Fuzzy Stopping of a Dynamic Fuzzy System

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Abstract

This paper discusses an optimal stopping problem of fuzzy systems with fuzzy rewards in a class of fuzzy stopping times, which are extended from usual stopping times. Some properties regarding the optimal fuzzy stopping times are presented.

0. Introduction

Multistage decision-making models under fuzzy environment are first studied by Bellman and Zadeh [1]. Kacprzyk [6] introduced a fuzzy stopping time, which is called fuzzy termination time in the paper, and discussed an optimization problem. By using α-cut technique in Kurano et al. [9, 10, 11] studied a stopping problem which maximizes the measure of the truth of the fuzzy stopped system and in which we find an optimal fuzzy stopping time in a class of fuzzy stopping times. It is a fuzzy extension of stopping rule in stopping problems ([Yoshida [15]) for dynamic fuzzy systems with fuzzy rewards. This paper discusses the further properties of the fuzzy stopping times for the fuzzy systems.

1 Fuzzy Stopping Times

In this paper, applying the idea of Kacprzyk [5, 6], we formulate a stopping problem for a dynamic fuzzy system ([9, 10]) with fuzzy rewards, which is thought of as a natural fuzzification of non-fuzzy stopping problems induced by determistic dynamic systems. And the validity of the approach by α-cuts of fuzzy sets will be discussed in constructing an optimal fuzzy stopping time. In remainder of this section, we will give some notations, by which a fuzzy stopping problem is formulated in the following section.

Let E be a convex compact subset of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x\in E}\tilde{u}(x) = 1$. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of E and by ρ_E the Hausdorff metric on $\mathcal{C}(E)$. Clearly, $\tilde{u} \in \mathcal{F}(E)$ means the α-cut $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0, 1]$, where

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\} \ (\alpha > 0)$$

and

$$\tilde{u}_0 := \mathrm{cl}\ \{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set.

Let $\mathbf{R}$ be the set of all real numbers. Especially, $\mathcal{C}(\mathbf{R})$ and $\mathcal{F}(\mathbf{R})$ are the set of all bounded closed intervals in $\mathbf{R}$ and all upper semi-continuous and convex fuzzy numbers on $\mathbf{R}$ with compact supports, respectively.

The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$,

$$(\tilde{m}+\tilde{n})(x) := \sup_{x_1, x_2 \in \mathbf{R}:\ x_1+x_2=x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\} \tag{1.1}$$

($x \in \mathbf{R}$) and

$$(\lambda\tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \tag{1.2}$$

($x \in \mathbf{R}$). And, hence

$$(\tilde{m}+\tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda\tilde{m})_\alpha = \lambda\tilde{m}_\alpha$$

($\alpha \in [0,1]$), where $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A + \emptyset = \emptyset + A := A$ and $\lambda\emptyset := \emptyset$ for any non-empty closed intervals $A, B \in (\mathbf{R})$.

We consider the fuzzy system (see Kurano et al. [9, 10]) with fuzzy rewards, which is characterized by the elements $(S, \tilde{q}, \tilde{r})$ as follows:

Definition 1.

(i) The state space S is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by a element of $\mathcal{F}(S)$.

(ii) The law of motion and the fuzzy reward for the system are denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$ and $\tilde{r} : S \times \mathbf{R} \mapsto [0,1]$ respectively. We assume that $\tilde{q} \in \mathcal{F}(S \times S)$ and $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a fuzzy reward $R(\tilde{s})$ is earned and the state is moved to a new fuzzy state $Q(\tilde{s})$, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ and $R : \mathcal{F}(S) \mapsto \mathcal{F}(\mathbf{R})$ are defined by

$$Q(\tilde{s})(y) := \sup_{x \in S}\{\tilde{s}(x) \wedge \tilde{q}(x, y)\} \quad (y \in S) \tag{1.3}$$

and

$$R(\tilde{s})(z) := \sup_{x \in S}\{\tilde{s}(x) \wedge \tilde{r}(x, z)\} \quad (z \in \mathbf{R}). \tag{1.4}$$

For the dynamic fuzzy system $(S, \tilde{q}, \tilde{r})$, if we give an initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \tag{1.5}$$

In the following section, a fuzzy stopping problem for $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$ is formulated.

