v$ for some $v \in V$.

Definition (V-directional). *Let $V \subset \mathbb{R}^n$. For $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}$ is called V-directional if $\widetilde{s}_\alpha \supset_V \widetilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.*

Corollary 7.1. *Let K be a closed convex cone of $\mathbb{R}^n$ and $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$. Suppose that K^+ is a determining class for $\widetilde{\mathfrak{L}}$. Let a sequence $\{\widetilde{s}_l\} \subset \mathcal{F}(\mathbb{R}^n)$ be satisfied that*

(a) *$\{\widetilde{s}_l\}$ is bounded and monotone w.r.t. $\preccurlyeq_K$,*
(b) *each $\widetilde{s}_l$ is V-directional for a finite set $V \subset \mathbb{R}^n$ and*
(c) *there exists a compact subset D of $\mathbb{R}^n$ such that $\widetilde{s}_{l0} \subset D$ for all $l \geq 1$, where $\widetilde{s}_{l0}$ is the support or 0-cut of $\widetilde{s}_l$.*

Then the sequence $\{\widetilde{s}_l\}$ converges w.r.t. ρ_n.

Proof. In view of (a) and (c), by the argument similar to the proof of Corollary 6.1, we have that for each $\alpha \in [0,1], \widetilde{s}_{l\alpha} \to \widetilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)$ as $l \to \infty$, where $\widetilde{s}_{l\alpha}$ is the α-cut of $\widetilde{s}_l$.

We define $\{\widetilde{s}_\alpha^+\}$ by $\widetilde{s}_\alpha^+ := \cap_{\alpha' < \alpha} \widetilde{s}_{\alpha'}$ if $\alpha \in (0,1]$ and $\widetilde{s}_0^+ := \widetilde{s}_0$. Obviously, the family $\{\widetilde{s}_\alpha^+\}$ satisfies the condition (ii) of the representative theorem in Section 2. So, if $\widetilde{s}(x) = \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{\widetilde{s}_\alpha^+}(x)\}$ for all $x \in \mathbb{R}^n$, it holds $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$.

Also, by the same discussion as that in the proof of Lemma 3.2 in [23], we get that $\widetilde{s}_\alpha^+ \supset_V \widetilde{s}_\alpha$ for all $\alpha \in [0,1]$ and $\{\alpha \in [0,1] \mid \widetilde{s}_\alpha^+ \neq \widetilde{s}_\alpha\}$ is at most countable, which implies that $\widetilde{s}_l \to \widetilde{s}$ as $l \to \infty$, as required. □

The following monotone convergence theorem is thought of as an extension of Theorem 6.2 to fuzzy sets.

Theorem 7.2. *Let K be a closed, acute convex cone of $\mathbb{R}^n$ and $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$. Suppose that K^+ is a determining class for $\widetilde{\mathfrak{L}}$. Then, any sequence $\{\widetilde{s}_l\}_{l=1}^\infty \subset \widetilde{\mathfrak{L}}$ which satisfies* (a) *and* (b) *in Corollary 7.1 converges w.r.t. ρ_n.*

Proof. From the definition of boundedness of $\{\widetilde{s}_l\}$, there exists $D, G \in \mathcal{C}(\mathbb{R}^n)$ such that $D \preccurlyeq_K \widetilde{s}_{l0} \preccurlyeq_K G$ for all $l \geq 1$.

By Lemma 6.4, there exists a compact subset X of $\mathbb{R}^n$ with $\widetilde{s}_{l0} \subset X$ for all $l \geq 1$, which implies that the condition (c) in Corollary 7.1 holds. Thus, applying Corollary 7.1, Theorem 7.2 follows. □

Now, for any closed convex cone K, we define $\widetilde{\mathfrak{L}}(K^+)$ by

$$\widetilde{\mathfrak{L}}(K^+) := \{\widetilde{s} \in \mathcal{F}(\mathbb{R}^n) \mid \widetilde{s}_\alpha \in \mathfrak{L}(K^+) \text{ for all } \alpha \in [0,1]\}.$$

The previous Lemma 6.6 is extended to that for $\mathcal{F}(\mathbb{R}^n)$ in the following lemma, whose proof is easily done by α-cuts of the corresponding fuzzy sets and leaved to a reader.

