Almost everywhere convergence of random set sequence on non-additive measure spaces *

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ABSTRACT: In this paper, we investigate the convergence and the pseudo-convergence of sequence of set-valued mappings. Egoroff type theorem for random set sequence on monotone non-additive measure spaces is presented.

Keyword: Set-valued mapping; Random set; Non-additive measure; Egoroff's theorem

1 Introduction

The convergences of random set sequence on measure spaces were studied by Zhang ([10]), and some important convergence theorems in random set theory were obtained. Liu ([6]) discussed the convergence of sequence of random set (it is called measurable set-valued function in [6]) on fuzzy measure spaces and some results, such as Egoroff's theorem, Lebesgue's theorem and Riesz's theorem in fuzzy measure theory ([9]) have been adapted to set-valued case.

In this paper, we discuss the convergence and the pseudo-convergence of sequence of random set sequence on monotone non-additive measure spaces. Egoroff type theorem for random set sequence with respect to monotone non-additive measure is shown. It is a generalization of the related results obtained by authors [3, 4, 5].

2 Preliminaries

Let Ω be a non-empty set, $\mathcal{A}$ be a σ-algebra of subsets of Ω, and let R^m be m-dimensional Euclidean space, and d a Euclidean metric on R^m. All concepts and signs not defined in this paper may be found in [1, 2, 10]

Definition 1 Let $\{A_n\}$ be a sequence of closed subsets of R^m. We say that the subset

$$\limsup_{n\to\infty} A_n = \{x \in R^m : \liminf_{n\to\infty} d(x, A_n) = 0\}$$

is the upper limit of the sequence $\{A_n\}$ and that the subset

$$\liminf_{n\to\infty} A_n = \{x \in R^m : \lim_{n\to\infty} d(x, A_n) = 0\}$$

is its lower limit. A subset A is said to be the set limit of the sequence $\{A_n\}$, denoted by $\lim_{n\to\infty} A_n = A$, if

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A$$

where $d(x, A_n) = \inf_{y\in A} d(x, y)$.

Definition 2 ([4]) A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called *monotone non-additive measure*, if it satisfies the following properties:

(1) $\mu(\emptyset) = 0$;

(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$ (monotonicity).

When μ is a monotone non-additive measure, the triple $(\Omega, \mathcal{A}, \mu)$ is called monotone non-additive measure space.

A set function $\mu : \mathcal{A} \to [0, +\infty]$ is said to be *continuous from below*, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$ ([7]); *strongly order continuous* ([4]), if $\lim_{n\to+\infty} \mu(A_n) = 0$ whenever $\{A_n\}_n \subset \mathcal{F}$, $A_n \searrow B$ and $\mu(B) = 0$; has *property (S)* (resp. *property (PS)*) ([8]), if for any $\{A_n\}_n$ with $\lim_{n\to\infty} \mu(A_n) = 0$ (resp. $\lim_{n\to\infty} \mu(A \setminus A_n) = \mu(A)$), there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\mu(\limsup A_{n_i}) = 0$ (resp. $\mu(A \setminus \limsup A_{n_i}) = \mu(A)$).

Definition 3 ([10]) Let $(\Omega, \mathcal{A})$ be a measurable space, and f a set-valued mapping from Ω to closed subsets of R^m. If for every closed subset F of R^m,

$$f^{-1}(F) = \{\omega \in \Omega : f(\omega) \cap F \neq \emptyset\} \in \mathcal{A},$$

then f is called random set (with respect to $\mathcal{A}$).

Let $\mu[\Omega]$ denote the class of all random set defined on Ω (with respect to $\mathcal{A}$), and let f_n $(n \in \mathbf{N}), f \in \mu(\Omega), E \in \mathcal{A}$. We say that

(1) $\{f_n\}$ *converges to f almost everywhere on E*, and denoted by $f_n \xrightarrow{a.e.} f$ on E, if there exists $N \in E \cap \mathcal{A}$, such that $\mu(N) = 0$ and for every $\omega \in E \setminus N$, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ (in the sense of Definition 1);

(2) $\{f_n\}$ *converges uniformly to f on E* , denoted by $f_n \xrightarrow{u} f$ on E, if for any $\varepsilon > 0$ and any compact subset K of R^m, there exists some positive integer $N_{(\varepsilon,K)}$, such that

$$E(\triangle_{\varepsilon n}^{-1}(K)) \triangleq \{\omega \in A : [(f_n \setminus \varepsilon f) \cup (f \setminus \varepsilon f_n)](\omega) \cap K \neq \emptyset\} = \emptyset$$

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whenever $n \ge N_{(\varepsilon,K)}$;

(3) $\{f_n\}$ *converges almost uniformly to* f *on* E, denoted by $f_n \xrightarrow{a.u} f$ on E, if there exists a sequence $\{E_m\}$ of measurable sets of $E\cap\mathcal{A}$ such that $\lim_{n\to\infty}\mu(E_m) = 0$, and $f_n \xrightarrow{u} f$ on $E\setminus E_m (m = 1,2\ldots)$;

