A note on properties for a complementary graph and its tree graph

Abulimiti Yiming*

Department of Mathematics

Institute of Mathematics & Informations

Xinjiang Normal University

China

Masami Yasuda†

Department of Mathematics & Informations

Faculty of Science

Chiba University

Chiba, 263-8522

Japan

Abstract

In this note a complementary graph in n-th $(n \geq 2)$ order is defined and discussed its property. Since it can not be reduced from n-th order to 2-nd, we must consider its property of n-th order graph separately. We investigate whether similar properties of the graph arising from 2-nd order case hold or not. A relation to the complete graph K_{2n} and also to the tree graph associated with the complementary graph are studied.

Keywords : *???*

0. Introduction

Let $G = (V, E)$ be a non-directed graph whose vertex set is V and edge is E. Assume its degree is p and the size is q, that is, $|V| = p$, $|E| = q$. Here we permit the graph G has multiple edges but there are

* *E-mail*: ablimit1@hotmail.com

† *E-mail*: yasuda@math.s.chiba-u.ac.jp

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no loops. The definition and notation is usual one so refer to the classical noted book [7], etc. If the connected graph G has two spanning trees T_1 and T_2, which satisfies $G = T_1 \cup T_2$ with $T_1 \cap T_2 = \emptyset$, we call the graph as a 2-nd order complementary tree or the graph has a structure of 2-nd order complementary.

It is known that Lee [3] applies this complementary graph to the theory of circuit network firstly and then Kajitani [1] discussed several dis-cussion in 2-nd order. There are few paper concerning with this topics however. In this note we will study this notion of complementary tree from 2-nd order to n-th$(n \geq 2)$ and investigate their properties.

1. Definition and notations

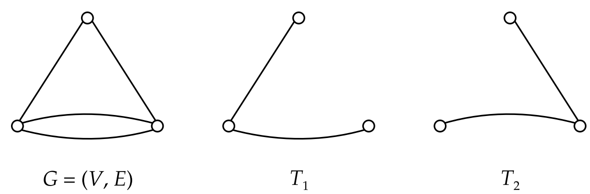
Definition 1.1. If there exist n spanning trees $T_1, T_2, \cdots, T_n$ in a given connecting graph G, and each satisfies the following conditions:

(i) $G = T_1 \cup T_2 \cup \cdots \cup T_n$

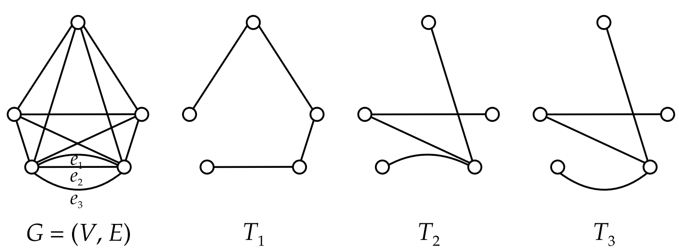
(ii) $T_i \cap T_j = \emptyset$, $i \neq j$; $i, j = 1, 2, \cdots, n$,

then the graph G is called as an n-th order *complementary graph*.

Example 1. The next graph G is a sample for 2-nd order complementary graph.



Example 2. The next graph G is a sample for 3-rd order complementary graph. We note that the 3-rd order graph cannot be reduced to 2-nd one because the definition require each graph must be a tree.



Now we will discuss several properties for the n-th order complimentary tree.

Theorem 1.1. *For an n-th order complementary graph $G = (V, E)$ with $|V| = p$, $|E| = q$, then it holds that the equality $q = n(p-1)$ and $\lambda(G) \geq n$ provided $\lambda(G)$ means the number of connected component for G.*

Proof. The concernment is clear from the definition. To prove the equality, we consider each edge. Since each spanning tree of G has same $p-1$ edges, so there are $n(p-1)$ edges in total. And every vertex connects to all spanning tree. The least number of cutting edge is n. Therefore the graph G has n-edges of cutting. Thus the edge connection of G is greater than n. □

Theorem 1.2. *The rank $r(G)$ of n-th order complementary graph G equals $r(G) = p - 1$ and the nullity $\mu(G)$ equals $\mu(G) = (n-1)(p-1)$.*