Lemma 7.1. *The following* (i) *to* (iii) *hold.*

(i) *K^+ is a determining class for $\widetilde{\mathfrak{L}}(K^+)$.*
(ii) *$\widetilde{\mathfrak{L}}(K^+)$ is closed w.r.t. the convergence defined in Section 6.*
(iii) *For any $\widetilde{s} \in \widetilde{\mathfrak{L}}(K^+)$, $\lambda \widetilde{s} + \mu \widetilde{s} \in \widetilde{\mathfrak{L}}(K^+)$ $(\lambda, \mu \geq 0)$.*

We have the following.

Theorem 7.3. *Let K^+ be a closed, acute convex cone of $\mathbb{R}^n$. Then, any sequence $\{\widetilde{s}_l\}_{l=1}^\infty \subset \widetilde{\mathfrak{L}}(K^+)$ which satisfies* (a) *and* (b) *in Corollary 7.1 converges.*

Proof. From Lemma 7.1, K^+ is a determining class for $\widetilde{\mathfrak{L}}(K^+)$ and $\widetilde{\mathfrak{L}}(K^+)$ is closed. So, applying Theorem 7.2, the results follow. □

8 Applications to Monotone Dynamic Fuzzy Systems

In this section, as an application of the results obtained in the preceding section, we consider a limit theorem for a sequence of fuzzy states defined by the dynamic fuzzy system (cf. [9], [10], [20], [21], [22], [23]) with a monotone fuzzy relation.

Let $\widetilde{q} : \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ be a continuous fuzzy relation such that $\widetilde{q}(x, \cdot) \in \mathcal{F}(\mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and $\widetilde{q}(\cdot,\cdot)$ is convex, that is,

$$\widetilde{q}(\lambda x^1 + (1-\lambda)x^2, \lambda y^1 + (1-\lambda)y^2) \geq \widetilde{q}(x^1, y^1) \wedge \widetilde{q}(x^2, y^2) \tag{8.1}$$

for any $x^1, x^2, y^1, y^2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$. From this fuzzy relation $\widetilde{q}$, we define $\widetilde{q} : \mathcal{F}(\mathbb{R}^n) \to \{$the set of fuzzy sets on $\mathbb{R}^n\}$ as follows.

$$\widetilde{q}(\widetilde{u})(y) := \sup_{x \in \mathbb{R}^n} \{\widetilde{u}(x) \wedge \widetilde{q}(x,y)\}, \ \in \mathbb{R}^n, \tag{8.2}$$

where $a \wedge b = \min\{a, b\}$. Also, for any $\alpha \in [0,1]$, $\widetilde{q}_\alpha : \mathcal{C}(\mathbb{R}^n) \to 2^{\mathbb{R}^n}$ will be defined by

$$\widetilde{q}_\alpha(D) := \begin{cases} \{\ y \mid \widetilde{q}(x,y) \geq \alpha \text{ for some } x \in D\}, & \text{for } \alpha > 0,\ D \in \mathcal{C}(\mathbb{R}^n) \\ \mathrm{cl}\{\ y \mid \widetilde{q}(x,y) > 0 \text{ for some } x \in D\}, & \text{for } \alpha = 0,\ D \in \mathcal{C}(\mathbb{R}^n), \end{cases} \tag{8.3}$$

where cl denotes the closure of a set and $2^{\mathbb{R}^n}$ the set of all closed subsets of $\mathbb{R}^n$. For simplicity, we put $\widetilde{q}(x) := \widetilde{q}(\{x\})$ for $x \in \mathbb{R}^n$.

The following facts are well-known (cf. [8], [9], [23]).

Lemma 8.1 *The following* (i) *to* (iii) *hold.*

(i) $\widetilde{q}_\alpha(D) \in \mathcal{C}(\mathbb{R}^n)$ *for any* $D \in \mathcal{C}(\mathbb{R}^n)$ *and* $\widetilde{q}_\alpha(\cdot)$ *is continuous in* $\mathcal{C}(\mathbb{R}^n)$ *for each* $\alpha \in (0,1]$.
(ii) $\widetilde{q}(\widetilde{u}) \in \mathcal{F}(\mathbb{R}^n)$ *for any* $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$.
(iii) $\widetilde{q}(\widetilde{u})_\alpha = \widetilde{q}_\alpha(\widetilde{u})_\alpha$ *for any* $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$ *and* $\alpha \in [0,1]$, *where* $\widetilde{q}(\widetilde{u})_\alpha$ *is the* α*-cut of* $\widetilde{q}(\widetilde{u})$.

