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GOLDEN DUALITY IN DYNAMIC OPTIMIZATION

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Abstract. This paper discusses the dual of infinite-variable quadratic minimization (primal) problems from a view point of Golden ratio. We consider two pairs of primal and dual (maximization) problems. One pair yields the Golden duality. While the minimum value function is Golden quadratic and the minimum point constitutes a Golden path, so is the maximum value function and the maximum point does such another. The other yields the inverse-Golden duality. While the minimum value function is inverse-Golden quadratic and the minimum point constitutes the same Golden path, so is the maximum value function and the maximum point does the same such another.

1. Introduction

The Golden ratio is one of the most beautiful numbers. The desire for optimality is inherent for humans. One minimization leads to the other maximization, which arrives at a duality. We direct our attention to both the Golden ratio and the duality. A duality of fine features is shown.

In this paper, we are concerned with dynamic optimization problems of infinitely many variables from a viewpoint of Golden duality. We take two typical quadratic minimization (primal) problems with initial condition and associate each problem with a quadratic maximization (dual) problem with transversality condition. The two pairs of primal and

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dual problems have interesting characteristics. As for the first pair, the minimum value function is Golden quadratic and the minimum point constitutes a Golden path, while so is the maximum value function and the maximum point does such another. As for the second, the minimum value function is inverse-Golden quadratic and the minimum point constitutes the same Golden path, while so is the maximum value function and the maximum point does the same such another.

2. Duality

A real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is called *Golden number* [3, 4, 17]. It is the larger of the two solutions to quadratic equation (QE)

$$x^2 - x - 1 = 0. \tag{1}$$

Sometimes QE (1) is called *Fibonacci* or *Golden*. The Golden QE has two real solutions: ϕ and its *conjugate* $\overline{\phi} := 1 - \phi$. We note that

$$\phi + \overline{\phi} = 1, \qquad \phi \cdot \overline{\phi} = -1.$$

Further we have

$$\begin{aligned} \phi^{-1} &= \phi - 1, & (\overline{\phi})^{-1} &= -\phi \\ \phi^{-1} + (\overline{\phi})^{-1} &= -1, & \phi^{-1} \cdot (\overline{\phi})^{-1} &= -1 \\ \phi^2 &= 1 + \phi, & \overline{\phi}^2 &= 2 - \phi \\ \phi^2 + \overline{\phi}^2 &= 3, & \phi^2 \cdot \overline{\phi}^2 &= 1. \end{aligned}$$

A quadratic function $v(x) = ax^2$ is called *Golden* (resp. *inverse-Golden*) if $a = \phi$ (resp. ϕ^{-1}). In this section, we consider two pairs of primal and dual problems. One pair yields a duality for the Golden quadratic function. The other pair yields a duality for the inverse-Golden quadratic function.

2.1. *Golden duality*

We take an interval $[0, x]$, where $x > 0$. Let us consider the set of all divisions of the interval $[0, x]$. Each division is specified by an inner point $y \in [0, x]$, which splits the interval $[0, x]$ into two intervals $[0, y]$ and $[y, x]$. A point $(2 - \phi)x$ splits the interval into two intervals $[0, (2-\phi)x]$ and $[(2-\phi)x, x]$. A point $(\phi - 1)x$ splits it into $[0, (\phi-1)x]$ and $[(\phi-1)x, x]$. In either case, the length constitutes the Golden ratio $(2-\phi) : (\phi-1) = 1 : \phi$. Thus both divisions are the *Golden section* [3, 4, 17].

DEFINITION 1. [14] A sequence $x : \{0, 1, \dots\} \to R^1$ is called *Golden path* if and only if either

$$\frac{x_{t+1}}{x_t} = \phi - 1 \quad \text{or} \quad \frac{x_{t+1}}{x_t} = 2 - \phi.$$

