

GOLDEN OPTIMAL VALUE IN DISCRETE-TIME DYNAMIC OPTIMIZATION PROCESSES

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Abstract. Aesthetics feature fascinates mathematician. Since ancient times, the Golden Ratio (ϕ) has been keeping to give a profound influence in various fields. We will show typical dynamic programming problems: Allocation problem, Linear-Quadratic control problem and Multi-variate stopping problem. For these problems, there appears the Golden Ratio in the solution of Bellman equation. This paper also considers a minimization problem of quadratic functions over an infinite horizon. We show that the Golden trajectory is optimal in the optimization. The Golden optimal trajectory is obtained through the corresponding Bellman equation, which in turn admits the Golden optimal policy. For Multi-variate stopping problem with three players on the unit interval $[0, 1]$, its related expected value could be obtained as ϕ^{-1} .

1. Introduction. The Golden Ratio ($\phi = 1.161803 \dots$) has been a profound influence since ancient times such as the Parthenon at Athens. The shape of the Golden Ratio is supposed to be interesting in a graphic forms for their sculptures and paintings. The beauty appears even in the ingredient of nature creatures. The most influential mathematics textbook by Euclid of Alexandria defines the proportion. These information presents a broad sampling of ϕ -related topics in an engaging and easy-to-understand format.

The Fibonacci sequence $(1, 1, 2, 3, 5, 8, 13, \dots)$ is closely related to the Golden Ratio, which is a limiting ratio of its two adjacent numbers. It is also known that the diagonal

2000 *Mathematics Subject Classification*: Primary 90C39; Secondary 11A99.

Key words and phrases: Golden Ratio, Dynamic Programming, Allocation Problem, LQ Control Problem, Monotone Stopping Problem.

The paper is in preliminary form and the final version of it will be published elsewhere.

summation produces the Fibonacci sequence in the Pascal's triangle. These repeated procedure or iteration have something in common.

The principle of Dynamic Programming is said to 'divide and conquer.' In fact, if it is not possible to work out directly, divide up a problem into smaller ones. The basic idea is aiming to provide the original problem an effective family of sub-problems. The Bellman's curse of dimensionality conquers the computational explosion with the problem dimension through the use of parametric representations. The more it's in a complex, the more it is divided. When a problem is in a multi-stage decision form, we should consider the problem repeatedly. If this reduction procedure gives a self-similar one, the methodology turns out to be effective. The Golden Ratio is created repeatedly by its own in a quite same form. A recurrence relation is ubiquitous. Let us say that a beautiful continued fraction represents the Golden number. It is interesting that this quite introductory problem of Dynamic programming produces the basic mathematical aspects.

In the following sections, we treat typical dynamic programming problems; Allocation problem and Linear-Quadratic Control problem. However problems are in a simple fashion, it figures out the essence of Dynamic Programming.

Let us consider a typical type of criterion in a deterministic optimization. We minimize the next quadratic criteria :

$$(1.1) \quad I(x) = \sum_{n=0}^{\infty} [x_n^2 + (x_n - x_{n+1})^2],$$

and

$$(1.2) \quad J(x) = \sum_{n=0}^{\infty} [(x_n - x_{n+1})^2 + x_{n+1}^2].$$

Let R^∞ be the set of all sequences of real values :

$$R^\infty = \{x = (x_0, x_1, \dots, x_n, \dots) \mid x_n \in R^1 \ n = 0, 1, \dots\}.$$

First we take the quadratic criterion (1.1).

Now we consider a mathematical programming problem for any given initial value c :

$$\text{MP}_1(c) : \text{ minimize } I(x) \text{ subject to (i) } x \in R^\infty, \text{ (ii) } x_0 = c.$$

Let us evaluate a few special trajectories :

EXAMPLE 1. *First all but on and after nothing* $y = \{c, 0, 0, \dots, 0, \dots\}$ yields $I(y) = 2c^2$.

EXAMPLE 2. *Always all constant* $z = \{c, c, \dots, c, \dots\}$ yields $I(z) = \infty$.