The sequence of fuzzy states, $\{\widetilde{s}_t\}_{t=1}^\infty \subset \mathcal{F}(\mathbb{R}^n)$, for the dynamic system with fuzzy transition $\widetilde{q}$ is defined as follows.

$$\widetilde{s}_{t+1} = \widetilde{q}(\widetilde{s}_t) \quad (t \geq 1), \tag{8.4}$$

where $\widetilde{s}_1 \in \mathcal{F}(\mathbb{R}^n)$ is the initial fuzzy state.

The problem in this section is to consider a convergence of the sequence $\{\widetilde{s}_t\}_{t=1}^\infty$ defined by (8.4), so that we derive the monotone property of the fuzzy relation $\widetilde{q}$ w.r.t. the pseudo order $\preccurlyeq_K$ defined by the ordering cone K in $\mathbb{R}^n$.

Definition ($\preccurlyeq_K$-monotone). *The fuzzy relation $\widetilde{q}$ is called $\preccurlyeq_K$-monotone if $x^1 \preccurlyeq_K x^2 \quad (x^1, x^2 \in \mathbb{R}^n)$ means $\widetilde{q}(x^1, \cdot) \preccurlyeq_K \widetilde{q}(x^2, \cdot)$.*

Remark. Yoshida et al [23] has introduced a monotone property concerning the fuzzy relation $\widetilde{q}$ whose definition is as follows: $\widetilde{q}_\alpha(y) \subset \widetilde{q}_\alpha(x) + \ell(x,y)$ for $x, y \in \mathbb{R}^n$, where $\ell(x,y) := \{\gamma(y-x) \mid \gamma \geq 0\}$. Obviously, if $\widetilde{q}$ is monotone in the sense of [23], then $\widetilde{q}$ is $\preccurlyeq_n$-monotone, but the converse is not necessarily true.

The following lemma is useful for our further discussion.

Lemma 8.2. *Suppose that $\widetilde{q}$ is $\preccurlyeq_K$-monotone. Then, for any $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R}^n)$ with $\widetilde{u} \preccurlyeq_K \widetilde{v}$, it holds that $\widetilde{q}(\widetilde{u}) \preccurlyeq_K \widetilde{q}(\widetilde{v})$.*

Proof. By Lemma 8.1(iii), for any $a \in K^+$, $a \cdot \widetilde{q}(\widetilde{u})_\alpha = a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha) \in \mathcal{C}(\mathbb{R}^1)$. So, we write it by the closed interval as follows, $a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha) := [\min a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha), \max a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha)]$ and $a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha) := [\min a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha), \max a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha)]$.

Obviously, for each $\alpha \in [0,1]$, there exists $y^1 \in \widetilde{v}_\alpha$ such that $\min a \cdot \widetilde{q}_\alpha(y^1) = \min a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha)$. From Lemma 4.3, $\widetilde{u} \preccurlyeq_K \widetilde{v}$ means $\widetilde{u}_\alpha \preccurlyeq_K \widetilde{v}_\alpha$ for all $\alpha \in [0,1]$, so that there exists $x^1 \in \widetilde{u}_\alpha$ with $x^1 \preccurlyeq_K y^1$. By $\preccurlyeq_K$-monotonicity of $\widetilde{q}$, we have $\widetilde{q}_\alpha(x^1) \preccurlyeq_K \widetilde{q}_\alpha(y^1)$, which from Lemma 6.5 implies $\min a \cdot \widetilde{q}_\alpha(x^1) \leq \min a \cdot \widetilde{q}_\alpha(y^1)$. This leads to $\min a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha) \leq \min a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha)$.

Now, by taking $x^2 \in \widetilde{u}_\alpha$ with $\max a \cdot \widetilde{q}_\alpha(x^2) = \max a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha)$, we can show by the same way as the above that $\max a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha) \leq \max a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha)$.