2 A fuzzy stopping problem

For the sake of brevity, we denote $\mathcal{F} := \mathcal{F}(S)$. The metric ρ on $\mathcal{F}$ is given as $\rho(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \rho_S(\tilde{u}_\alpha, \tilde{v}_\alpha)$ for $\tilde{u}, \tilde{v} \in \mathcal{F}$. Let $\mathcal{B}(\mathcal{F})$ be the set of all Borel measurable subsets of $\mathcal{F}$ with respect to ρ. Putting by $\Omega_t := \mathcal{F}^t$ the t times product of $\mathcal{F}$ and by $\mathcal{B}_t := \mathcal{B}(\mathcal{F}^t)$ the set of Borel measurable subsets of $\mathcal{F}^t$ with a metric ρ^t on $\mathcal{F}^t$ defined by

$$\rho^t(\{\tilde{s}_l\}_{l=1}^t, \{\tilde{s}'_l\}_{l=1}^t) := \sum_{l=1}^{t} 2^{-(l-1)} \rho(\tilde{s}_l, \tilde{s}'_l), \tag{2.1}$$

($\{\tilde{s}_l\}_{l=1}^t, \{\tilde{s}'_l\}_{l=1}^t \in \mathcal{F}^t$) for $1 \leq t \leq \infty$, we can interpret $\{\tilde{s}_t\}_{t=1}^{\infty} \in \Omega_\infty$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.5) with any given initial fuzzy state $\tilde{s}_1 = \tilde{s} \in \mathcal{F}$. Here, applying the idea of fuzzy termination time in Kacprzyk [5, 6], we will define a fuzzy stopping time. Let $\mathbf{N}$ be the set of all natural numbers.

Definition 2. A fuzzy stopping time is a fuzzy relation $\tilde{\sigma} : \Omega_\infty \times \mathbf{N} \mapsto [0,1]$ such that

(i) for each $t \geq 1$, $\tilde{\sigma}(\cdot, t)$ is $\mathcal{B}_t$-measurable, and

(ii) for each $\overline{\omega} \in \Omega_\infty$, $\tilde{\sigma}(\overline{\omega}, \cdot)$ is non-increasing and there exists $t_{\overline{\omega}} \in \mathbf{N}$ with $\tilde{\sigma}(\overline{\omega}, t) = 0$ for all $t \geq t_{\overline{\omega}}$.

In the grade of membership of stopping times, '0' and '1' represent 'stop' and 'continue' respectively. We denote by Σ the set of all fuzzy stopping times.

Lemma 2.1. *Let any $\tilde{\sigma} \in \Sigma$. Define a map $\tilde{\sigma}_\alpha : \Omega_\infty \mapsto \mathbf{N}$ by*

$$\tilde{\sigma}_\alpha(\varpi) = \min\{t \geq 1 \mid \tilde{\sigma}(\varpi, t) < \alpha\} \quad (\varpi \in \Omega_\infty) \tag{2.2}$$

for $\alpha \in (0,1]$. Then, we have:

(i) $\{\tilde{\sigma}_\alpha \leq t\} \in \mathcal{B}_t \quad (t \geq 1)$;

(ii) $\tilde{\sigma}_\alpha(\varpi) \leq \tilde{\sigma}_{\alpha'}(\varpi) \quad (\varpi \in \Omega_\infty)$ *if* $\alpha \geq \alpha'$;

(iii) $\lim_{\alpha' \uparrow \alpha} \tilde{\sigma}_{\alpha'}(\varpi) = \tilde{\sigma}_\alpha(\varpi)$ $(\varpi \in \Omega_\infty)$ *if* $\alpha > 0$.