Proof. Since G is connected, the incident matrix of G has rank $n-1$. So the rank of G is $n-1$. This is also seen from the definition in directly. For any graph, it is well known that the nullity $\mu(G) = q - p + \omega(G)$ where $\omega(G)$ denotes the number of components of G. In this note the complementary graph is assumed to be connected so $\omega(G) = 1$. Hence $\mu(G) = q - p + 1$. Substitute for $q = n(p-1)$, the relation $\mu(G) = (n-1)(p-1)$ is obtained. □

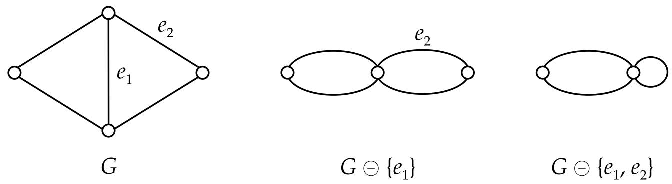
Theorem 1.3. *Every edge of n-th order complementary graph G is contained a cycle of G:*

$$\forall\, e_i \in G \Rightarrow \exists\, C_i,\ e_i \in C_i\,.$$

Proof. Every edge of G is contained by a spanning tree T_i for some i. There exists other edge which connecting to it. By adding the edge to the tree, it becomes loop, that is, a cycle. □

Definition 1.2. Let $e = xy$ be an edge connecting between the vertex x and y in a graph G. The length of edge e becomes shrinking as a vertex. The graph obtained by this manipulation, it is called a *contraction* of G and denoted by $G \ominus \{e\}$. Similarly $G \ominus \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\}$ is called the contraction of n-th order. Alternatively by cutting edges e from the graph G, it is a *removal* denoted by expressed as $G - \{e\}$ and also $G - \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ is the removal of n-th order.

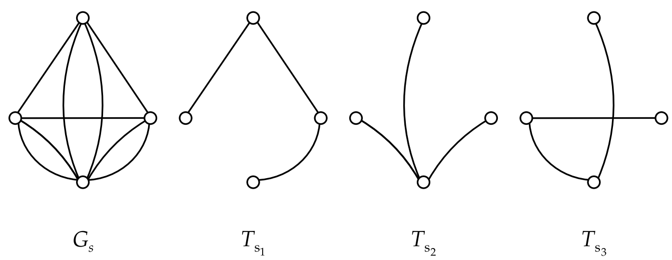
Example 3. The next figure is a sample of reduction.



Theorem 1.4. *For n-th order complementary graph G with multiple edges; $e_{i1}, e_{i_2}, \cdots, e_{i_n}$, the reduction $G \ominus \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\}$ of G is also n-th order complementary graph G.*

Proof. By the assumption, let $G = T_1 \cup T_2 \cup \cdots \cup T_n$ with $T_i \cap T_j = \emptyset$, $i \neq j;\ i, j = 1, 2, \cdots, n$. Since the number of two connecting vertices is less than or equal to n, $G \ominus \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\}$ has no loop. Therefore, for the edges $e_{i_j} \in T_j$, $j = 1, 2, \cdots, n$ the contraction $T_j^* = T_j \ominus \{e_{ij}\}$ has the property $\cup_j T_j^* = (\cup_j T_j) \ominus \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\} = G \ominus \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\}$ and $T_i \cap T_j = \emptyset$, $i \neq j;\ i, j = 1, 2, \cdots, n$. Thus the assertion holds from Definition 1.1. □

Example 4. The graph G of Example 3 is 3-rd order complementary graph. The contraction subset G_s of its multiple edge $\{e_1, e_2, e_3\}$ is also 3-rd order complementary graph:

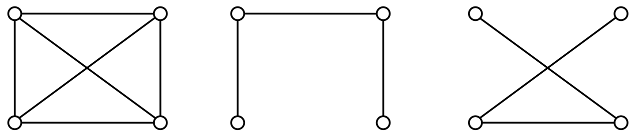


2. The case of the complete graph

Now we will discuss the important class of the complete graph is included by the complementary graph.

Theorem 2.1. *The complete graph K_{2n} $(n \geq 2)$ is also n-th order complementary graph.*

Proof. We will prove it by a mathematical induction. For $n = 2$, the graph $K_{2\times 2} = K_4$ is seen to be a 2-nd order complementary graph immediately.



