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Randomized stopping games and Markov market games

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Abstract We study nonzero-sum stopping games with randomized stopping strategies. The existence of Nash equilibrium and ε-equilibrium strategies are discussed under various assumptions on players random payoffs and utility functions dependent on the observed discrete time Markov process. Then we will present a model of a market game in which randomized stopping times are involved. The model is a mixture of a stochastic game and stopping game.

Keywords Stopping games · Stochastic games · Nash equilibrium · Markov chain

1 Introduction and preliminaries

The paper is concerned with two types of games: m-person nonzero-sum noncooperative sequential games in which randomized stopping times are players strategies and some specific stochastic games interpreted as market games (or some econometric models). In the former, players payoffs are their utility functions (in particular case) dependent on a state of the observed sequentially discrete—time Markov process at the random moment of stopping. These games are generalizations of the stopping game formulated by Dynkin (1969) as an example of optimal stopping of random sequences. In the latter, players control transition probability law of a Markov chain so as to stop it at a random moment with the aim to maximize their expected utilities dependent on the current state of the process and on the collection of players who have decided to stop it.

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First we will quote main results on stopping games with players strategies belonging to a class of nonrandomized stopping times. Let $(\Omega, \mathcal{F}, P)$ be some fixed probability space on which all considered random variables are defined and $\{\mathcal{F}_n, n \in \mathcal{N}\}$ an increasing sequence of sub-σ-fields of $\mathcal{F}$, $\mathcal{N} = \{1, 2, \dots\}$. $\mathcal{F}_n$ is a σ-field of events observed by the players till the moment n, $n \in \mathcal{N}$, inclusively. Let M denote the set of stopping times τ with respect to $\{\mathcal{F}_n, n \in \overline{\mathcal{N}}\}$, $\overline{\mathcal{N}} = \mathcal{N} \cup \{\infty\}$, $\mathcal{F}_\infty = \sigma(\{\mathcal{F}_n, n \in \mathcal{N}\}$, i.e. $\tau : \Omega \to \overline{\mathcal{N}}$ and $\{\omega \in \Omega : \tau(\omega) = n\} \in \mathcal{F}_n$, for any $n \in \overline{\mathcal{N}}$. Dynkin considered the following zero-sum stopping game. Two players observe a bivariate sequence of random variables $\{(X_n, \varphi_n), n \in \mathcal{N}\}$ - $\{\mathcal{F}_n, n \in \mathcal{N}\}$ adapted. The player 1 and the player 2 have the strategy sets $M \cap \{\tau : \varphi_\tau \leq 0\}$ and $M \cap \{\tau : \varphi_\tau > 0\}$, respectively. Let (τ_1, τ_2) be a pair of players strategies. The game terminates at $\tau = \min\{\tau_1, \tau_2\}$ and X_τ is the reward (loss) for Player 1 (Player 2). The objective of Player 1 (Player 2) is to maximize (minimize) the mean $E(X_\tau)$. Neveu considered a slight modification of this zero-sum stopping game in which the game strategy is a pair of stopping times $(\tau_1, \tau_2) \in M \times M$ and the reward (loss) $R(\tau_1, \tau_2)$ for Player 1 (Player 2) is determined by a bivariate sequence of random variables $\{(X_n, Y_n), n \in \mathcal{N}\} - \{\mathcal{F}_n, n \in \mathcal{N}\}$ adapted, so that

$$R(\tau_1, \tau_2) = X_{\tau_1} I_{\{\tau_1 < \tau_2\}} + Y_{\tau_2} I_{\{\tau_2 \leq \tau_1\}}, \tag{1}$$

where I_A denotes the indicator function of the set A in $\mathcal{F}$. The game payoff is $V(\tau_1, \tau_2) = E\left(R(\tau_1, \tau_2)\right)$. The game has the value and max–min strategy under suitable integrability assumptions and $X_n \leq Y_n$, a.s., $n \in \mathcal{N}$. Zero-sum stopping games with more general forms of the reward (1) were investigated by Yasuda (1986) and Rosenberg et al. (2001). Let $\{(X_n, Y_n, W_n), n \in \mathcal{N}\}$ be $\{\mathcal{F}_n, n \in \mathcal{N}\}$ adapted sequence of the observed rewards satisfying the following condition

$$E\left(\sup_{n \in \mathcal{N}} \max\{|X_n|, |Y_n|, |W_n|\}\right) < \infty. \tag{2}$$