(4) $\{f_n\}$ *converges to* f *pseudo−almost everywhere on* E, denote by $f_n \xrightarrow{p.a.e.} f$ on E, if there exists $N \in E\cap\mathcal{A}$, such that $\mu(N) = \mu(E)$ and for every $\omega \in E\setminus N$, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$;

(5) $\{f_n\}$ *pseudo−converges almost uniformly to* f *on* E, denoted by $f_n \xrightarrow{p.a.u} f$ on E, if there exists a sequence $\{E_m\}$ of measurable sets of $E\cap\mathcal{A}$ such that $\lim_{n\to\infty}\mu(E\setminus E_m) = \mu(E)$, and $f_n \xrightarrow{u} f$ on $E\setminus E_m (m = 1,2\ldots)$ (cf. [6], [9] and [10]).

3 Egoroff's theorem for random set sequence

In this section, we show Egoroff type theorem for random set sequence on monotone non-additive measure space.

Theorem 1 *(Egoroff's theorem) Let* μ *be a monotone non-additive measure on* $(\Omega,\mathcal{A})$ *and* $f_n (n\in\mathbb{N}), f \in \mu(\Omega)$ *and* $E \in \mathcal{A}$.

(1) *If* μ *is strongly order continuous and has property (S), then on E*

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u} f.$$

(2) *If* μ *is continuous from below and has property (PS), then on E*

$$f_n \xrightarrow{p.a.e.} f \Longrightarrow f_n \xrightarrow{p.a.u} f.$$

Proof: (1) Let $0 < \varepsilon_l \downarrow 0$, and $R^m = \bigcup_{l=1}^{\infty} U_l$, where U_l is a bounded open subset of R^m, and its closure $\overline{U_l} \subset U_{l+1} (l = 1,2,\ldots)$.

Since $f_n \xrightarrow{a.e.} f$, there exists $E \in \mathcal{A}$, such that $\mu(X-E) = 0$, and f_n converges to f everywhere on E.

For each $l > 0$, denote

$$E_m^{(l)} = \bigcup_{n=m}^{\infty}\left\{\omega\in X : [(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap\overline{U_l} = \emptyset\right\},$$

then $E_m^{(l)}$ is increasing in n for each fixed l, and we get $\bigcup_{m=1}^{\infty} E_m^{(l)} = E (l = 1,2,\ldots)$. In fact, for any $\omega \in \cup_{m=1}^{\infty} E_m^{(l)}$, there exists m_0, such that

$$\omega\in E_{m_0}^{(l)} = \bigcup_{n=m_0}^{\infty}\left\{\omega\in X : [(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap\overline{U_l} = \emptyset\right\},$$

that is,

$$\left[(f_n\setminus\varepsilon_l f)\bigcup(f\setminus\varepsilon_l f_n)\right](\omega)\bigcap\overline{U_l} = \emptyset$$

whenever $n \ge m_0$. Then for any $\varepsilon > 0$ and any compact subset $K$of R^m, there exists some positive integer $l_0(\varepsilon,K)$, such that $\varepsilon_{l_0} < \varepsilon$, $K \subset \overline{U_{l_0}}$, and $[(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap K = \emptyset$. So we have $\cup_{m=1}^{\infty} E_m^{(l)} = E (l = 1,2,\ldots)$, and for each fixed l

$$X - E_m^{(l)} \searrow X - \bigcup_{m=1}^{\infty} E_m^{(l)} = X - E.$$

By using the strong order continuity of μ, we have

$$\lim_{m\to\infty}\mu(X-E_m^{(l)}) = \mu(X-E) = 0 \quad (l\ge 1).$$

Thus, there exists a subsequence $\{X\setminus E_{m_l}^{(l)}\}$ of $\{X\setminus E_m^{(l)}\}$ satisfying

$$\mu(X\setminus E_{m_l}^{(l)}) \le \frac{1}{l} \quad (\forall l\ge 1),$$

and hence

$$\lim_{l\to\infty}\mu(X\setminus E_{m_l}^{(l)}) = 0.$$

By applying the property (S) of μ to the sequence $\{X\setminus E_{m_l}^{l}\}$, then there exists a subsequence $\{X\setminus E_{m_{l_i}}^{l_i}\}$ of $\{X\setminus E_{m_l}^{l}\}$ such that

$$\mu\left(\limsup_i (X\setminus E_{m_{l_i}}^{(l_i)})\right) = 0$$

and $l_1 < l_2 < \ldots$. On the other hand,

$$\bigcup_{j=k}^{\infty}(X\setminus E_{m_{l_j}}^{(l_j)}) \searrow \limsup_i (X\setminus E_{m_{l_i}}^{(l_i)}) \quad (j\to\infty),$$

therefore, by using the strong order continuity of μ, we have

$$\lim_{k\to\infty}\mu\left(\bigcup_{j=k}^{\infty}(X\setminus E_{m_{l_j}}^{(l_j)})\right) = 0.$$

Put $E_k = \bigcap_{j=k}^{\infty} E_{m_{l_j}}^{(l_j)}$, then $\lim_{n\to\infty}\mu(X\setminus E_k) = 0$.