EXAMPLE 3. *A proportional (geometrical)* $w = \{c, \rho c, \dots, \rho^n c, \dots\}$ yields

$$(1.3) \quad \begin{aligned} I(w) &= \{c^2 + (1 - \rho)^2 c^2\} (1 + \rho^2 + \dots + \rho^{2n} + \dots) \\ &= \frac{1 + (1 - \rho)^2}{1 - \rho^2} c^2 \quad (0 < \rho < 1). \end{aligned}$$

Since

$$\min_{0 \leq \rho < 1} \frac{1 + (1 - \rho)^2}{1 - \rho^2}$$

is attained at $\hat{\rho} = 2 - \phi$, we have the minimum value

$$\frac{1 + \{1 - (2 - \phi)\}^2}{1 - (2 - \phi)^2} = \phi.$$

EXAMPLE 4. The propotional $\hat{u} = \{c, (2 - \phi)c, \dots, (2 - \phi)^n c, \dots\}$, with ratio $(2 - \phi)$, yields

$$(1.4) \quad I(\hat{u}) = \phi c^2.$$

Thus $\hat{u} = (\hat{u}_n)$ gives a Golden optimal trajectory, because \hat{u}_{n+1} is always a Golden section point of interval $[0, \hat{u}_n]$.

Next let us now consider a control process with an additive transition $T(x, u) = x + u$.

$$\begin{aligned} & \text{minimize} \quad \sum_{n=0}^{\infty} (x_n^2 + u_n^2) \\ & \text{subject to} \quad \begin{aligned} & \text{(i)} \quad x_{n+1} = x_n + u_n, \quad n \geq 0 \\ & \text{(ii)} \quad -\infty < u_n < \infty \\ & \text{(iii)} \quad x_0 = c. \end{aligned} \end{aligned}$$

Then the value function v satisfies Bellman equation :

$$(1.5) \quad v(x) = \min_{-\infty < u < \infty} [x^2 + u^2 + v(x + u)].$$

Eq. (1.5) has a quadratic form $v(x) = \phi x^2$.

Second we take the following quadratic criterion

$$J(x) = \sum_{n=0}^{\infty} [(x_n - x_{n+1})^2 + x_{n+1}^2].$$

We consider a problem of form:

$$\text{MP}_2(c) : \text{minimize } J(x) \text{ subject to (i) } x \in R^\infty, \text{ (ii) } x_0 = c.$$

Since $J(x) = I(x) - c^2$, the minimum value is $J(\hat{u}) = (\phi - 1)c^2$ at the Golden trajectory

$$\hat{u} = \{c, \mu c, \dots, \mu^n c, \dots\}; \quad \mu = 2 - \phi.$$

In fact, a proportional $w = \{c, \rho c, \dots, \rho^n c, \dots\}$ yields

$$\begin{aligned} J(w) &= \{\rho^2 c^2 + (1 - \rho)^2 c^2\} (1 + \rho^2 + \dots + \rho^{2n} + \dots) \\ &= \frac{\rho^2 + (1 - \rho)^2}{1 - \rho^2} c^2 \quad (0 < \rho < 1). \end{aligned}$$

Figure 1 in the Appendix shows that

$$\min_{0 \leq x < 1} \frac{x^2 + (1 - x)^2}{1 - x^2}$$

is attained at $\hat{x} = 2 - \phi$ with the minimum value

$$\frac{(2 - \phi)^2 + \{1 - (2 - \phi)\}^2}{1 - (2 - \phi)^2} = \phi - 1.$$

2. An Illustrative Graph. Let us now describe a graph which has dual Golden extremum points in the previous section. The graph is $x = f(u) = \frac{u^2 + (1-u)^2}{1-u^2}$. See Figure 1 in Appendix. For this function, it is seen that two equalities:

$$(2.1) \quad \min_{0 < u < 1} f(u) = \min_{0 < u < 1} \frac{u^2 + (1-u)^2}{1-u^2} = -1 + \phi$$

and $\max_{1 < u < \infty} f(u) = \max_{1 < u < \infty} \frac{u^2 + (1-u)^2}{1-u^2} = -\phi$ hold. Equivalently, the latter equality is that

$$(2.2) \quad \min_{1 < u < \infty} \{-f(u)\} = \min_{1 < u < \infty} \frac{u^2 + (1-u)^2}{u^2 - 1} = \phi$$

The minimum in (2.1) attains iff $\hat{u} = 2 - \phi$, and the minimum in (2.2) attains iff $u^* = 1 + \phi$. Thus we have the inequality

$$f(u) \geq -1 + \phi \quad \text{on } (-1, 1) \quad \text{and} \quad f(u) \leq -\phi \quad \text{on } (-\infty, -1) \cup (1, \infty).$$

Refer to the shape for this graph in the Figure 1 of Appendix.