Let the reward (loss) for Player 1 (Player 2) be as follows

$$R(\tau_1, \tau_2) = X_{\tau_1} I_{\{\tau_1 < \tau_2\}} + Y_{\tau_2} I_{\{\tau_2 < \tau_1\}} + W_{\tau_1} I_{\{\tau_1 = \tau_2\}}, \tag{3}$$

and the discounted reward with $\lambda \in (0, 1)$:

$$R_\lambda(\tau_1, \tau_2) = \lambda^{\tau_1} X_{\tau_1} I_{\{\tau_1 < \tau_2\}} + \lambda^{\tau_2} Y_{\tau_2} I_{\{\tau_2 < \tau_1\}} + \lambda^{\tau_1} W_{\tau_1} I_{\{\tau_1 = \tau_2\}}. \tag{4}$$

Unless the inequalities $X_n \leq W_n \leq Y_n$, a.s., $n \in \mathcal{N}$, are fulfilled these games may not have the values. It turns out that without this assumption one may get existence of the game value in the class of randomized stopping times. Existence of the value of a zero-sum game with randomized stopping times and discounted reward $R_\lambda(\tau_1, \tau_2)$ was proved by Yasuda (1986). Rosenberg et al. (2001) showed that in the class of randomized stopping times game with a reward (3) has the value. Moreover, they showed that the game value $V(\tau_1, \tau_2)$ is the limit of game values $V_\lambda(\tau_1, \tau_2)$ as $\lambda \to 0^+$ of games with discounted rewards $R_\lambda(\tau_1, \tau_2)$.

Nonzero-sum stopping games with general rewards (5), given below, satisfying the integrability condition (2), were investigated by Ohtsubo (1987, 1991). Let the reward for Player $i, i = 1, 2$, under the game strategy $(\tau_1, \tau_2) \in M \times M$ be as follows

$$R^i(\tau_1, \tau_2) = X^i_{\tau_i} I_{\{\tau_i < \tau_j\}} + Y^i_{\tau_j} I_{\{\tau_j < \tau_i\}} + W^i_{\tau_i} I_{\{\tau_i = \tau_j < \infty\}} + \limsup_{n\to\infty} W^i_n I_{\{\tau_1 = \tau_2 = \infty\}}, \tag{5}$$

where sequences of trivariate random variables $\{(X^i_n, Y^i_n, W^i_n), n \in \mathcal{N}\}$ are $\{\mathcal{F}_n, n \in \mathcal{N}\}$ adapted. They represent players' rewards associated with their appropriate decisions. The aim of each of the players is to make his mean reward as large as possible. So, they look for a Nash equilibrium strategy $(\widehat{\tau}_1, \widehat{\tau}_2) \in M \times M$ of the game $\mathcal{G} = (M \times M, V^1, V^2)$, presented in a normal form, where players' payoff functions V^i are their mean rewards, i.e. $V^i(\tau_1, \tau_2) = E(R^i(\tau_1, \tau_2))$, $i = 1, 2$. Thus, for any strategy (τ_1, τ_2) from $M \times M$ we have

$$V^1(\widehat{\tau}_1, \widehat{\tau}_2) \geq V^1(\tau_1, \widehat{\tau}_2) \quad \text{and} \quad V^2(\widehat{\tau}_1, \widehat{\tau}_2) \geq V^2(\widehat{\tau}_1, \tau_2).$$

Similarly as in the case of zero-sum stopping games the game $\mathcal{G}$ may not have a Nash equilibrium strategy unless sequences of rewards satisfy some inequalities. For instance, Ohtsubo (1987) proved existence of Nash equilibrium assuming that sequences $\{(X^i_n, Y^i_n, W^i_n), n \in \mathcal{N}\}$ satisfy the integrabilty condition (2) and the inequalities $X^i_n \geq W^i_n \geq Y^i_n$, a.s., for $i = 1, 2$, $n \in \mathcal{N}$. Other types of constraints on reward sequences assuring existence of Nash equilibrium in the class of nonrandomized stopping times were considered in Ohtsubo (1991), Ferenstein (1992, 1993), Bobecka and Ferenstein (2001). Finite-horizon stopping games were investigated in a series of papers, often in the context of best choice problems. A broad survey of stopping games is given in Nowak and Szajowski (1999), Neumann et al. (2002).

