the next two propositions. The subset B of S in (1.2) is closed if
\[ P(j, i) = 0 \quad \text{for } i \notin B, \quad j \in B \]
where \( B \) denotes the complement of the set B. The process is stable if the sequence defined by \( v_n(i) = \tau(i) \),
\[ v_n(i) = \max_{j \in N} (v_{n-1}(j) - c(i)) \]
converges uniformly and \( \lim_{n \to \infty} v_n(i) = v(i) \) for \( i \in B \).

Proposition 2.1 (Ross[10]). If the process is stable and the set B is closed, then the OLA policy is optimal.

Although this situation occurs in many applications and is useful to determine the stopping region, the proposition does not state the optimal value. The stability assumption is somewhat less satisfactory to check in the application because the optimal value is unknown. So we impose assumption on a potential of the chain. Our aim is to calculate the optimal expected value under the closedness and the following equalization assumption in stead of the stability assumption, and express it explicitly by using a potential. We call the problem where the OLA policy is optimal as the OLA-optimal stopping problem.

Proposition 2.2 (Hordijk[5]). Suppose that
\[ v(i) = P F(i) - c(i) \]
when \( \tau(i) = c(i) \) or \( \tau(i) = r(i) \).
If the value function is a potential, then the hitting time of set \( \Gamma \) becomes optimal.

Proof. This is a special case in Theorem 4.1 of Hordijk[5]. Q.E.D.

Let \( P_A \) denote the restriction to a subset \( A \) of the transition probability \( P \). i.e.,
\[ P_A(i, j) = P(i, j) \chi_A(j) \]
where \( \chi_A \) denotes the indicator of set \( A \).

When the stability is dropped, as Ross shows, a stopping problem does not imply the optimality of the OLA policy. So we must impose a condition so as to preserve the OLA principle. The condition of equalizing for the reward function due to a potential notion is considered.

Assumption 2.1. We assume that
\[ \lim_{n \to \infty} (P^n_B) r(i) = 0 \quad \text{for } i \in B \]
where \( B \) denotes the complement of the set B defined by (1.2), and that the potential for \( P_A - c \) is finite-valued with respect to \( P_B \), that is,

\[ v(i) = \max_{j \in N} (v_{n-1}(j) - c(i)) \]
converges uniformly and \( \lim_{n \to \infty} v_n(i) = v(i) \) for \( i \in B \).

The assumption (2.5) is equivalent to
\[ \lim_{k \to \infty} \left\{ \sum_{k=1}^\infty (P(x, y) - 1)^k \right\} \chi_{EB} = 0 \]
where \( \tau \) is the hitting time of \( B \). The property \( \lim_{k \to \infty} (P^n_B) r(i) = 0 \) for the optimal value is called equalizing in Optimal Gambling (Rubins / Savage[3] or Nordjik[5]). One might say that how the actually received in the time period up to \( N \) and the promised earnings equalize as \( N \) tends to infinity. If the optimal value satisfies this property, the assumption (2.5) holds since \( v(i) \geq r(i) \), i.e., by (2.1).

Theorem 2.1. Under the assumptions (2.5) and (2.6), the OLA policy \( T_B \) is optimal, that is, the optimal policy is the hitting time of \( B \). The optimal value \( v(i) = v(i; T_B) \) is given by
\[ v(i) = r(i) + H_P(x, e) \]
for any policy \( \tau \). Hence, we get the first assertion immediately.

Next we show that it satisfies the optimality equation (2.1).

\[ v(i) = P F(i) - c(i) = P F_B(i) + P F_B(i) - c(i) \]
on \( B \). Hence the proof is complete.
max[r(i), P_v(i) - c(i)]
  = max[r(i), P_v(i) - c(i)]
  = r(i) for i ∈ B.

On the other hand, by substitution of (2.7),

P_v(i) - c(i) = P_B v(i) + P_{B|c} - c(i)
  = [P_B P_{B|c}] v(i) + P_{B|c} - c(i)
  = [P_B P_{B|c}] v(i), i ∈ S.

On i ∈ S, that
c(i) < P_v(i) - c(i) = P_B v(i) + P_{B|c} - c(i)
implies r(i) < P_B v(i) < P_{B|c} - c(i). And, by the assumption (2.6),

v(i) ≤ H_B P_{B|c} - c(i) for i ∈ S. Combining the above assertions,

max[r(i), P_v(i) - c(i)]
  = max[r(i), H_B P_{B|c} - c(i)]
  = H_B P_{B|c} - c(i) for i ∈ S.

Thus the value v(i) = v(i; c) satisfies the optimality equation.

It now remains to calculate the potential H(Pr-c) v(i), i ∈ S.

From the definition of the set B in (1.2), we have (Pr-c) v(i) = 0 on B. So the support of charge is the complement of B and hence

H(Pr-c) v(i) = 0 on B.