3. Dynamic Programming of the discrete-time system. The conceptual cluster of Dynamic Programming are investigated throughout the mathematics. Not only the analytical aspect of optimization method, but also the investigate problem with a repeated structure. To give a useful explanation and an interesting implication, we show some explicitly solvable problems.

First the general setting of Dynamic Programming problem are illustrated. It is composed as (S, A, r, T) . Let S be a state space in the Euclidean space R and $A = (A_x)$, $A_x \subset R$, $x \in S$ means a feasible action space depending on a current state $x \in S$. The immediate reward is a function of $r = r(x, a, t)$, $x \in S$, $a \in A_x$, $t = 0, 1, 2, \dots$. And the terminal reward $K = K(x)$, $x \in S$ is given. The transition law from the current x to the new $y = m(x, a, t)$ by the action or decision $a \in A_x$ at a time t . If the transition law $m(x, a, t)$ does not depend t , it is called a stationary $m(x, a, t)$ and we treat it in this paper. Here we consider additive costs and the optimal value of a_t will be depend on the decision history. Assume its value at time t denoted by x_t , which enjoy the following properties:

- (a) The value of x_t is observable at time t .
- (b) The sequence $\{x_t\}$ follows a recurrence in time:

$$(3.1) \quad x_{t+1} = m(x_t, a_t, t).$$

It is termed that the function $y = m(x, \pi(x, t), t)$ means a move from the current x to the next y at t so the law of motion or the plant equation by adapting a policy $a = \pi(x, t)$.

- (c) The set of a_t may adopt depends on x_t and t .
- (d) The cost function $C_\pi(x, t)$ starting a state x at time-to-go $T_t = T - t$ to optimize over all policies π has the additive form;

$$(3.2) \quad C_\pi(x_0, t_0) = \sum_{t=t_0}^{T-1} r(x_t, a_t, t) + K(x_T)$$

with $x_0 = x_{t_0}$ and

$$x_{t+1} = m(x_t, \pi(x_t, t), t), \quad a_t = \pi(x_t, t)$$

where T is a given finite planning horizon.

Let

$$F(x, t) = \inf_{\pi} C_{\pi}(x, t).$$

It is well known that the sequence $\{F(\cdot, t)\}$ satisfies the optimality equation (DP equation):

$$(3.3) \quad F(x, t+1) = \inf_{a \in A_x} [r(x, a, t) + F(m(x, a, T_t), t)]$$

with the boundary $F(x, T) = K(x)$ where $T_t = T - t$ for $x \in S$, $0 \leq t < T$.

All of these are referred from text books by Bertsekas [2], Whittle [15], Sniedovich [14], etc.

The relation between The Golden ratio formula and Fibonacci sequence is known as [4] etc. To produce the Fibonacci sequence, it is a good example in a recursive programming [13]. Also the Fibonacci sequence are related with continued fraction. For the notation of continued fraction, we adopt ourself to the following notations:

$$b_0 + \frac{c_1}{b_1 + \frac{c_2}{b_2 + \frac{c_3}{b_3 + \frac{c_4}{b_4 + \dots}}}} = b_0 + \frac{c_1}{b_1 +} \frac{c_2}{b_2 +} \frac{c_3}{b_3 +} \frac{c_4}{b_4 +} \dots$$

Note that the Golden number satisfies $\phi^2 = 1 + \phi$. By using this relation repeatedly, $\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$. Similarly the

reciprocal (or sometimes called as a dual Golden number) is denoted $\phi^{-1} = 0.618\dots = \phi - 1 = \frac{1}{\phi} = \frac{1}{1 + \phi} = \frac{1}{1 + \frac{1}{1 + \phi}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$.

This reproductive property suggests our fundamental claim for the following typical example of Dynamic Programming. Before we solve the problem, let us induce a sequence $\{\phi_n\}$ as

$$(3.4) \quad \phi_{n+1} = 1 + \frac{1}{1 + 1/\phi_n} = 1 + \frac{1}{1 + \frac{1}{\phi_n}} \quad (n \geq 1), \quad \phi_1 = 1.$$

Also let $\{\hat{\phi}_n\}$ as

$$(3.5) \quad \begin{aligned} \hat{\phi}_{n+1} &= \frac{1}{1 + \frac{1}{\hat{\phi}_n}} \quad (n \geq 1), \quad \hat{\phi}_1 = 1, \\ i.e. \\ \hat{\phi}_{n+1}^{-1} &= \frac{1}{\hat{\phi}_{n+1}} = 1 + \frac{1}{1 + \hat{\phi}_n} = 1 + \frac{1}{1 + \hat{\phi}_n} \quad (n \geq 1). \end{aligned}$$