Then, we have $a \cdot \widetilde{q}_\alpha(\widetilde{u}_\alpha) \preccurlyeq_1 a \cdot \widetilde{q}_\alpha(\widetilde{v}_\alpha) \quad (\alpha \in [0,1])$. Applying Lemma 4.3, we get $\widetilde{q}(\widetilde{u}) \preccurlyeq_K \widetilde{q}(\widetilde{v})$, as required. □

Assumption A. *The following* (i) *to* (iii) *hold.*

(i) *The ordering cone K is a closed, acute convex one in $\mathbb{R}^n$.*
(ii) *The fuzzy relation $\widetilde{q}$ is $\preccurlyeq_K$-monotone.*
(iii) *There exists a finite subset $V \subset U$ such that, for any $D, D' \in \mathcal{C}(\mathbb{R}^n)$ $(D' \supset D)$, if $D' \supset_V D$ then $\widetilde{q}_{\alpha'}(D') \supset_V \widetilde{q}_\alpha(D)$ for all α, α' $(0 \leq \alpha' \leq \alpha \leq 1)$).*

For any given $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$, putting $\widetilde{s}_1 := \widetilde{u}$, we define the sequence $\{\widetilde{s}_t\}_{t=1}^{\infty}$ by (8.4). Then, we have the following.

Theorem 8.1. *In addition to Assumption A, suppose that the following* (iv) *to* (vi) *hold.*

(iv) *$\widetilde{u} \in \widetilde{\mathfrak{L}}(K^+)$ and $\widetilde{u} \preccurlyeq_K \widetilde{q}(\widetilde{u})$.*
(v) *$\widetilde{u}_{\alpha'} \supset_V \widetilde{u}_\alpha$ for all α, α' $(0 \leq \alpha' \leq \alpha \leq 1))$, where V is as in Assumption* A(iii).
(vi) *$\{\widetilde{s}_t\} \subset \widetilde{\mathfrak{L}}(K^+)$ and bounded from above.*

Then, the sequence $\{\widetilde{s}_t\}$ *converges and the limit* $\widetilde{s} := \lim_{t\to\infty} \widetilde{s}_t$ *satisfies the following fuzzy relational equation:*

$$\widetilde{s} = \widetilde{q}(\widetilde{s}). \tag{8.5}$$

Proof. From Lemma 8.2 and Assumption, we observe that the sequence $\{\widetilde{s}_t\}$ is bounded and $\preccurlyeq_K$-monotone. Also, we can check that all the assumption in Theorem 7.3 are satisfied. Thus, from Theorem 7.3, we can prove the theorem. □

Theorem 8.2. *In addition to Assumption A, suppose that the following* (iv'), (v) *and* (vi') *hold.*

(iv') $\widetilde{u} \in \widetilde{\mathfrak{L}}(K^+)$ *with* $\widetilde{u}_0 \subset K$ *and* $\widetilde{q}(\widetilde{u}) \preccurlyeq_K \widetilde{u}$.
(v) $\widetilde{u}_{\alpha'} \supset_V \widetilde{u}_\alpha$ *for all* α, α' $(0 \le \alpha' \le \alpha \le 1))$, *where* V *is as in Assumption* A(iii).
(vi') $\{\widetilde{s}_t\} \subset \widetilde{\mathfrak{L}}(K^+)$.

Then, the sequence $\{\widetilde{s}_t\}$ *converges and the limit* $\widetilde{s} := \lim_{t\to\infty} \widetilde{s}_t$ *satisfies the fuzzy relational equation* (8.5).

Proof. From Lemma 8.2, $\widetilde{s}_{t+1} \preccurlyeq_K \widetilde{s}_t$ and $\mathbf{1}_{\{0\}} \preccurlyeq_K \widetilde{s}_t \preccurlyeq_K \widetilde{s}_1$ for all $t \ge 1$. Applying Theorem 7.3 to the $\preccurlyeq_K$-decreasing case, we can prove the theorem. □

As an example of $\preccurlyeq_K$-monotone fuzzy relation, we put the fuzzy relation $\widetilde{q}$ by

$$\widetilde{q}(x,y) := \widetilde{r}(y) + \beta \mathbf{1}_{\{x\}} \quad (x, y \in \mathbb{R}^n), \tag{8.6}$$

where $\widetilde{r} \in \widetilde{\mathfrak{L}}(k^+)$ with $\widetilde{r}_{\alpha'} \supset_V \widetilde{r}_\alpha$ for some finite set $V \subset U$ and α, α' $(0 \le \alpha' \le \alpha \le 1)$ and $0 < \beta < 1$.