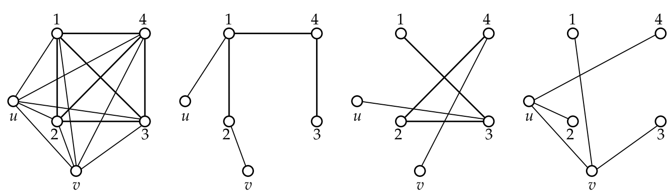
$K_{2\cdot 2} = K_4$ T_1 T_2

Assume the case of $n = k$ holds. That is, there exist k spanning tree $T_1, T_2, \cdots, T_k$ with the property $K_{2k} = T_1 \cup T_2 \cup \cdots \cup T_k$ and they are satisfy $T_i \cap T_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \cdots, k$. Let u, v be two vertices in the complete graph $K_{2(k+1)}$ and others are simply denoted by $1, 2, \cdots, 2k$. Since the graph is complete, each u, v connects to other $2k$ vertices. So the edges connecting between u and $1, 2, \cdots, 2k$ are denoted by $e_{u,1}, e_{u,2}, \cdots, e_{u,2k}$, and similarly the edges connecting between v and $1, 2, \cdots, 2k$ are denoted by $e_{v,1}, e_{v,2}, \cdots, e_{v,2k}$. The edge $e_{u,v}$ is between u and v.

By the assumption the removed graph $K_{2k} = K_{2(k+1)} - \{u, v\}$ is k-th order complementary graph, there exists k spanning tree $T_1', T_2', \cdots, T_k'$ and these satisfy that $T_1' \cup T_2' \cup \cdots \cup T_k' = K_{2k} = K_{2(k+1)} - \{u, v\}$ and $T_i' \cap T_j' = \emptyset$, $i \neq j$; $i, j = 1, 2, \cdots, k$.

If we introduce a new tree $T_i = T_i' \cup \{e_{u,2i-1}, e_{v,2i}\}$, $i = 1, 2, \cdots, k$ and by letting $T_{k+1} = \{e_{u,2}, e_{v,3}, \cdots, e_{u,2k-2}, e_{v,2k-1}, e_{u,2k}, e_{v,1}, e_{u,v}\}$, then Definition 1.1 imply that $K_{2(k+1)}$ is a $k+1$-th order complementary tree. Therefore, for arbitrary $n \in N$, we have proved that K_{2n} is n-th order complementary graph. $\square$

The next graph denotes a complete graph corresponding 3-rd order complementary graph.

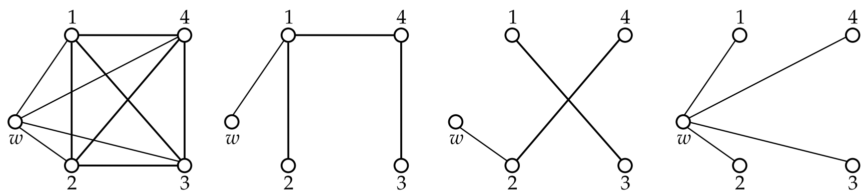


$K_{2\cdot 3} = K_6$ T_1 T_2 T_3

Corollary 2.2. *For the complete graph K_{2n+1}, there exist n spanning trees $T_1, T_2, \cdots, T_n$ with $T_i \cap T_j = \emptyset$, $i \neq j$; $i, j = 1, 2, \cdots, n$. Also there exists a spanning tree T which distance equals $d(T \cap T_i) = p - 2$ for each T_i and $|T \cap T_i| = 1$ simultaneously.*

Proof. Let ω be an arbitrary vertex in the complete graph $K_{2(n+1)}$ and other $2n$ vertices are denoted by $1, 2, \cdots, 2n$. In this case the removal $T_{2n+1} - \{\omega\} = K_{2n}$ is a complete graph. By the previous Theorem 2.1, there exist n spanning trees $T_1', T_2', \cdots, T_n'$ which satisfy $T_1' \cup T_2' \cup \cdots \cup T_n' = K_{2n}$ and $T_i' \cap T_j' = \emptyset$, $i \neq j$; $i, j = 1, 2, \cdots, n$.