The sequence $\{\phi_n\}$ of (3.4) satisfies

$$\begin{aligned}\phi_{n+1} &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\phi_{n-1}}}}} \\ &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\phi_{n-2}}}}}}\end{aligned}$$

Similarly $\{\hat{\phi}_n\}$ of (3.5) satisfies

$$\begin{aligned}\frac{1}{\hat{\phi}_{n+1}} &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\hat{\phi}_{n-1}}}} \\ &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\hat{\phi}_{n-2}}}}}\end{aligned}$$

From this definition, it is seen easily that

$$(3.6) \quad \lim_{n \rightarrow \infty} \phi_n = \phi = (\sqrt{5} + 1)/2$$

$$(3.7) \quad \lim_{n \rightarrow \infty} \hat{\phi}_n = 1/\phi = (\sqrt{5} - 1)/2$$

4. Linear-Quadratic Control problem. The Linear Quadratic (LQ) control problem is to minimize the quadratic cost function over the linear system. If the state of system $\{x_t\}$ moves on

$$(4.1) \quad x_{t+1} = x_t + a_t, \quad t = 0, 1, 2, \dots$$

with $x_0 = 1$ by an input control $\{a_t, -\infty < a_t < \infty\}$. The cost incurred is

$$(4.2) \quad \sum_{t=0}^{T-1} (x_t^2 + a_t^2) + x_T^2.$$

Then DP equation of LQ is

$$(4.3) \quad v_{t+1}(x) = \min_{a \in A_x} \{r(a, x) + v_t(a + x)\}$$

where

$$\begin{aligned}r(a, x) &= a^2 + x^2, \\ a &\in A_x = (-\infty, \infty), x \in (-\infty, \infty)\end{aligned}$$

THEOREM 4.1. *The solution of (4.3) is given by*

$$\begin{cases} v_0(x) = \phi_T x^2 \\ v_t(x) = \phi_{T-t} x^2, t = 1, 2, \dots \end{cases}$$

using the Golden number related sequence $\{\phi_n\}$ by (3.4).

(Proof) The proof is immediately obtained by an elementary quadratic minimization and then the mathematical induction. \square

5. Allocation problem. Allocation problem or sometime called as partition problem is of the form

$$(5.1) \quad v_{t+1}(x) = \min_{a \in A_x} \{r(a, x) + v_t(a)\}$$

for $t = 0, 1, 2, \dots, T$, where

$$\begin{aligned} r(a, x) &= a^2 + (x - a)^2, \\ a &\in A_x = [0, x], x \in (-\infty, \infty). \end{aligned}$$

THEOREM 5.1. *The solution of (5.1) is given by using the dual golden number as*

$$\begin{cases} v_0(x) = \hat{\phi}_T x^2 \\ v_t(x) = \hat{\phi}_{T-t} x^2, t = 1, 2, \dots \end{cases}$$

(Proof) Using the Schwartz inequality, the following holds immediately: For given positive constants A and B with a fixed x ,

$$\min_{0 \leq a \leq 1} \{Aa^2 + B(x - a)^2\} = \frac{x^2}{1/A + 1/B}.$$

So the proof could be done inductively. \square

Remark 1 : We note here that the number $\phi^{-1} = 0.618\dots$ of reciprocal of the Golden number is called sometimes *Dual Golden number*. The above two problems are closely related.

Remark 2 : It is seen that the same quadratic function of the form; $v(x) = cx^2$ where c is a constant, becomes a solution if the DP equation is, for Allocation and LQ,

$$(5.2) \quad v_{t+1}(x) = \min_{a \in A_x} \left\{ r(a, x) + 2 \int_0^a v_t(y)/y \, dy \right\}$$

$$(5.3) \quad v_{t+1}(x) = \min_{a \in A_x} \left\{ r(a, x) + 2 \int_0^{a+x} v_t(y)/y \, dy \right\}$$

respectively. Refer to [9].