Two-person nonzero-sum stopping games with rewards (5) and randomized stopping times were analyzed in Ferenstein (2005). Special type of m-person randomized stopping games with nonrandom rewards, called quitting games, was considered by Solan and Vieille (2001). They obtained existence of Nash ϵ-equilibrium strategies under some inequality constraints on rewards approaching infinity. In Sect. 2 we generalize results obtained in Ferenstein (2005) for two-person games and compare them with those in Solan and Vieille (2001). In Sect. 3 we analyze Markov randomized stopping game as a special type of a new nonzero-sum discounted stochastic game introduced by Nowak (2003). In Sect. 4 we construct a new model of m-person sequential game which is a mixture of some stochastic game and stopping game. The considered form of players rewards suggest that it may be used as a market game.

Section 5 is devoted to some proofs of results presented in Sect. 2.

2 Dynkin game with randomized stopping times

In this section we will present a generalized model of Dynkin stopping game. This is m-person nonzero-sum noncooperative game in which randomized stopping times are players' strategies. We assume that m players (decision makers) observe sequentially elements of a sequence of random vectors $\{R_n, n \in \mathcal{N}\}$ defined on

$(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_n,\ n \in \mathcal{N}\}$ adapted, which determine players' rewards depending on the moment of stopping the game and on the collection of players who have decided to stop it. Precisely, at any moment n each player makes one of the two possible decisions: either to quit (stop) the game or to continue it. The game is over (stopped) if at least one of the players decides to quit it. The vector of rewards at the $n-th$ stage of the game $R_n = (R^i_{n,D})_{i\in\mathcal{M},\, D\in 2^{\mathcal{M}}\setminus\varnothing}$, $\mathcal{M} = \{1, \ldots, m\}$, is interpreted so that the random variable $R^i_{n,D}$ is the reward for the player i if the game is stopped at the moment n and D is a collection of players who have decided to quit the game. Players' decisions are independent and they are based on update observations, so they are random variables $\{\mathcal{F}_n, n \in \mathcal{N}\}$ adapted. Precisely, we admit two representations of players' strategies: randomized stopping strategies or randomized stopping times. We assume in the sequel, without loss of generality, that the underlying probability space $(\Omega, \mathcal{F}, P)$ on which all considerded random variables are defined is rich enough to allow randomization.

Denote by Λ the set of randomized stopping times. Let us recall the notion of randomized stopping time as in Chow et al. (1971), similarly in Ferguson (1967). Namely, $\tau \in \Lambda$ iff there exists a filtration $\{\mathcal{U}_n\}_{n\in\overline{\mathcal{N}}}$ such that below conditions (a)–(c) are satisfied.

(a) $\mathcal{F}_n \subset \mathcal{U}_n$, for $n \in \overline{\mathcal{N}}$,
(b) $P(A|\mathcal{F}_n) = P(A|\mathcal{U}_n)$, a.s., $A \in \mathcal{F}_\infty$, $n \in \overline{\mathcal{N}}$,
(c) τ is a stopping time with respect to $\{\mathcal{U}_n\}_{n\in\overline{\mathcal{N}}}$.

Players observe subsequently events in $\mathcal{F}_n,\ n \in \mathcal{N}$, and their decisions either to quit the game or to continue it are independent. Thus, we assume that the set of game strategies $\widetilde{\Lambda}$ consists of vectors $\widetilde{\tau} = (\tau_1, \ldots, \tau_m)$ of randomized stopping times $\tau_1 \in \Lambda, \ldots, \tau_m \in \Lambda$ which are conditionally independent given $\mathcal{F}_n$, for any $n \in \mathcal{N}$. τ_i is the strategy of the player i, $i = 1, \ldots, m$. Under the strategy $\widetilde{\tau} \in \widetilde{\Lambda}$ the payoff for the player i is $V^i(\widetilde{\tau})$, defined as follows

$$V^i(\widetilde{\tau}) = E\left(R^i_{t(\widetilde{\tau}),D(\widetilde{\tau})}\right),$$

where $t\,(\widetilde{\tau}) = \min\{\tau_1, \ldots, \tau_m\},\ D(\widetilde{\tau}) = \{1 \le j \le m : \tau_j = t(\widetilde{\tau})\}$.