Denote each edge between a vertex ω and vertices $1, 2, \cdots, 2n$ by $e_{\omega,1}, e_{\omega,2}, \cdots, e_{\omega,2n}$ respectively, each graph $T_i = T_i' \cup \{e_{\omega,i-1}\}$ is n spanning trees of $K_{2(n+1)}$. Also $T_i \cap T_j \neq \emptyset$, $i \neq j$; $i, j = 1, 2, \cdots, n$ hold and $T = \{e_{\omega,1}, e_{\omega,2}, \cdots, e_{\omega,2n}\}$ is a spanning trees of $K_{2(n+1)}$ too. See the following figure.



$K_{2\cdot3+1} = K_5$ $\quad T_1 \quad T_2 \quad T_3$

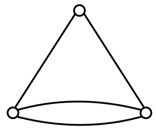
Note that $T \cap T_i = \{e_{2i-1}\}$, $i = 1, 2, \cdots, n$. Thus $d(T, T_i) = 2n - 1 = p - 2$ holds, where $d(T, T_i) = \frac{1}{2}N(T \oplus T_i)$, where $N(T \oplus T_i) = |T \triangle T_i| = |(T \cup T_i) \setminus (T \cap T_i)|$ means a symmetrical difference T and T_i. $\square$

3. Relation to a tree graph

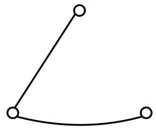
The relation between a tree graph and a complementary tree is known [7]. First the definition of a tree graph corresponding to a graph is given.

Definition 3.1. The *tree graph* corresponding to a given graph $G = (V, E)$ is a graph denoted by $T(G) = (V_t, E_t)$ and their vertices and edges are defined as follows: Each vertex v_{tk} of $T(G)$ has one to one correspond to each of a spanning tree T_k in G, and two vertices $v_{ti}, v_{tj} \in V_t$ has a distance $d(v_{ti}, v_{tj}) = 1$ provided that these are adjoining and non-directed. The distance between vertices is defined by $d(T, T_i) = \frac{1}{2}N(T \oplus T_i)$, where $N(T \oplus T_i) = |T \triangle T_i| = |(T \cup T_i) \setminus (T \cap T_i)|$ means a symmetrical difference T and T_i.

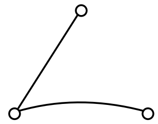
Example 5. The next T_i, $i = 1, 2, \cdots, 5$ illustrates all of spanning trees for $G = (V, E)$ and its tree graph $T(G) = (V_t, E_t)$.



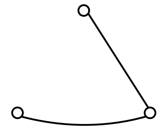
$G = (V, E)$



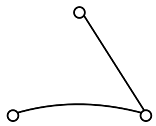
T_1



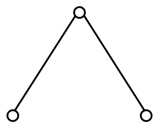
T_2



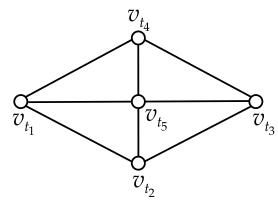
T_3



T_4



T_5



$G = (V_t, E_t)$

The next theorem is a characterization of 2-nd order complementary graph by its tree graph. However we see that it is unknown for case of n-th $(n \geq 3)$ order. Example 6 is a counterexample of general case.

Theorem 3.1. *Let there are p vertices and $2(p-1)$ edges in the graph $G = (V, E)$. The necessarily and sufficient condition to be $G = (V, E)$ a 2-nd order complementary graph is that there exists pairing vertices v_i, v_j in the tree graph $T(G) = (V_t, E_t)$ for $G = (V, E)$ and the shortest length between them is equal to $p - 1$.*

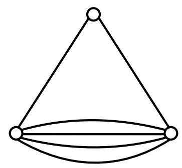
Proof. "*Sufficiency*": Let v_i, v_j are adjoin two vertices in a tree graph $T(G)$ which is corresponding to a graph G. Since the shortest length of path is $p-1$, their connecting path $v_i - v_j$ is written as $P = v_i v_{i_1}, \cdots, v_{i_{p-2}} v_j$. From Definition 3.1, there exist p spanning trees $T_i, T_{i_1}, \cdots, T_{i_{p-2}} T_j$ and they correspond to $v_i, v_{i_1}, \cdots, v_{i_{p-2}}, v_j$ of the vertices in $P(G)$. Also, by the same Definition 3.1, $d(T_i, T_{i_1}) = 1$, $d(T_{i_1}, T_{i_2}) = 1, \cdots, d(T_{i_{p-n}}, T_{i_j}) = 1$ hold. Since the length from v_i to v_j is $p-1$, $d(T_i, T_{i_k}) \geq k$, $k = 1, 2, \cdots, p-2$. Thus we have $d(T_i, T_j) = p-1$. Clearly the number of edges in G equals $2(p-1)$, so $T_i \cap T_j = G$ and $T_i \cap T_j = \emptyset$. Therefore $G = (V, E)$ is a 2-complementary tree.