In order to complete the description of an optimal fuzzy stopping problem, we will specify a function which measures the system's performance when a fuzzy stopping time $\tilde{\sigma} \in \Sigma$ and an initial fuzzy state $\tilde{s} \in \mathcal{F}$ are given. We define $\omega_\infty(\cdot) : \mathcal{F} \mapsto \Omega_\infty$ by

$$\omega_\infty(\tilde{s}) := \{\tilde{s}_t\}_{t=1}^\infty, \tag{2.3}$$

and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\tilde{s}_1 = \tilde{s}$. Let $g : \mathcal{C}(\mathbf{R}) \mapsto \mathbf{R}$ be a continuous and monotone function. Using this g as a weighting function (see Fortemps and Roubens [4]), the scalarization of the total fuzzy reward will be done by

$$\begin{aligned} G(\tilde{s}, \tilde{\sigma}) &:= \int_0^1 g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha)\, d\alpha \\ &= \int_0^1 g\left(\sum_{t=1}^{\tilde{\sigma}_\alpha - 1} R(\tilde{s}_t)_\alpha\right) d\alpha, \end{aligned} \tag{2.4}$$

where $\tilde{\sigma}_\alpha := \tilde{\sigma}_\alpha(\omega_\infty(\tilde{s}))$ and $\varphi(\tilde{s}, \tilde{\sigma})_\alpha := \sum_{t=1}^{\tilde{\sigma}_\alpha - 1} R(\tilde{s}_t)_\alpha$ (We define $\sum_1^0 := \{0\}$). Note that $\varphi(\tilde{s}, \tilde{\sigma})_\alpha \in \mathcal{C}(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha)$ is left-continuous on $(0,1]$, so that the right-hand integral of (2.4) is well-defined. Now, our objective is to maximize (2.4) over all fuzzy stopping times $\tilde{\sigma} \in \Sigma$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}$.

Definition 3. For $\tilde{s} \in \mathcal{F}$, a fuzzy stopping time $\tilde{\sigma}^*$ is called $\tilde{s}$-optimal if $G(\tilde{s}, \tilde{\sigma}) \leq G(\tilde{s}, \tilde{\sigma}^*)$ for all $\tilde{\sigma} \in \Sigma$. If $\tilde{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}$, $\tilde{\sigma}^*$ is called optimal.

In the following section, the α-cuts of fuzzy stopping time will be investigated, whose results are used to construct an optimal fuzzy stopping time in Section 4.

3 The α-cut of fuzzy stopping times

First, we establish several notations that will be used in the sequel. Associated with the fuzzy relations $\tilde{q}$ and $\tilde{r}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ and $R_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(\mathbf{R})$ $(\alpha \in [0,1])$ are defined, respectively, as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{y \in S \mid \tilde{q}(x,y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \tag{3.1}$$

and

$$R_\alpha(D) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x,z) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{z \in \mathbf{R} \mid \tilde{r}(x,z) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \tag{3.2}$$

By $\tilde{q} \in \mathcal{F}(S \times S)$ and $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$, the maps Q_α and R_α $(\alpha \in [0,1])$ are well-defined. The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α-cuts of $Q(\tilde{s})$ and $R(\tilde{s})$ defined by (1.3) and (1.4) are specified using the maps Q_α and R_α.

Lemma 3.1 ([9, 10]). *For any $\alpha \in [0,1]$ and $\tilde{s} \in \mathcal{F}$, we have:*

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $R(\tilde{s})_\alpha = R_\alpha(\tilde{s}_\alpha)$;

(iii) $\tilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha) \quad (t \geq 1)$,

where $\tilde{s}_{t,\alpha} := (\tilde{s}_t)_\alpha$ and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\tilde{s}_1 = \tilde{s}$.

Here we need the following assumption which is assumed to hold henceforth.

Assumption A (Lipschitz condition). There exists a constant $K > 0$ such that

$$\rho_S(Q_\alpha(D_1), Q_\alpha(D_2)) \leq K \rho_S(D_1, D_2) \tag{3.3}$$

for all $\alpha \in [0,1]$ and $D_1, D_2 \in \mathcal{C}(S)$.