Obviously, Assumption A is satisfies for $\widetilde{q}$ of (8.6). Also, we observe from Lemma 7.1 that the assumptions (iv) to (vi) in Theorem 8.1 hold for $\widetilde{u} = \widetilde{r}$. So that by Theorem 8.1, the sequence $\{\widetilde{s}_t\}$ defined by (8.4) with $\widetilde{s}_1 = \widetilde{r}$ converges.

Remark. Note that the fuzzy relation $\widetilde{q}$ of (8.6) satisfies the contraction property introduced in [8]. Thus, we see that the limit $\widetilde{s} = \lim_{t\to\infty} \widetilde{s}_t$ is a unique solution of the fuzzy relational equation (8.5) and given by $\widetilde{s} = (1-\beta)^{-1}\widetilde{r}$.

Example. We give a one-dimensional numerical example whose fuzzy relation $\widetilde{q}$ is given by

$$\widetilde{q}(x,y) = (1 - 2|y - (3 - x^{-2})|) \vee 0 \quad (x > 0).$$

For $\alpha \in [0,1]$, it holds that by (8.3)

$$\widetilde{q}_\alpha(x) = [3 - (1-\alpha)2^{-1} - x^{-2}, 3 + (1-\alpha)2^{-1} - x^{-2}].$$

This is illustrated in Figure 11. So, we observe that $\widetilde{q}$ is $\preccurlyeq_1$-monotone in $(0,\infty) \times (0,\infty)$, also that $\mathbf{1}_{\{1\}} \preccurlyeq_1 \widetilde{q}(\mathbf{1}_{\{1\}})$ and $\widetilde{q}(x,\cdot) \preccurlyeq_1 \mathbf{1}_{\{7/2\}}(x)$.

Applying Theorem 8.1, the sequence $\{\widetilde{s}_t(x)\}$ defined by (8.4) with $\widetilde{s}_1(x) = \mathbf{1}_{\{1\}}(x)$ converges. The convergence is shown in Figure 11 and 12 with the limit $\widetilde{s}(x) = \lim_{t\to\infty} \widetilde{s}_t(x)$, where the α-cut $\widetilde{s}_\alpha$ of the limit $\widetilde{s}(x)$ for $\alpha = 0$ and $\alpha = 1$ are $\min \widetilde{s}_0 = 2.313099034$, $\max \widetilde{s}_0 = 3.414213562$ and $\widetilde{s}_1 = 2.879385242$.

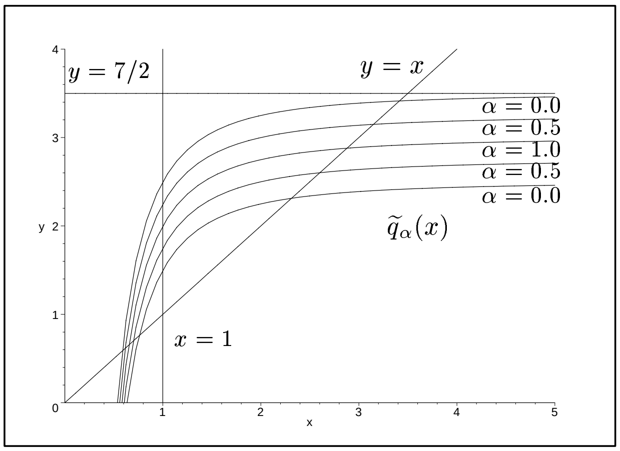

Figure 11: $\widetilde{q}_\alpha(x)$ and the limit $\widetilde{s}(x)$ of $\{\widetilde{s}_t(x)\}$