"*Necessity*": If $G = (V, E)$ is 2-nd order complementary graph, there exist two spanning trees T_1, T_2 such that $T_1 \cap T_2 = G$ and $T_1 \cap T_2 = \emptyset$. By converting the edge of trees, we associate with other $p-2$ spanning trees $T_{i_1}, \cdots, T_{i_{p-2}}$. Thus $T_1, T_{i_1}, \cdots, T_{i_{p-2}}, T_2$ and $d(T_1, T_{i_1}) = 1, d(T_{i_1}, T_{i_2}) = 1$, $d(T_1, T_{i_2}) = 1, d(T_{i_{p-2}}, T_2) = 1, d(T_1, T_2) = 1$ are hold. For a tree graph

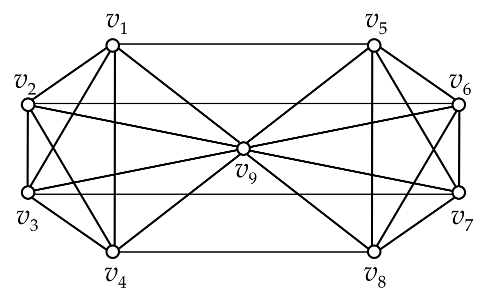
$T(G)$ of G, there is a pairing vertex v_1, v_2. Thus there is a shortest path of length $p-1$ from v_1 to v_2. □

As a remark of this theorem, the above assertion does not hold in general because the following example shows.

Example 6. The figure G has $p = 3$ vertices and $3(p-1)$ edges. This graph is not 3-complementary. But its tree graph $T(G)$ has the shortest length $p-1=2$ from v_1 to v_8.



G



$T = (G)$

The next theorem is a partial answer for characterization of n-th order complementary graph by using the tree graph.

Theorem 3.2. *There is an n-th order complementary graph for the tree graph $T(C_{2n})$ where C_{2n} is a cycle of order $2n$.*

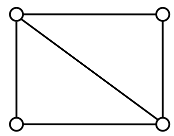
Proof. Since C_{2n} has $2n$ spanning trees, and the distance with each others equal 1, so $T(C_{2n})$ and K_{2n} is equivalent. Theorem 2.1 implies that $T(C_{2n})$ is n-th order complementary graph. □

From this Theorem 3.2, we can prove the following corollary:

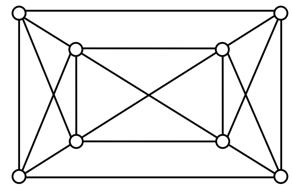
Corollary 3.3. *If G is simple, the tree graph $T(G)$ which number of vertices is $n(\geq 3)$ contains K_n as its subgraph.*

Proof. The graph G is connected however it is not tree, so it contains a cycle of the length at least three. Similarly the distance between them in the cycle equals one. Therefore at least three vertices of $T(G)$ are adjoined with each other. □

The next figure shows eight vertices of a tree graph $T(G)$ which contain K_3 and K_4.



G



$T = (G)$

Corollary 3.4. *If G is a simple graph, there are no tree graphs which consists of only two vertices.*

Proof. Assume that if there exist a tree graph with two vertices. Then the graph G has two spanning trees. This is impossible. $\square$

Corollary 3.5. *For $n > 3$, the tree graph $T(C_n)$ contains $\left\lfloor \dfrac{n}{2} \right\rfloor$-th order complementary graph where the notation $\lfloor x \rfloor$ denotes the gamester integer $\leq x$.*

Proof. When $n = 2k$, a tree graph $T(G)$ has k-complementary graph by Theorem 3.2. If $n = 2k + 1$, $T(G)$ contains k-complementary graph by Corollary 3.3. These complete the proof. $\square$

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