LEMMA 1. [14] A Golden path x is either

$$x_t = x_0(\phi - 1)^t \quad \text{or} \quad x_t = x_0(2 - \phi)^t.$$

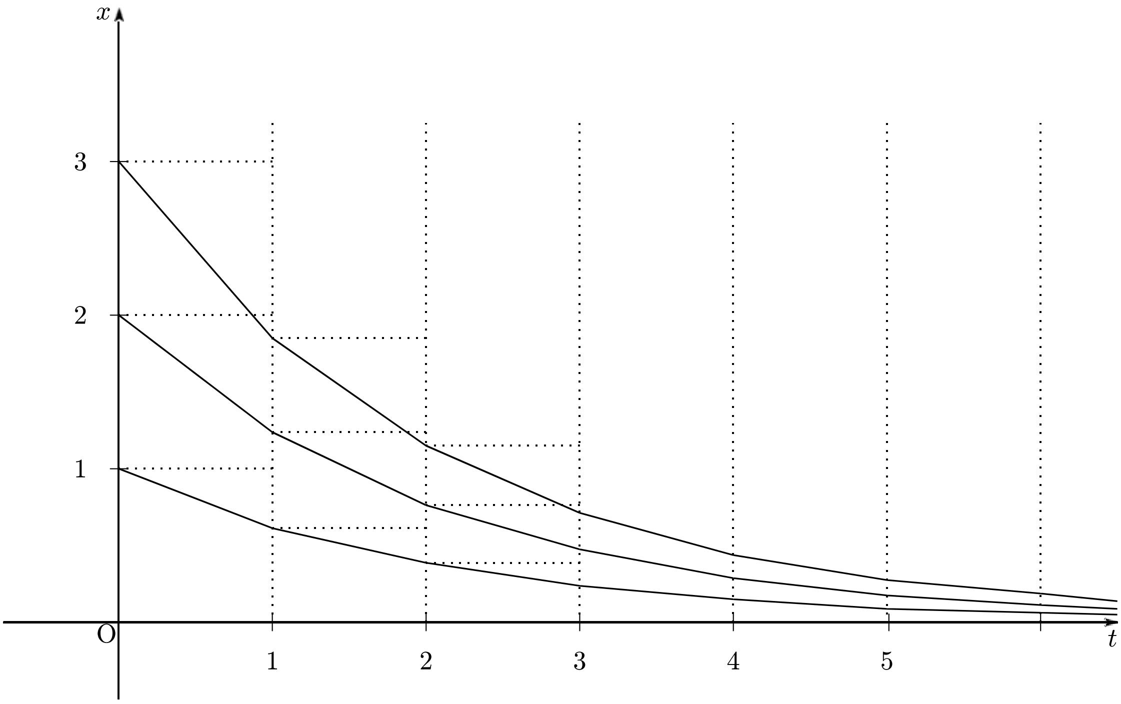

Fig. 1 Golden paths $x = c(\phi - 1)^t \quad c = 1, 2, 3$

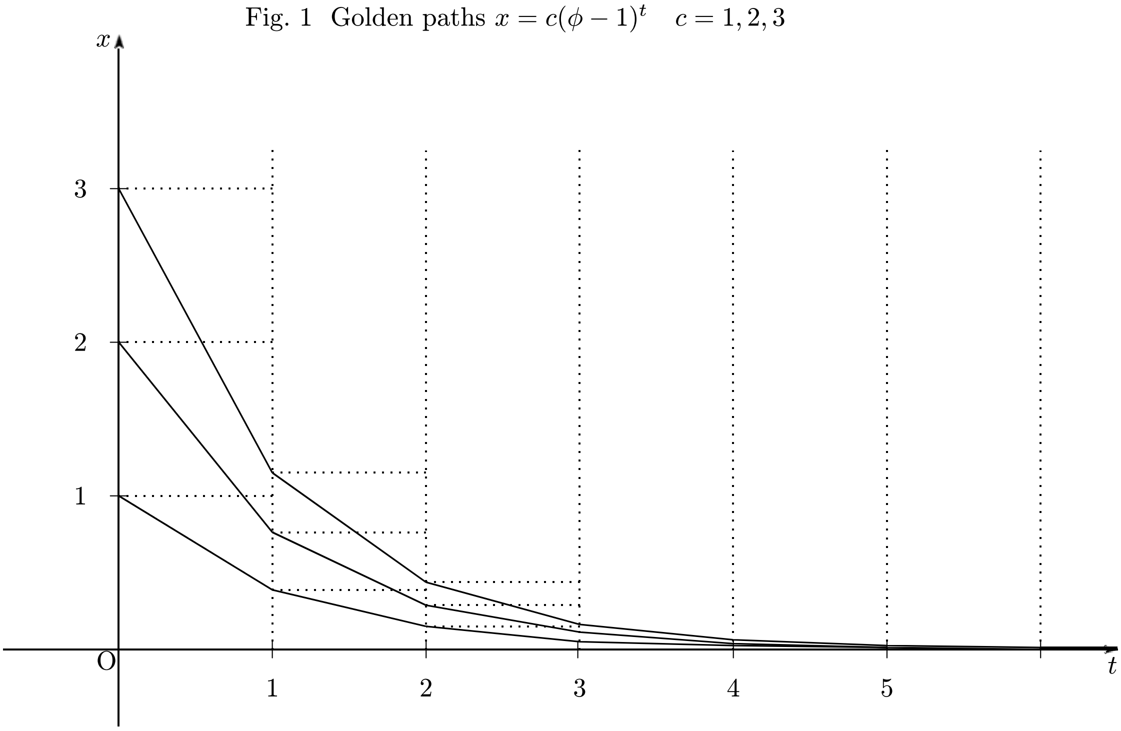

Fig. 2 Golden paths $x = c(2 - \phi)^t \quad c = 1, 2, 3$

We remark that

$$(\phi-1)^t=\phi^{-t}, \qquad (2-\phi)^t=(1+\phi)^{-t}$$

where

$$\phi-1\,=\,\phi^{-1}\,\approx\,0.618, \qquad 2-\phi\,=\,(1+\phi)^{-1}\,\approx\,0.382$$

Let R^∞ be the set of all sequences of real values :

$$R^\infty=\{x=(x_0,x_1,\ldots,x_n,\ldots)\,|\,x_n\in R^1 \quad n=0,1,\ldots\}.$$

We consider a primal problem[1] on R^∞ :

$$\begin{array}{lll} & \text{minimize} & \sum_{n=0}^{\infty}\left[x_n^2+(x_n-x_{n+1})^2\right] \\ (\mathrm{P}_1) & \text{subject to} & \text{(i)} \;\; x\in R^\infty \\ & & \text{(ii)} \;\; x_0=c \end{array}$$

where $c\in R^1$. A dual problem is a maximization problem of $\lambda=(\lambda_0,\ \lambda_1,\ \ldots,\ \lambda_n,\ldots)\in R^\infty$:

$$\begin{array}{lll} & \text{Maximize} & c^2+c\lambda_0-\frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2+(\lambda_n-\lambda_{n+1})^2\right] \\ (\mathrm{D}_1) & \text{subject to} & \text{(i)} \;\; \lambda\in R^\infty \\ & & \text{(ii)} \;\; \lim_{n\to\infty}\lambda_n=0. \end{array}$$