Let S be the set of randomized stopping strategies, i.e. the set of $\{\mathcal{F}_n\}_{n\in\overline{\mathcal{N}}}$ adapted random sequences $s = \{p_n\}_{n\in\overline{\mathcal{N}}}$ such that

$$0 \le p_n \le 1 \quad \text{and} \quad \sum_{n\in\overline{\mathcal{N}}} p_n = 1, \qquad \text{a.s.} \tag{6}$$

For a randomized stopping strategy $s = \{p_n\}_{n\in\overline{\mathcal{N}}}$, p_n is to be interpreted as conditional probability that stopping occurs at time n, given the observations in $\mathcal{F}_n$.

Let $\widetilde{s} = (s_1, \ldots, s_m) \in \widetilde{S} = \Pi^m_{i=1} S$, $s_i = \{p^i_n\}_{n\in\overline{\mathcal{N}}}$, $i \in \mathcal{M}$. Then, let us define the player's i payoff

$$H_i\,(\widetilde{s}) = E\left(\sum_{n\in\mathcal{N}} \sum_{D\in 2^{\mathcal{M}}\setminus\varnothing} R^i_{n,D} \prod_{l\in D} p^l_n \prod_{j\notin D} \left(1 - p^j_1 - \cdots - p^j_n\right)\right). \tag{7}$$

Proposition 1 (i) *For* $\widetilde{s} = (s_1, \ldots, s_m) \in \widetilde{S}$ *there exists* $\widetilde{\tau} = (\tau_1, \ldots, \tau_m) \in \widetilde{\Lambda}$ *such that*

$$H_i(\widetilde{s}) = V^i(\widetilde{\tau}), \quad i \in \mathcal{M}. \tag{8}$$

(ii) *For* $\widetilde{\tau} = (\tau_1, \ldots, \tau_m) \in \widetilde{\Lambda}$ *there exists* $\widetilde{s} = (s_1, \ldots, s_m) \in \widetilde{S}$ *such that the equality* (8) *is fulfilled.*

Because of Proposition 1 we may analyze games with players' strategies either from $\widetilde{\Lambda}$ or $\widetilde{S}$ depending on the covenience. Now, we will obtain results on existence of Nash equilibrium strategies of the games $\mathcal{G}\left(\widetilde{\Lambda}\right) = \left(\widetilde{\Lambda}, V^1, \ldots, V^m\right)$ and $\mathcal{G}\left(\widetilde{S}\right) = \left(\widetilde{S}, H_1, \ldots, H_m\right)$. We will need the below conditions

(A1) $E(\sup_{n\in\mathcal{N}} |R^i_{n,D}|) < \infty$, $i \in \mathcal{M}$, $D \in 2^{\mathcal{M}} \setminus \varnothing$,

(A2) For any $n \in \mathcal{N}$, $\mathcal{F}_n$ is generated by at most countable set of events $\{B^n_1, \ldots, B^n_{k_n}\}$ from $\mathcal{F}$, $k_n \leq \infty$.

Theorem 1 *Suppose that Assumptions* (A1), (A2) *are satisfied. and the sequences* $\{R^i_{n,D}\}_{n\in\mathcal{N}}$ *tend to* 0 *as* $n \to \infty$, *in probability,* $i \in \mathcal{M}$, $D \in 2^{\mathcal{M}} \setminus \varnothing$. *Then, the game* $\mathcal{G}\left(\widetilde{S}\right)$ *has a Nash equilibrium strategy.*

Suppose that in the above theorem we will neglect assumption that reward sequences approach zero at infinity. Then to assure existence of Nash equilibrium strategies still we will have to put some constraints on length of the game which may be done in two ways: either restrictions on quitting probabilities (randomized stopping strategies) or restrictions on rewards which appear for instance in the case of random horizon independent on the observed filtration. The below theorem concerns random-horizon stopping game.