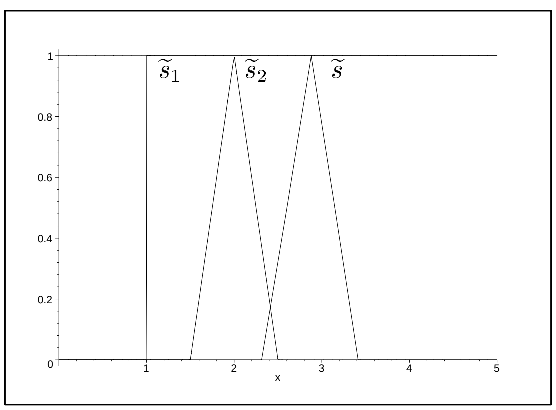

Figure 12: The sequence $\{\widetilde{s}_t(x)\}$

References

1. Congxin, W., Cong, C. (1997) The supremum and infimum of the set of fuzzy numbers and its application. J. Math. Anal. Appl. **210**, 499–511
2. Diamond, P., Kloeden, P. (1994) Metric Spaces of Fuzzy Sets, Theory and Applications. World Scientific
3. Dubois, D., Prade, H. (1980) Fuzzy Sets and Systems: Theory and Applications. Academic Press
4. Furukawa, N. (1993) An order relation and fundamental operations on fuzzy numbers and their applications to fuzzy linear programming. Bulletin Infor. Cybernetics **25**, 161-179
5. Furukawa, N. (1993) A parametric total order on fuzzy numbers and a fuzzy shortest route problem. Optimization **30**, 367-377
6. Furukawa, N. (1997) Parametric orders on fuzzy numbers and their roles in fuzzy optimization problems. Optimization **40**, 171–192
7. Kaufmann, A., Gupta, M. M. (1988) Fuzzy Mathematical Models in Engineering and Management Science. North-Holland
8. Kurano, M., Yasuda, M., Nakagami, J., Yoshida, Y. (1992) A limit theorem in some dynamic fuzzy systems. Fuzzy Sets and Systems **51**, 83–88
9. Kurano, M., Yasuda, M., Nakagami, J., Yoshida, Y. (1996) Markov-type fuzzy decision processes with a discounted reward on a closed interval. European Journal of Operations Research **92**, 649-662
10. Kurano, M., Yasuda, M., Nakagami, J., Yoshida, Y. (1999) The time average reward for some dynamic fuzzy systems. Computers and Mathematics with Applications **37**, 77-86
11. Kurano, M., Yasuda, M., Nakagami, J., Yoshida, Y. (2000) Ordering of fuzzy sets – A brief survey and new results. Journal of Operations Research Society of Japan **43**, 138-148
12. Kuroiwa, D., Tanaka, T., Ha, T. X. D. (1997) On cone convexity of set-valued maps. Nonlinear Analysis, Theory and Applications **30**, 1487-1496
13. Li, H. X., Yen, V. C. (1995) Fuzzy Sets and Fuzzy Decision-making. CRC Press
14. Nanda, S. (1989) On sequences of fuzzy numbers. Fuzzy Sets and Systems **33**, 123-126
15. Novák, V. (1989) Fuzzy Sets and Their Applications. Adam Hilder, Bristol-Boston
16. Ramík, J., Řimánek, J. (1985) Inequality relation between fuzzy numbers and its use in fuzzy optimization. Fuzzy Sets and Systems **16**, 123-138
17. Stoer, J., Witzgall, C. (1970) Convexity and Optimization in Finite Dimensions I. Springer-Verlag, Berlin and New York
18. Stowinski, R. :editor (1998) Fuzzy Sets in Decision Analysis, Operations Research and Statistics. Kluwer Academic Publishers
19. Syau, Y. R. (1997) Sequences in a fuzzy metric space. Computers and Mathematics with Applications **33**, 73-76
20. Yoshida, Y., Yasuda, M., Nakagami, J., Kurano, M. (1993) A potential of fuzzy relations with a linear structure: The contractive case. Fuzzy sets and Systems **60**, 283-294
21. Yoshida, Y., Yasuda, M., Nakagami, J., Kurano, M. (1994) A potential of fuzzy relations with a linear structure: The unbounded case. Fuzzy sets and Systems **66**, 83-96

22. Yoshida, Y. (1998) A time-average fuzzy reward criterion in fuzzy decision processes. Information Science **110**, 103-112
23. Yoshida, Y., Yasuda, M., Nakagami, J., Kurano, M. (1998) A limit theorem in dynamic fuzzy systems with a monotone property. Fuzzy sets and Systems **94**, 109-119
24. Zadeh, L. A. (1965) Fuzzy sets. Information and Control**8**, 338-353
25. Zhang, K., Hirota, K. (1997) On fuzzy number lattice $(\tilde{R}, \preccurlyeq)$. Fuzzy sets and Systems **92**, 113-122