We note that both problems contain a common series $\sum_{n=0}^{\infty}\left[y_n^2+(y_n-y_{n+1})^2\right]$. In either problem, we are concerned with the finite convergence case : $\sum_{n=0}^{\infty}\left[y_n^2+(y_n-y_{n+1})^2\right]<\infty$. This implies that $\lim_{n\to\infty}y_n=0$. In Section 3, we will see that the additional transversality condition (ii) enables us to make dual of (P_1) without difficulty. Therefore, the transversality condition may be removed from the constraints.

THEOREM 1. (Golden duality) (i) The primal problem (P_1) has the minimum value $m=\phi c^2$ at the point

$$\hat{x}=c\,(1,\ (2-\phi),\ \ldots,\ (2-\phi)^n,\ldots)\,.$$

(ii) The dual problem (D_1) has the maximum value $M=\phi c^2$ at the point

$$\lambda^*\,=\,2\phi^{-1}c\,(1,\ (2-\phi),\ \ldots,\ (2-\phi)^n,\ldots)\,.$$

We make an observation about the two optimal solutions. Both the minimum point and the maximum point constitute a *Golden path* and both the minimum value function and the maximum value function are the identical *Golden quadratic value function*. What

[1] As for corresponding finite variable problems see [8], and as for their dual and others see [5, 6, 7, 8, 15].

a beautiful duality this is! Thus, the duality is called *Golden duality*. The proof will be given throughout the discussion in Section 3.