Now we prove that f_n converges to f on E_k uniformly for any fixed $k = 1,2,\ldots$. In fact, for any $\varepsilon > 0$ and any compact subset K of R^m, there exists some positive integer $l_{0(\varepsilon,K)}$ such that $\varepsilon_{l_0} < \varepsilon$ and $K \subset \overline{U_{l_0}}$. Therefore we have

$$E_k \subset E_{m_{l_0}}^{(l_0)} \subset \bigcap_{n=m_{l_0}}^{\infty}\{\omega\in X : [(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap K = \emptyset\}$$

that is,

$$\begin{aligned} & E_k(\triangle_{\varepsilon n}^{-1}(K)) \\ = \; & \{\omega\in X \mid [(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap K \ne \emptyset\} \\ = \; & \emptyset \end{aligned}$$

whenever $n \ge N_{(\varepsilon,K)} = m_{l_0}$. This shows $f_n \xrightarrow{a.u} f$.

(2) Let $\varepsilon_l > 0$, $\varepsilon_l \downarrow 0$ and $R^m = \bigcup_{l=1}^{\infty} U_l$, where U_l is a bounded open subset of R^m, and its closure $\overline{U_l} \subset U_{l+1}$ $(l = 1,2,\ldots)$.

Since $f_n \xrightarrow{p.a.e.} f$ on A, there exists $E \in A\cap\mathcal{A}$, such that $\mu(E) = \mu(A)$ and f_n converges to f everywhere on E. For each $l > 0$, letting

$$E_m^{(l)} = \bigcap_{n=m}^{\infty}\{\omega\in A : [(f_n\setminus\varepsilon f)\cup(f\setminus\varepsilon f_n)](\omega)\cap K = \emptyset\},$$

then $E_1^{(l)} \subset E_2^{(l)} \subset \ldots$, and $\bigcup_{m=1}^{\infty} E_m^{(l)} = E$ $(l = 1,2,\ldots)$. By using the continuity from below of μ, we have

$$\lim_{m\to\infty}\mu(E_m^{(l)}) = \mu(E) = \mu(A).$$

Thus there exists a subsequence $\{E_{m_l}^{(l)}\}_l$ of $\{E_m^{(l)}; l, m \geq 1\}$ satisfying:
(i) if $\mu(A) < \infty$, then

$$\mu(A) - \mu(E_{m_l}^{(l)}) < \frac{1}{l}, \ \ \forall\, l \geq 1.$$

(2) if $\mu(A) = \infty$, then

$$\mu(E_{m_l}^{(l)}) > l, \ \ \forall\, l \geq 1.$$

Therefore, we have

$$\lim_{l\to\infty} \mu(E_{m_l}^{(l)}) = \mu(A).$$

By applying the property (PS) of μ to the sequence $\{E_{m_l}^{(l)}\}$, then there exists a subsequence $\{E_{m_{l_i}}^{(l_i)}\}$ of $\{E_{m_l}^{(l)}\}$, such that $l_1 < l_2 < \ldots$ and

$$\mu\left(\bigcup_{k=1}^{\infty}\bigcap_{i=k}^{\infty} E_{m_{l_i}}^{(l_i)}\right) = \mu(A).$$

It follows from the continuity from below of μ that

$$\lim_{k\to\infty} \mu\left(\bigcap_{i=k}^{\infty} E_{m_{l_i}}^{(l_i)}\right) = \mu(A).$$

Put $F_k = A \setminus \bigcap_{i=k}^{\infty} E_{m_{l_i}}^{(l_i)}$ $(k = 1, 2\ldots)$, then

$$\lim_{k\to\infty} \mu(A \setminus F_k) = \mu(A).$$

Now we prove that f_n converges to f on $A \setminus F_k$ uniformly for any fixed $k = 1, 2, \ldots$ In fact, for any $\varepsilon > 0$, and any compact subset K of R^m, there exists some positive integer $l_{0(\varepsilon,K)}$, such that $\varepsilon_{l_0} < \varepsilon$ and $K \subset \overline{U_{l_0}}$, So we have

$$A \setminus F_k \subset \bigcap_{j=m_{l_0}}^{\infty} \{\omega \in A : [(f_n \setminus \varepsilon f) \cup (f \setminus \varepsilon f_n)](\omega) \cap K = \emptyset\}.$$

That is

$$\begin{aligned} & A \setminus F_k(\triangle_{\varepsilon n}^{-1}(K)) \\ = \ & \{\omega \in A : [(f_n \setminus \varepsilon f) \cup (f \setminus \varepsilon f_n)](\omega) \cap K \neq \emptyset\} \\ = \ & \emptyset \end{aligned}$$

whenever $n \geq N_{(\varepsilon,K)} = m_{l_0}$. Therefore, we have $f_n \xrightarrow{p.a.u} f$. $\square$

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