6. Monotone Stopping Game. A monotone rule is introduced to sum up individual declarations in a multi-variate stopping problem [16]. The rule is defined by a monotone logical function and is equivalent to the winning class of Kadane [12]. There given p -dimensional random process $\{X_n; n = 1, 2, \dots\}$ and a stopping rule π by which the group decision determined from the declaration of p players at each stage. The stopping rule is p -variate $\{0, 1\}$ -valued monotone logical function. We consider two cases of rules with $p = 3$ as follows:

$$(6.1) \quad \pi(x_1, x_2, x_3) = x_1 + x_2$$

and

$$(6.2) \quad \pi(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3.$$

That is, in case of the case (6.1), if either of player 1 and 2 declares stop, then the system stops neglecting of x_3 . In case of (6.1), we have the system stops when either of player 1 and 2 declares stop accompanying with player 1. Without loss of generality, we can assume that each X_n takes the uniformly distribution on $[0, 1]$. Then equilibrium expected values for each player is given as

x_1	x_2	$\pi(x_1, x_2, x_3)$
0	0	0
0	1	1
1	0	1
1	1	1

Figure 1. $\pi(x_1, x_2, x_3) = x_1 + x_2$ for any x_3

x_1	x_2	x_3	$\pi(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Figure 2. $\pi(x_1, x_2, x_3) = x_1x_2 + x_1x_3$

	Player 1	Player 2	Player 3
$\pi(x_1, x_2, x_3) = x_1x_2 + x_1x_3$	$\frac{\sqrt{5}-1}{2}$	$\frac{\sqrt{5}-1}{2}$	0.5
$\pi(x_1, x_2, x_3) = x_1x_2 + x_1x_3$	1	$\frac{\sqrt{5}-1}{2}$	$\frac{\sqrt{5}-1}{2}$

Table 1. The equilibrium expected value for each players.

In order to derive the value $\phi^{-1} = \frac{\sqrt{5}-1}{2}$, we consider an equilibrium stopping strategy of threshold type in the form $\{X_n > a\}$ for some a . Bellman type equation for this game version will be given as [16]. That is, each player declares “stop” or “continue” if the observed value exceeds some a or not. The event of the occurrence is denoted by $D_n^i = \{\text{Player } i \text{ declares stop}\}$. Two trivial cases are the whole event Ω and the empty event \emptyset .

In generally a logical function is assumed “monotone” so its function can be written as

$$\pi(x^1, \dots, x^p) = x^i \cdot \pi(x^1, \dots, \overset{i}{1}, \dots, x^p) + \overline{x^i} \cdot \pi(x^1, \dots, \overset{i}{0}, \dots, x^p) \quad x^i \in \{0, 1\} \quad \forall i$$

where $\overline{x^i} = 1 - x^i$. Corresponding to this expression,

$$\Pi(D^1, \dots, D^p) = D^i \cdot \Pi(D^1, \dots, \overset{i}{\Omega}, \dots, D^p) + \overline{D^i} \cdot \Pi(D^1, \dots, \overset{i}{\emptyset}, \dots, D^p)$$

where $\overline{D^i}$ is the complement of the event D^i . The general equation for the expected for player i equals

$$(6.3) \quad E \left[(X_n^i - v^i)^+ \mathbf{1}_{\Pi(D_n^1, \dots, \overset{i}{\Omega}, \dots, D_n^p)} \right] + E \left[(X_n^i - v^i)^- \mathbf{1}_{\Pi(D_n^1, \dots, \overset{i}{\emptyset}, \dots, D_n^p)} \right]$$

where $D_n^i = \{X_n^i \geq v^i\}$ and $(x)^+ = \max\{x, 0\}$, $(x)^- = \min\{x, 0\}$. If we assume an independence case between player's random variable X_n^i for each i . The equation (6.3) becomes as

$$(6.4) \quad \beta_n^{\Pi(i)} E[(X_n^i - v^i)^+] + \alpha_n^{\Pi(i)} E[(X_n^i - v^i)^-]$$

where rule(a)Our objection is to find an equilibrium strategy and values of playes for a given monotone rule as the rule (6.1) and (6.2). A sequence of expected value (a net gain) under the situation formulated in the section is obtained as

$$(6.5) \quad v_{n+1}^i = v_n^i + \beta_n^{\Pi(i)} E[(X_n^i - v_n^i)^+] + \alpha_n^{\Pi(i)} E[(X_n^i - v_n^i)^-]$$

for player $i = 1, \dots, p$ and n denotes a time-to-go. The details refer to Theorem 2.1 in YKN [16]. Under these derivation, now we are able to calculate the optimal (equilibrium) value $v^i = \lim_n v_n^i$ for player $i = 1, 2, 3$ for the rule (6.1) and (6.2) in the table .