We see that the minimum solution of (P_1) yields the maximum solution of (D_1). Let (P_1) have the minimum value $m = \phi c^2$ at the minimum point $\hat{x}$. Then we have

$$\begin{aligned}
&\operatorname{Max}\left[c^2 + c\lambda_0 - \frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right]\right] \\
&= \operatorname*{Max}_{\lambda_0}\left[c^2 + c\lambda_0 - \frac{1}{4}\min_{\{\lambda_n\}_{n\ge 1}}\sum_{n=0}^{\infty}\left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right]\right] \\
&= \operatorname*{Max}_{\lambda_0}\left[c^2 + c\lambda_0 - \frac{1}{4}\phi\lambda_0^2\right] \\
&= \phi c^2 \qquad \text{for} \quad \lambda_0 = 2\phi^{-1}c
\end{aligned}$$

where the minimum is attained at

$$\left(\hat{\lambda}_1,\ \hat{\lambda}_2,\ \ldots,\ \hat{\lambda}_n, \ldots\right) = \lambda_0\left(2-\phi,\ (2-\phi)^2,\ \ldots,\ (2-\phi)^n, \ldots\right).$$

THEOREM 2. *The functional equation*

$$f(c) = \operatorname*{Max}_{\lambda\in R^1}\left[c^2 + c\lambda - \frac{1}{4}f(\lambda)\right] \qquad c \in R^1$$

has the Golden quadratic maximum value function $f(c) = \phi c^2$ for maximum point function $\hat{\lambda}(c) = 2\phi^{-1}c$. This is a unique continuous function.

Proof. It is easily verified that f with $\hat{\lambda}$ satisfies the functional equation. The uniqueness is shown through the well known method in dynamic programming [1, Ch.4], because of $\left|-\frac{1}{4}\right| < 1$ (see also [18]). ■

COROLLARY 3. *The functional equation*

$$f(c) = \operatorname*{Max}_{\lambda\in R^1}\left[c\lambda + \frac{1}{4}\lambda^2 - \frac{1}{4}f(\lambda)\right] \qquad c \in R^1$$

has the Golden quadratic maximum value function $f(c) = \phi c^2$ for maximum point function $\check{\lambda}(c) = 2\phi c$. This is a unique continuous function.

COROLLARY 4. *The functional equation*

$$\frac{1}{4}f(c) = \operatorname*{Max}_{\lambda\in R^1}\left[\lambda^2 + c\lambda - f(\lambda)\right] \qquad c \in R^1$$

has the Golden quadratic maximum value function $f(c) = \phi c^2$ for maximum point function $\tilde{\lambda}(c) = \dfrac{\phi}{2}c$.

2.2. *Inverse-Golden duality*

Second we consider a primal problem

$$(\mathrm{P}_2) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty}\left[(x_n - x_{n+1})^2 + x_{n+1}^2\right] \\ \text{subject to} & \text{(i)} \;\; x \in R^{\infty} \\ & \text{(ii)} \;\; x_0 = c \end{array}$$

and a dual problem

$$(\mathrm{D}_2) \qquad \begin{array}{ll} \text{Maximize} & c\lambda_0 - \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right] \\ \text{subject to} & \text{(i)} \;\; \lambda \in R^{\infty} \\ & \text{(ii)} \;\; \displaystyle\lim_{n\to\infty} \lambda_n = 0 \end{array}$$

where $c \in R^1$. We note that a difference between (P_1) and (P_2) is constant :

$$\sum_{n=0}^{\infty}\left[x_n^2 + (x_n - x_{n+1})^2\right] = x_0^2 + \sum_{n=0}^{\infty}\left[(x_n - x_{n+1})^2 + x_{n+1}^2\right].$$

The difference $x_0^2 = c^2$ is also preserved between (D_1) and (D_2). This enables us to obtain a duality in terms of inverse-Golden number $\phi^{-1} = \phi - 1$ as follows.

THEOREM 5. (Inverse-Golden duality) (i) The primal problem (P_2) has the minimum value $m = \phi^{-1}c^2$ at the point

$$\hat{x} = c\,(1,\ (2-\phi),\ \ldots,\ (2-\phi)^n, \ldots).$$

(ii) The dual problem (D_2) has the maximum value $M = \phi^{-1}c^2$ at the point

$$\lambda^* = 2\phi^{-1}c\,(1,\ (2-\phi),\ \ldots,\ (2-\phi)^n, \ldots).$$

We remark that both optimal points constitute a *Golden path* and that both optimal value functions are the same — the inverse-Golden quadratic value function —. This is a Golden duality, too. The Golden paths $\hat{x}$ and λ^* are the same ones in (P_1) and (D_1), respectively.