On the other hand, since (Pr-c) v(i) = (Pr-c) v(i) = (P_B P_{B|c}) v(i) for i ∈ S, we have that

P(Pr-c) v(i) = P_B P_{B|c} v(i)
  = (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i) + (P_B) v(i), i ∈ S.

Repeating this procedure to take the expectation up to k times and adding those,

v(i) ≤ H_B P_{B|c} - c(i) ≤ H_B P_{B|c} - c(i) ≤ H_B P_{B|c} - c(i) ≤ r(i), i ∈ S.

Hence

H(Pr-c) v(i) = H_B P_{B|c} - c(i) - r(i), i ∈ S
follows immediately from the assumptions (2.5) and (2.6). Q.E.D.

We remark that the upper bound on the optimal value in Theorem 3.6 of Darling[2] equals exactly the optimal value in this case. That is, the bound holds with equality when the OLA policy is optimal and it is equalizing. This explicit solution and the proposition 2.1, 2.2 determine the optimal value and policy completely in the OLA-optimal stopping problem.

3. The Best Choice Problem. In this section we apply the previous method to the typical stopping problem known as the best choice problem. By taking a scale limit, the asymptotic form of the problem is considered in order to get an analytical explicit solution. In the asymptotic form the state space of the problem is not countable but the unit interval. However immediately the previous method can be applied to the case. Some of results are well known and the optimal values are already obtained by some method, for example, by using differential equation. Here we intend to illustrate it for the application which optimal value can be determined by the unified previous method. Each of those are cases whose problem is the OLA-optimal stopping one.

3.1. The Classical Secretary Problem. The secretary problem, variously called dowry problem or Gougal, is an optimal stopping problem based on relative ranks for objects arriving in a random fashion; the objective is to find the stopping rule that maximizes the probability of stopping at the best object of the sequence. The optimality equation for the optimal value v(i), i ∈ S, the maximal probability of win, on the state space S = {1, 2, ..., n}, is as follows:

v(i) = max [i/n, 1, v(j)/(j-1)], i ∈ S, n ∈ N, v(n) = 1

where there are no costs. This formulation of the problem as Markov Process is given by Dynkin and Yushkevich[4]. Also refer to Chow/Koblis/Sieg mund[1] or Shiryayev[12]. To consider the problem in the asymptotic form, take the scale limit by i/n → x as i, n → ∞. It becomes that

(3.1) v(x) = max{[1 - x]^[1/x] v(y), x ∈ S}, x ∈ [0, 1].

The solution of (3.1) is well known as

(3.2) v(x) = \begin{cases} 0 & \text{on } [0, 0^+), \\ x & \text{on } [0^+, 1] \end{cases}

Now we extend the problem by changing the reward function r(x) = x, which means probability of choosing the best object in the classical secretary problem, to a general reward r(x). To ensure the OLA policy is still optimal, we assume, for a function h(x) defined by

(3.3) h(x) = r(x) - x ∫ r(y)y^{-2}dy on [0, 1],

that

\begin{cases} 1 - x & \text{on } [0, 0^+), \\ x & \text{on } [0^+, 1]\end{cases}

x ∈ [0, 1].
1) each term of the function \( h(x) \) is finite-valued on \([0,1]\), and
2) it changes its sign from - to + only once as \( x \) increases \( \theta \) to \( 1 \) and so the equation \( h(x) = 0 \) has a unique solution \( \omega \).

The optimality equation (3.1) with the rowed function \( v(x) \) is

\[
(3.4) \quad v(x) = \max\{\omega(x), x \int v(y) y^{-2} dy, \chi \}, \quad x \in [0,1].
\]

provided that the underlying Markov chain is unchanged. The solution of this equation (3.4) is given by

\[
(3.5) \quad v(x) = \begin{cases} \omega(x) & \text{on } [0,\omega), \\ \chi & \text{on } [\omega,1] \end{cases}
\]

where \( \omega \) is defined by (3.1).11

In fact, one can show this (3.5) by applying the previous result. Straightforward calculation yields that the set \( B \) becomes \([\omega,1]\) and it is closed with respect to the transition probability:

\[
P(x, dy) = \begin{cases} xy^{-2} dy & \text{for } y \geq x, \\ \theta & \text{otherwise}. \end{cases}
\]

We have \( P_{\theta}^n(x, [\omega,1]) = \omega(x) / \omega \) and \( P_{\theta}^n(x, [\omega,1]) = \omega(x) / \omega \) for \( x \in [0,\omega), \ n = 0,1,2,\ldots \). Hence

\[
\begin{align*}
N_{\theta}^n(x) &= \omega(x) \omega(1) \log_n(n+1) + 2\log^2(n) + \ldots \\
&= \omega(x) \\
&\text{for } x \in [0,\omega) \text{ and thus the assumption (2.6) is satisfied. Also we can check the condition of (2.5)}:
\end{align*}
\]

\[
(\theta P_{\theta}^{n})^{\infty}(x, dy) = \int_0^\infty \omega(y) y^{-2} \log n (y) dy \\
\int_0^\omega dy
\]

tends to zero as \( n \to \infty \) for \( x \in [0,\omega) \).