Theorem 3.1. *Let a fuzzy stopping time $\tilde{\sigma} \in \Sigma$. Then, the map $\tilde{\sigma}'(\cdot,\cdot) : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ defined by $\tilde{\sigma}'(\tilde{s},t) := \tilde{\sigma}(\omega_\infty(\tilde{s}),t)$ $(\tilde{s} \in \mathcal{F}, t \in \mathbf{N})$ has the following properties (i) and (ii):*

(i) *$\tilde{\sigma}'(\cdot,t)$ is $\mathcal{B}(\mathcal{F})$-measurable for each $t \geq 1$.*

(ii) *For each $\tilde{s} \in \mathcal{F}$, $\tilde{\sigma}'(\tilde{s},\cdot)$ is non-increasing and there exists $t_{\tilde{s}} \in \mathbf{N}$ such that $\tilde{\sigma}'(\tilde{s},t) = 0$ for all $t \geq t_{\tilde{s}}$.*

Observing (2.4) and the form of the objective function $G(\tilde{s},\tilde{\sigma})$ for our stopping problem, we can confine ourselves to the class of fuzzy stopping times $\tilde{\sigma}'(\cdot,\cdot) : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ satisfying (i) and (ii) in Theorem 3.1, and so the class of such fuzzy stopping times will be denoted by Σ'. The following theorem is useful in constructing an optimal fuzzy time which is done in Section 4.

Theorem 3.2. *Suppose that, for each $\alpha \in [0,1]$, there exists a $\mathcal{B}(\mathcal{C}(S))$-measurable map $\sigma_\alpha : \mathcal{C}(S) \mapsto \mathbf{N}$. Using this family $\{\sigma_\alpha\}_{\alpha\in[0,1]}$, define the map $\tilde{\sigma} : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ by*

$$\tilde{\sigma}(\tilde{s},t) := \sup_{\alpha\in[0,1]} \{\alpha \wedge 1_{\{\sigma_\alpha(\tilde{s}_\alpha)>t\}}\}, \quad \tilde{s} \in \mathcal{F},\ t \geq 1. \tag{3.4}$$

Then, if for each $\tilde{s} \in \mathcal{F}$, $\sigma_\alpha(\tilde{s}_\alpha)$ is non-increasing and left-continuous in $\alpha \in [0,1]$, it holds that

(i) *$\tilde{\sigma} \in \Sigma'$, and*

(ii) *$\sigma_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid \tilde{\sigma}(\tilde{s},t) < \alpha\}$ $(\alpha \in (0,1])$.*

4 Optimal fuzzy stopping times

In this section, we try to construct an optimal fuzzy stopping time, by applying an approach by α-cuts. Now, we define a non-fuzzy stopping problem specified by $\mathcal{C}(S)$, Q_α and R_α $(\alpha \in [0,1])$, associated with the fuzzy stopping problem considered in the preceding section. For each $\alpha \in [0,1]$ and any initial subset $c \in \mathcal{C}(S)$, a sequence $\{c_t\}_{t=1}^\infty \subset \mathcal{C}(S)$ is defined by

$$c_1 := c \quad \text{and} \quad c_{t+1} := Q_\alpha(c_t) \ \ (t \geq 1). \tag{4.1}$$

Let Σ_1 be the family of all maps $\sigma : \mathcal{C}(S) \mapsto \mathbf{N}$ such that

$$\{\sigma = t\} \in \mathcal{B}(\mathcal{C}(S)) \quad \text{for each } t \geq 1. \tag{4.2}$$

Using this sequence $\{c_t\}_{t=1}^\infty$ given by (4.1) with $c_1 := c$, let

$$\varphi^\alpha(c,t) := \sum_{l=1}^{t-1} R_\alpha(c_l) \quad \text{for } c \in \mathcal{C}(S). \tag{4.3}$$

Note that $\varphi^\alpha(c,\sigma(c)) = \sum_{l=1}^{\sigma(c)-1} R_\alpha(Q_\alpha^{t-1}(c)) \in \mathcal{C}(\mathbf{R})$ for all $\sigma \in \Sigma_1$. The non-fuzzy stopping problem considered here is to maximize $g(\varphi^\alpha(c,\sigma(c)))$ over all $\sigma \in \Sigma_1$, where g is the weighting function given in Section 2. A map $\tau_\alpha \in \Sigma_1$ is called an α-optimal stopping time if

$$g(\varphi^\alpha(c,\tau_\alpha(c))) \geq g(\varphi^\alpha(c,\sigma(c))) \quad \text{for all } \sigma \in \Sigma_1.$$