Further the minimum solution of (P_2) yields the maximum solution of (D_2). Let (P_2) have the minimum value $m = \phi^{-1}c^2$ at the minimum point $\hat{x}$. Then we have

$$
\begin{aligned}
&\text{Max}\left[c\lambda_0 - \frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right]\right] \\
&= \underset{\lambda_0}{\text{Max}}\left[c\lambda_0 - \frac{1}{4}\lambda_0^2 - \frac{1}{4}\min_{\{\lambda_n\}_{n\geq 1}}\sum_{n=0}^{\infty}\left[(\lambda_n - \lambda_{n+1})^2 + \lambda_{n+1}^2\right]\right] \\
&= \underset{\lambda_0}{\text{Max}}\left[c\lambda_0 - \frac{1}{4}\lambda_0^2 - \frac{1}{4}\phi^{-1}\lambda_0^2\right] \\
&= \phi^{-1}c^2 \qquad \text{for} \quad \lambda_0 = 2\phi^{-1}c
\end{aligned}
$$

where the minimum is attained at

$$
\left(\hat{\lambda}_1,\ \hat{\lambda}_2,\ \ldots,\ \hat{\lambda}_n, \ldots\right) = \lambda_0\left(2-\phi,\ (2-\phi)^2,\ \ldots,\ (2-\phi)^n, \ldots\right).
$$

THEOREM 6. *The functional equation*

$$
g(c) = \underset{\lambda \in R^1}{\text{Max}}\left[c\lambda - \frac{1}{4}\lambda^2 - \frac{1}{4}g(\lambda)\right] \qquad c \in R^1
$$

has the inverse-Golden quadratic maximum value function $g(c) = \phi^{-1}c^2$ *for maximum point function* $\check{\lambda}(c) = 2\phi^{-1}c$. *This is a unique continuous function.*

COROLLARY 7. *The functional equation*

$$
g(c) = \underset{\lambda \in R^1}{\text{Max}}\left[-c^2 + c\lambda - \frac{1}{4}g(\lambda)\right] \qquad c \in R^1
$$

has the inverse-Golden quadratic maximum value function $g(c) = \phi^{-1}c^2$ *for maximum point function* $\hat{\lambda}(c) = 2\phi c$. *This is a unique continuous function.*

COROLLARY 8. *The functional equation*

$$
\frac{1}{4}g(c) = \underset{\lambda \in R^1}{\text{Max}}\left[-c\lambda - \lambda^2 - g(\lambda)\right] \qquad c \in R^1
$$

has the inverse-Golden quadratic maximum value function $g(c) = \phi^{-1}c^2$ *for maximum point function* $\tilde{\lambda}(c) = \dfrac{\phi}{2}c$.

3. Lagrangean Method

In this section we show how the Lagrangean method derives a maximization (dual) problem from the minimization (primal) problem.

Let us reconsider the primal problem

$$
\begin{array}{lll}
 & \text{minimize} & \displaystyle\sum_{n=0}^{\infty}\left[x_n^2 + (x_n - x_{n+1})^2\right] \\
(\mathrm{P}_1) & \text{subject to} & \text{(i)}\ \ x \in R^{\infty} \\
 & & \text{(ii)}\ \ x_0 = c.
\end{array}
$$

We introduce a sequence of variables $u = \{u_0,\ u_1,\ \dots,\ u_n,\ \dots\}$ by

$$u_k = x_{k+1} - x_k.$$

Then (P_1) is formulated into a quadratic minimization under a linear constraint

$$(\mathrm{P}'_1)\qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right) \\ \text{subject to} & \text{(i)}\ \ x_{n+1} = x_n + u_n \quad n \geq 0 \\ & \text{(ii)}\ \ x_0 = c. \end{array}$$

Let us now solve this problem through a Lagrangean multiplier's method. We introduce a sequence of variables $\lambda = \{\lambda_0,\ \lambda_1,\ \dots,\ \lambda_n,\ \dots\}$ with the property $\lim_{n\to\infty} \lambda_n = 0$, which is called a *Lagrange multiplier*. Let us construct the Lagrangean

$$L(x, u, \lambda) \ = \sum_{n=0}^{\infty} \left[x_n^2 + u_n^2 - \lambda_n(x_{n+1} - x_n - u_n)\right].$$

Then it has the partial derivatives

$$\begin{aligned} L_{x_n} &= 2x_n + \lambda_n - \lambda_{n-1} \qquad n \geq 1 \\ L_{u_n} &= 2u_n + \lambda_n \qquad n \geq 0 \\ L_{\lambda_n} &= x_{n+1} - x_n - u_n \qquad n \geq 0. \end{aligned}$$

Here we notice that the Lagrangean is not one for a *regular* (a finite n variables with a finite m constraints) extremal problem. The problem has a countably infinte variables under a countably infinite linear constraints.