3.3. A Problem with Refusal Probability. A variant of the best choice problem is a case with a refusal probability discussed by Smith[12]. We can also formulate the problem as optimal stopping on a Markov chain. The asymptotic form of the transition probability is obtained immediately and we have

\[
P(x, dy) = \begin{cases} p y^{-1}(x/y) P_{\omega}^\infty(y) & y \leq x < 1, \\ \theta & \text{otherwise} \end{cases}
\]

where \( p \) is a given parameter \( 0 < p \leq 1 \), which quantity \( 1-p \) means a probability of the refusal. When there is no refusal, i.e., \( p=1 \), it reduces to the classical secretary problem discussed in the section 3.1. Similarly as before, the optimality equation for the stopping problem with refusal probability \( p \) is

\[
(3.6) \quad v(x) = \max\{\omega(x), \int_0^\omega (x/y) P_{\omega}^\infty(y) dy, \chi \}, \quad x \in [0,1].
\]

Define a function \( h(x) \) by

\[
(3.7) \quad h(x) = x - \int_0^\omega (x/y) P_{\omega}^\infty(y) dy, \quad x \in [0,1].
\]

Under the same assumptions as (3.3) and (3.31), we obtain the optimal value with refusal probability as follows.

\[
(3.8) \quad v(x) = \begin{cases} \omega(x) & \text{on } [0,\omega), \\ \chi & \text{on } [\omega,1] \end{cases}
\]

where \( \omega \) is a unique solution of \( h(x) = 0 \) in (3.7).

Another method to solve the best choice problem is given by Hucic[7], which method reduces the value to the solution of a differential equation.

Let

\[
\begin{align*}
\frac{dV(x)}{dx} &= p x^{-1} (x-y) V(y) dy, \quad x \in [0,1] \\
V(1) &= \omega,
\end{align*}
\]

which means a conditional optimal value. This satisfies

\[
(3.9) \quad \begin{cases} \frac{dV(x)}{dx} = -p x^{-1} (x-y) V(y) dy, & x \in [0,1] \\
V(1) &= \omega,
\end{cases}
\]

The optimal value at the beginning \( v^* = V(0) \) equals \( \omega(x) \).

3.3. A Problem with Random Number of Objects. The discussion of the problem for variant on the random number of objects is given by Pramman/Sonin[9]. The random environment in the problem means that there is a distribution \( \pi(x, \omega) \) over \( x \in [0,1] \) which denotes the random number of objects. If we adopt the approach by the differential equation (3.9), the following functional equation is obtained by Yasuda[11]. For \( x \in [0,1] \),

\[
(3.10) \quad \begin{cases} \frac{dV(x)}{dx} = V(x) [1-\pi(x)]^{-1} d \pi(x) - x^{-1} (h(x) - V(x)) dy, \\
V(1) &= \omega,
\end{cases}
\]

where we set
When the distribution is absolutely continuous, (3.10) reduces to a differential equation such as (3.9). Let us define
\[ q(x) = \frac{1}{\Gamma(y)} \frac{d\theta(y)}{dy} \quad \text{and} \quad h(x) = q(x) - \frac{1}{\Gamma(y)} \int_{0}^{x} q(y) dy \text{ for } x \in [0,1]. \]

We assume conditions on the distribution \( q(x) \) so that these functions \( R(x), g(x) \) and \( h(x) \) are well defined. The following result is obtained already by Yasuda [13].

If \( h(x) \) changes its sign only once from - to + as \( x \) varies from 0 to 1, and if \( q(x) \) is continuous for \( 0 < x < 1 \), then the optimal value at the beginning \( v^* = V(\theta) \) is given by
\[ v^* = (1-\delta(\theta)) v(\theta) = a g(a) \]
where \( a \) is a unique solution of \( h(a) = 0 \) for \( x \in [0,1] \).

This can be also obtained by applying the previous method. Since the optimality equation with random number of objects in the asymptotic form is, for \( x \in [0,1] \),
\[ v(x) = \max \{ R(x), x(1-\theta(x))^{-1} \int_{x}^{1} (1-\delta(y)) y^{-2} v(y) dy \} \]
we have
\[ v(x) = \begin{cases} R(x) & \text{on } [0,a), \\ x & \text{on } [a,1]. \end{cases} \]

In fact, it is equivalent to the classical secretary problem
\[ w(x) = \max \{ x, \frac{1}{x} \int_{x}^{1} w(y) dy \} \]
with the function
\[ w(x) = (1-\delta(x)) v(x) \]
and with the reward function
\[ \gamma(x) = placebo + \Gamma(y) \]
\[ \gamma(x) = placebo + \Gamma(y) \]
\[ \gamma(x) = placebo + \Gamma(y) \]
\[ \gamma(x) = placebo + \Gamma(y) \]

Hence, the solution (3.14) is immediately obtained by the result of (3.5). This method is simpler than the ad hoc treatment of the functional equation (3.10).