In order to characterize α-optimal stopping times, let

$$\gamma_t^\alpha(c) := \sup_{\sigma\in\Sigma_t} g(\varphi^\alpha(c,\sigma(c))) \tag{4.4}$$

for $t \geq 1$ and $c \in \mathcal{C}(S)$, where $\Sigma_t := \{\sigma \vee t \mid \sigma \in \Sigma_1\}$ $(t \geq 1)$. Then, the next lemma is given as deterministic versions of the results for stochastic stopping problems in Chow et al. [3].

Lemma 4.1 (c.f. [3, Theorems 4.1]). *It holds that*

$$\gamma_t^\alpha(c) = \max\{g(\varphi^\alpha(c,t)), \gamma_{t+1}^\alpha(c)\}$$

$(t \geq 1, c \in \mathcal{C}(S))$.

For $c \in \mathcal{C}(S)$, let $\tau_\alpha^*(c)$ be the first time $t \geq 1$ such that $g(\varphi^\alpha(c,t)) \geq \gamma_{t+1}^\alpha(c)$.

Lemma 4.2 (c.f. [3, Theorem 4.5]). *Let $\alpha \in [0,1]$. Suppose that $\lim_{t\to\infty} g(\varphi^\alpha(c,t)) = -\infty$ and $\sup_{t\geq1} g(\varphi^\alpha(c,t)) < \infty$ for each $c \in \mathcal{C}(S)$. Then, τ_α^* is α-optimal and $\gamma_1^\alpha(\cdot) = g(\varphi^\alpha(\cdot,\tau_\alpha^*(\cdot)))$.*

Recently, Kadota et al. [8] discusses the validity of the one-step look ahead (OLA) policy for Markov chains with general utility, whose results are applicable to our problem. For the OLA stopping time, refer to Ross [14].

Assumption B (Closedness). For any $\alpha \in [0,1]$, if $(\varphi^\alpha(\tilde{s}_\alpha, t), \tilde{s}_{t,\alpha}) \in K^\alpha(g)$ for some t, then $(\varphi^\alpha(\tilde{s}_\alpha, t'), \tilde{s}_{t',\alpha}) \in K^\alpha(g)$ for all $t' > t$.

Lemma 4.3. *Suppose Assumption B holds. Then, under the assumption of Lemma 4.2, α-optimal $\tau^*_\alpha(c)$ is represented as*

$$\tau^*_\alpha(c) := \min\{t \in \mathbf{N} \mid (\varphi^\alpha(c,t), c_t) \in K^\alpha(g)\}, \tag{4.5}$$

where $K^\alpha(g) := \{(h,c) \in \mathcal{C}(\mathbf{R}) \times \mathcal{C}(S) \mid g(h) \geq g(h + R_\alpha(c))\}$.

For each $\alpha \in [0,1]$, applying the above lemmas, we can find an α-optimal stopping time τ^*_α. Assuming the existence of α-optimal stopping times for each $\alpha \in [0,1]$, let $\{\tau^*_\alpha\}_{\alpha\in[0,1]}$ be the family of such stopping times. Here, we try to construct an optimal fuzzy stopping time from $\{\tau^*_\alpha\}_{\alpha\in[0,1]}$. For this purpose, we need a regularity condition. In order to derive our main result, we introduce the following Assumption C.

Assumption C (Regularity). $\tau^*_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$.

We can assume the left-continuity of the map $\alpha \mapsto \tau^*_\alpha(\tilde{s}_\alpha)$, by considering $\lim_{\alpha'\uparrow\alpha} \tau^*_{\alpha'}(\tilde{s}_{\alpha'})$ instead of $\tau^*_\alpha(\tilde{s}_\alpha)$. Define a map $\tilde{\tau}^* : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ by

$$\tilde{\tau}^*(\tilde{s}, t) := \sup_{\alpha\in[0,1]} \left\{\alpha \wedge 1_{\{\tau^*_\alpha(\tilde{s}_\alpha) > t\}}\right\} \tag{4.6}$$

for all $\tilde{s} \in \mathcal{F}$ and $t \in \mathbf{N}$.