LEMMA 2. Let (x, u) be an extremum point. Then there exists a λ satisfying the condition that all the partial derivatives at point (x, u, λ) vanish:

$$L_{x_n} = 0 \quad n \geq 1, \qquad L_{u_n} = L_{\lambda_n} = 0 \quad n \geq 0. \tag{2}$$

Proof. Let $\hat{x} = (\hat{x}_n)_{n\geq 0}, \hat{u} = (\hat{u}_n)_{n\geq 0}$ be an extremum point for (P'_1). We take any large positive integer N and consider a finite-truncated conditional minimization problem of $x = (x_n)_0^N, u = (u_n)_0^N$:

$$(\mathrm{T}_N)\qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{N} \left(x_n^2 + u_n^2\right) \\ \text{subject to} & \text{(i)}\ \ x_{n+1} = x_n + u_n \quad 0 \leq n \leq N-1 \\ & \text{(ii)}\ \ \hat{x}_{N+1} = x_N + u_N \\ & \text{(iii)}\ \ x_0 = c. \end{array}$$

This has $(2N+1)$ variables $x_1, \dots, x_N$, $u_0, \dots, u_N$ and $(N+1)$ linear constraints. Let us construct the Lagrangean L^N by

$$\begin{aligned} L^N(x, u, \lambda) \ = & \sum_{n=0}^{N-1} \left[x_n^2 + u_n^2 - \lambda_n(x_{n+1} - x_n - u_n)\right] \\ & + x_N^2 + u_N^2 - \lambda_N(\hat{x}_{N+1} - x_N - u_N) \quad \text{for} \ \ \lambda = (\lambda_n)_0^N. \end{aligned}$$

Then the point $(\hat{x}_n)_1^N, (\hat{u}_n)_0^N$ is also an extremum point for the truncated problem. It satisfies the linear independent constraint qualification [16]. Therefore, Lagrange Multiplier Theorem (for a regular problem) implies that there exists a $(\lambda_n^*)_0^N$ such that $(\hat{x}_n)_1^N, (\hat{u}_n)_0^N; (\lambda_n^*)_0^N$ satisfies

$$L_{x_n}^N = 0 \quad 1 \le n \le N, \qquad L_{u_n}^N = L_{\lambda_n}^N = 0 \quad 0 \le n \le N.$$

Thus we have

$$\begin{aligned} 2\hat{x}_n + \lambda_n^* - \lambda_{n-1}^* &= 0 \qquad 1 \le n \le N \\ 2\hat{x}_n + \lambda_n^* &= 0 \qquad 0 \le n \le N \\ \hat{x}_{n+1} - \hat{x}_n - \hat{u}_n &= 0 \qquad 0 \le n \le N. \end{aligned}$$

Since N is arbitrarily large, we conclude that there exists a $(\lambda_n^*)_{n\ge0}$ such that $(\hat{x}_n)_{n\ge1}, (\hat{u}_n)_{n\ge0}$; $(\lambda_n^*)_{n\ge0}$ satisfies (2). This completes the proof. ■

Then (2) is equivalent to

$$\begin{aligned} 2x_n &= -(\lambda_n - \lambda_{n-1}) \quad n \ge 1 \\ 2u_n &= -\lambda_n \qquad n \ge 0 \\ u_n &= x_{n+1} - x_n \qquad n \ge 0. \end{aligned}$$

Now we solve this equivalent system in the following. Deleting u and λ, we get a system of linear equations:

$$\begin{aligned} x_0 &= c \\ x_{n-1} - 3x_n + x_{n+1} &= 0 \quad n \ge 1. \end{aligned} \tag{3}$$

Thus we have

$$\begin{aligned} u_n &= x_{n+1} - x_n \\ \lambda_k &= -2(x_{n+1} - x_n). \end{aligned} \tag{4}$$

We see that Eq.(3) has the solution[2]

$$x_n = c(2-\phi)^n \quad n \ge 0. \tag{5}$$

where ϕ is the Golden number (ratio). Thus we have

$$\begin{aligned} u_n &= -\frac{c}{\phi}(2-\phi)^n \\ \lambda_n &= \frac{2c}{\phi}(2-\phi)^n. \end{aligned} \tag{6}$$

LEMMA 3. The solution (x, u) in (5),(6) is a minimum point for (P_1'). Hence, x is a minimum point for (P_1).