7. Appendix.

$$f(u) = \frac{u^2 + (1-u)^2}{1-u^2}$$

$$f'(u) = (-2) \frac{u^2 - 3u + 1}{(u^2 - 1)^2}$$

$$f''(u) = 2 \frac{2u^3 - 9u^2 + 6u - 3}{(u^2 - 1)^3}$$

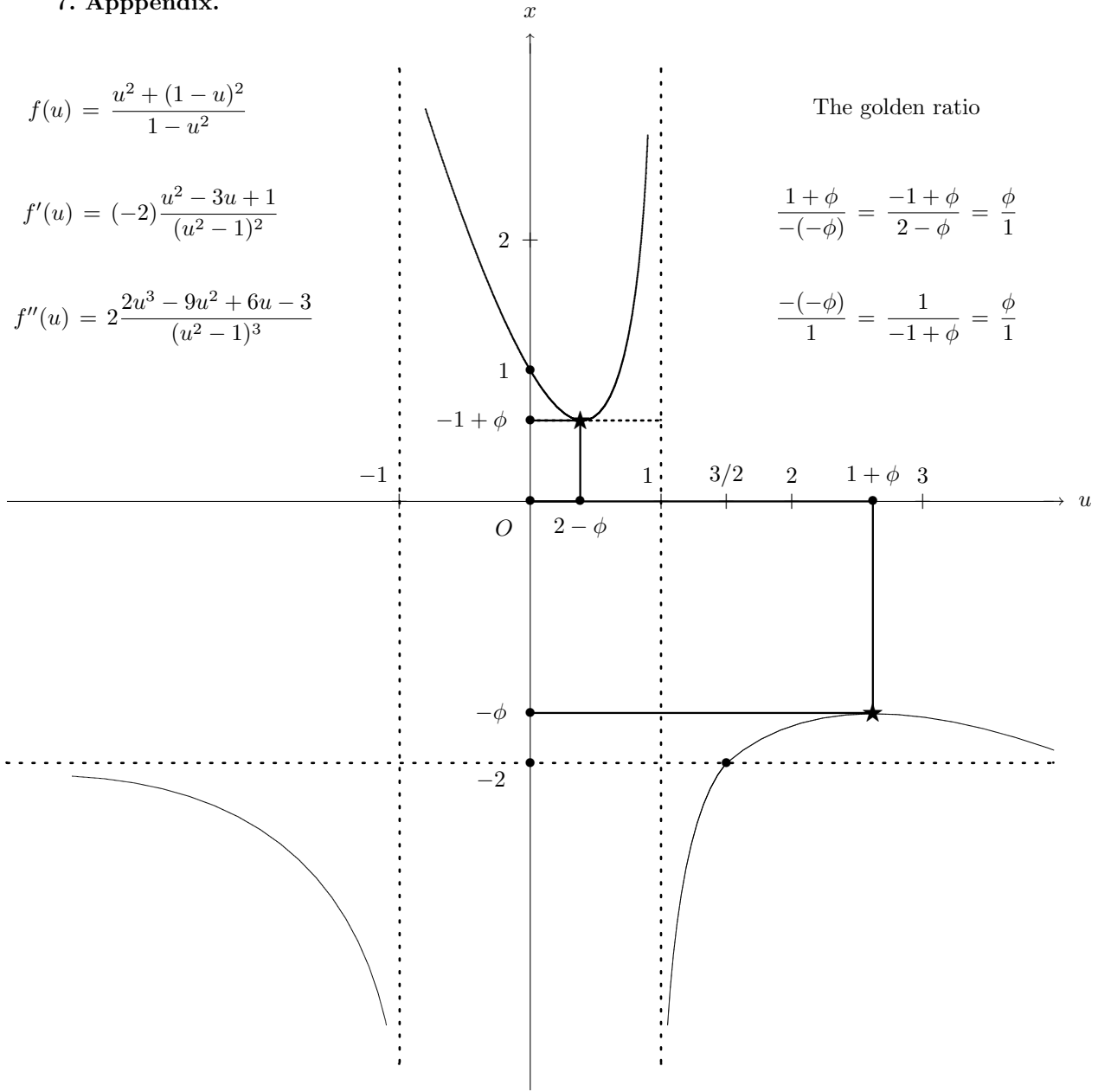


Figure 1. The curve $x = f(u)$ has dual golden extremum points with a marked ★.

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