Theorem 4.1. *Suppose Assumptions B and C hold. Then, $\tilde{\tau}^*$ defined by (4.6) is an $\tilde{s}$-optimal fuzzy stopping time. Further, if g is additive, i.e.*

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in \mathcal{C}(S), \tag{4.7}$$

then it holds that

$$\tilde{\tau}^*(\tilde{s}, t + r) = \tilde{\tau}^*(\tilde{s}, t) \wedge \tilde{\tau}^*(\tilde{s}_{t+1}, r) \tag{4.8}$$

for each $\tilde{s} \in \mathcal{F}$ and $t, r \in \mathbf{N}$.

If the regularity does not hold for some $\tilde{s} \in \mathcal{F}$, the $\tilde{s}$-optimality of $\tilde{\tau}^*$ does not follow. But, $\tilde{\tau}^*$ defined by (4.6) is thought of as a good fuzzy stopping time. We note that weighting functions g are usually additive in the sense of (4.7) (see Fortemps and Roubens [4]), and (4.8) gives a concrete representation of the non-increase in Definition 2(ii).

5 A numerical example

In this section, an example is given to illustrate the theoretical results. Let $S := [0,1]$ and $0 < \beta < 0.98$. the fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x,y) = (1 - |y - \beta x|/100) \vee 0, \quad x, y \in [0,1]$$

and

$$\tilde{r}(x,z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise} \end{cases}$$

for $x \in [0,1]$, $z \in \mathbf{R}$, where λ is an observation cost satisfying $\lambda > 1/100(1-\beta)$. Then, Q_α and R_α defined by (3.1) and (3.2) are as follows:

$$Q_\alpha([a,b]) = [\beta a - (1-\alpha), \beta b + (1-\alpha)]$$

and

$$R_\alpha([a,b]) = [a - \lambda, b - \lambda]$$

for $0 \leq a \leq b \leq 1$. Now, let $c = [a,b]$ $(0 \leq a \leq b \leq 1)$ and $g(c) = b$. Then

$$\begin{aligned} g(\varphi^\alpha(c,t)) &= g\left(\sum_{l=1}^{t-1} R_\alpha(c_l)\right) \\ &= \frac{(1-\beta^{t-1})b_\alpha}{1-\beta} - \lambda_\alpha(t-1), \end{aligned}$$

where $b_\alpha := b - (1-\alpha)/100(1-\beta)$ and $\lambda_\alpha := \lambda - (1-\alpha)/100(1-\beta)$ for $\alpha \in [0,1]$. Let

$$\tilde{s}(x) = (1 - |8x - 4|) \vee 0 \quad \text{for } x \in [0.1].$$

Then we see

$$\tilde{s}_\alpha = \left[\frac{3+\alpha}{8}, \frac{5-\alpha}{8}\right].$$

Therefore

$$\tau^*_\alpha(\tilde{s}_\alpha) = \left[\log \frac{\lambda_\alpha(1-\beta)}{-b_\alpha \log\beta} \Big/ \log\beta\right] + 1,$$

where $[z]$ is the largest integer equal to or less than a real number z. Since $\tilde{s}$ is regular with respect to $\{\tau^*_\alpha\}_{\alpha\in[0,1]}$, Theorem 4.1 gives the $\tilde{s}$-optimal fuzzy stopping time $\tilde{\tau}^*$.

6 Concluding Remarks

The aim of this paper is to consider an application of dynamic fuzzy systems in mathematical economics. Then we need to deal with fuzzy rewards which is a particular notion in fuzzy mathematical economic models, and the fuzzy rewards for fuzzy stopped systems are estimated by a scalarization. In this paper, we move fuzzy stopping times themselves in a class of fuzzy stopping times to maximize the fuzzy rewards, and we discuss the properties of optimal fuzzy stopping times. This approach is one step of the studies about the fuzzy stopping. We hope that the optimization problem will be discussed from various approaches and will be applied to many fields of practical applications.

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