[2] A general solution of (3) is $x_n = A(2-\phi)^n + B(1+\phi)^n$, where $A + B = c$. The case $A = c, B = 0$ attains a minimum value for (P_1).

Proof. We show that the $(x,\, u)$ is a minimum point. Let $(X,\, U)$ be any solution satisfying

$$X_{n+1} = X_n + U_n \qquad n \geq 0$$
$$X_0 = c.$$

Here we take the λ in (6). First we see that for any n-stage process

$$\begin{aligned}
\sum_{k=0}^{n} \left(x_k^2 + u_k^2\right) &= \sum_{k=0}^{n} \left[x_k^2 + u_k^2 - \lambda_k(x_{k+1} - x_k - u_k)\right] \\
&= c^2 + c\lambda_0 - \lambda_0 x_1 + \sum_{k=1}^{n-1} \left[x_k^2 - \lambda_n(x_{k+1} - x_k)\right] + x_n^2 + \lambda_n x_n \\
&\quad + \sum_{k=0}^{n} \left(u_k^2 + \lambda_k u_k\right) - \lambda_n x_{n+1} \\
&= c^2 + c\lambda_0 + \sum_{k=1}^{n} \left[x_k^2 + (\lambda_k - \lambda_{k-1})x_k\right] + \sum_{k=0}^{n} \left(u_k^2 + \lambda_k u_k\right) - \lambda_n x_{n+1} \\
&= c^2 + c\lambda_0 - \frac{1}{4}\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})^2 - \frac{1}{4}\sum_{k=0}^{n} \lambda_k^2 - \lambda_n x_{n+1} \\
&\quad + \sum_{k=1}^{n} \left[x_k + \frac{1}{2}(\lambda_k - \lambda_{k-1})\right]^2 + \sum_{k=0}^{n} \left(u_k + \frac{1}{2}\lambda_k\right)^2 . \qquad (7)
\end{aligned}$$

We may consider the set of all feasible x satisfying $\lim_{n\to\infty} \lambda_n x_{n+1} = 0$ in (P_1), because of the quadratic minimization and $\lim_{n\to\infty} \lambda_n = 0$. This set in turn includes the set of all feasible x satisfying $\lim_{n\to\infty} x_n = 0$. Thus letting $n \to \infty$ in (7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right) &= c^2 + c\lambda_0 - \frac{1}{4}\sum_{n=0}^{\infty} \left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right] \\
&\quad + \sum_{n=1}^{\infty} \left[x_n + \frac{1}{2}(\lambda_n - \lambda_{n-1})\right]^2 + \sum_{n=0}^{\infty} \left(u_n + \frac{1}{2}\lambda_n\right)^2 . \qquad (8)
\end{aligned}$$

Similarly, we have for $(X,\, U)$

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(X_n^2 + U_n^2\right) &= c^2 + c\lambda_0 - \frac{1}{4}\sum_{n=0}^{\infty} \left[\lambda_n^2 + (\lambda_n - \lambda_{n+1})^2\right] \\
&\quad + \sum_{n=1}^{\infty} \left[X_n + \frac{1}{2}(\lambda_n - \lambda_{n-1})\right]^2 + \sum_{n=0}^{\infty} \left(U_n + \frac{1}{2}\lambda_n\right)^2 . \qquad (9)
\end{aligned}$$

Since $(x,\, u)$ satisfies

$$x_n + \frac{1}{2}(\lambda_n - \lambda_{n-1}) = 0, \quad u_n + \frac{1}{2}\lambda_n = 0, \quad \lim_{n\to\infty} \lambda_n = 0$$

a comparison between (8) and (9) yields

$$\sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right) \leq \sum_{n=0}^{\infty} \left(X_n^2 + U_n^2\right).$$

This completes the proof. ■

From (9) we have a basic inequality as follows.

LEMMA 4. It holds that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(x_n^2+u_n^2\right) &= c^2+c\lambda_0-\frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2+(\lambda_n-\lambda_{n+1})^2\right]\\ &\quad+\sum_{n=1}^{\infty}\left[x_n+\frac{1}{2}(\lambda_n-\lambda_{n-1})\right]^2+\sum_{n=0}^{\infty}\left(u_n+\frac{1}{2}\lambda_n\right)^2 \qquad (10)\\ &\geq c^2+c\lambda_0-\frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2+(\lambda_n-\lambda_{n+1})^2\right]\end{aligned}$$

for any (x, u) satisfying (i), (ii) and any λ satisfying $\lim_{n\to\infty}\lambda_n = 0$. The equality holds if and only if

$$2x_n = -(\lambda_n-\lambda_{n-1}) \quad n\geq 1 \quad \text{and} \quad 2u_n=-\lambda_n \quad n\geq 0.$$

This lemma states that

$$L(\hat{x},\hat{u}:\lambda)\leq L(\hat{x},\hat{u}:\lambda^*)\leq L(x,u:\lambda^*) \qquad (11)$$

where

$$L(x,u:\lambda) = \sum_{n=0}^{\infty}\left[x_n^2+u_n^2-\lambda_n(x_{n+1}-x_n-u_n)\right]$$

and

$$\begin{aligned}\hat{x}_n &= c(2-\phi)^n\\ \hat{u}_n &= -\frac{c}{\phi}(2-\phi)^n \qquad n\geq 0\\ \lambda_n^* &= \frac{2c}{\phi}(2-\phi)^n.\end{aligned}$$

In fact, we have the equality between left-hand side and middle side:

$$L(\hat{x},\hat{u}:\lambda) = L(\hat{x},\hat{u}:\lambda^*) \qquad \forall\lambda\ ;\ \lim_{n\to\infty}\lambda_n=0.$$

Hence we have a maximization problem for $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots)$ as follows:

$$(\mathrm{D}_1)\qquad \begin{aligned}&\text{Maximize} \quad c^2+c\lambda_0-\frac{1}{4}\sum_{n=0}^{\infty}\left[\lambda_n^2+(\lambda_n-\lambda_{n+1})^2\right]\\ &\text{subject to} \quad \text{(i)} \quad \lambda\in R^{\infty}\\ &\qquad\qquad\quad \text{(ii)} \quad \lim_{n\to\infty}\lambda_n=0.\end{aligned}$$

Thus we have derived the desired dual problem together with the optimum solution.

LEMMA 5. The point λ^* with $\lambda_n^* = \dfrac{2c}{\phi}(2-\phi)^n$ is a maximum point for (D_1). It yields the maximum value $M=\phi c^2$.

4. Dynamic Programming

This section discusses a dynamic programming aspect of the two primal problems [1, 2, 8, 10]. It is shown that both optimal value function and optimum point function are Golden. The optimal value function is Golden quadratic. The optimum point yields what we call Golden section.

Let $f(c)$ be the minimum value of (P_1) for $c \in R^1$. Then we have the following results.

Theorem 9. *The minimum value function f satisfies the Bellman equation*

$$\begin{cases} f(0) = 0 \\ f(c) = \min_{x \in R^1} \left[c^2 + (c-x)^2 + f(x) \right]. \end{cases}$$

This has the Golden quadratic minimum value function $f(c) = \phi c^2$ for a minimum point function $\hat{x}(c) = \phi^{-2} c$.

Let $g(c)$ be the minimum value of (P_2) for $c \in R^1$. Then we have the following results.

Theorem 10. *The minimum value function g satisfies the Bellman equation*

$$\begin{cases} g(0) = 0 \\ g(c) = \min_{x \in R^1} \left[(c-x)^2 + x^2 + g(x) \right]. \end{cases}$$

This has the inverse-Golden quadratic minimum value function $g(c) = \phi^{-1} c^2$ for a minimum point function $\hat{x}(c) = \phi^{-2} c$.

We remark that the minimum point function $\hat{x}(c) = \phi^{-2} c$ is called a *Golden* decision function. Because it divides an interval $[0, c]$ into $[0, \hat{x}(c)] = [0, (2-\phi)c]$ and $[\hat{x}(c), c] = [(2-\phi)c, c]$, which is the so-called *Golden section*. Thus $\hat{x}$ is Golden (and) optimal. The Golden (Golden optimal) decision function leads to the *Golden (Golden optimal) policy* in dynamic programming. Recently the Golden optimal policy for dynamic optimization problems has been discussed in [9, 11, 12, 14]. It has a strong connection with duality, as we have shown in this paper.

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