Theorem 2 *Suppose that Assumptions* (A1), (A2) *are fulfilled. Let* K *be an non-negative integer valued random variable, independent on* $\{\mathcal{F}_n\}_{n\in\mathcal{N}}$, *and* $E(K) < \infty$. *Suppose that players' rewards and payoffs, for* $\widetilde{\tau} = (\tau_1, \ldots, \tau_m) \in \widetilde{\Lambda}$, *are defined as follows*

$$V^i_K(\widetilde{\tau}) = E\left(R^i_{t(\widetilde{\tau}),D(\widetilde{\tau})} I_{\{t(\widetilde{\tau})\leq K\}}\right),$$

where

$$t\left(\widetilde{\tau}\right) = \min\{\tau_1, \ldots, \tau_m\}, \quad D(\widetilde{\tau}) = \{1 \leq j \leq m : \tau_j = t(\widetilde{\tau})\}.$$

Then, the game $\mathcal{G}_K\left(\widetilde{\Lambda}\right) = \left(\widetilde{\Lambda}, V^1_K, \ldots, V^m_K\right)$ *has a Nash equilibrium strategy.*

Next theorem deals with quasi-finite-horizon game, where we use the below definition.

Definition 1 Let $r = \{r_n\}_{n\in\overline{\mathcal{N}}}$ be a sequence of random variables $\{\mathcal{F}_n\}_{n\in\overline{\mathcal{N}}}$ adapted with finite second moments and such that $E(\sum_{n=1}^{\infty} r^2_n) < \infty$. Let, for some given natural L, the subset of randomized stopping strategies $S^L_r \subset S$ be as follows

$$S^L_r = \left\{s = \{p_n\}_{n\in\overline{\mathcal{N}}} \in S : p_n \leq r_n, \text{ a.s., for } n \geq L\right\}.$$

For the game $\mathcal{G}\left(\widetilde{S}^L_r\right) = \left(\Pi^m_{i=1} S^L_r, H_1, \ldots, H_m\right)$ called quasi-finite-horizon randomized stopping game we have the following theorem.

Theorem 3 *Suppose that Assumptions* (A1), (A2) *are fulfilled. Then, the game* $\mathcal{G}\left(\widetilde{S}_r^L\right)$ *has a Nash equilibrium strategy.*

In the above theorems we use assumption that the σ-fields of the observed filtration are countably generated. For instance, in the case when rewards are functions of states of a Markov process it means that the state space of the process is countable. It would be interesting to get existence type results for uncounable state space. Solan and Vieille (2001) obtained existence of ε-equilibria without any restricions on the observed filtration but they had to assume some additional conditions on rewards. They proved the following

Theorem 4 *Assume that the sequence* $\{R_n, n \in \mathcal{N}\}$ *converges to* R_∞*, a.s. Assume that* (A1) *is satisfied. Then, for every* $\varepsilon > 0$*, the game* $\mathcal{G}\left(\widetilde{S}\right)$ *admits an* ϵ*-Nash equilibrium provided that* R_∞ *satisfies the following*

C1. $R^i_{\infty,\{i\}} = 1$, *a.s., for* $i \in \mathcal{M}$,
C2. $R^i_{\infty,D} \leq 1$, *a.s., for* $i \in \mathcal{M}$ *and* $D \in 2^{\mathcal{M}} \setminus \varnothing$.

Corollary 1 *Suppose that Assumption* (A1) *is satisfied and the sequences* $\{R^i_{n,D}\}_{n\in\mathcal{N}}$ *tend to* 0 *as* $n \to \infty$*, a.s.,* $i \in \mathcal{M}$, $D \in 2^{\mathcal{M}} \setminus \varnothing$*. Then, the game* $\mathcal{G}\left(\widetilde{S}\right)$ *has an* ϵ*-Nash equilibrium strategy.*

Similarly, theorems on existence of ϵ-Nash equilibrium of the games $\mathcal{G}_K\left(\widetilde{\Lambda}\right)$, $\mathcal{G}\left(\widetilde{S}_r^L\right)$ are true without Assumption (A2) which was needed to obtain existence of Nash equilibria in Theorems 2 and 3.

Theorem 5 *Suppose that Assumption* (A1) *is fulfilled. Let* K *be an nonnegative integer valued random variable, independent on* $\{\mathcal{F}_n\}_{n\in\mathcal{N}}$*, and* $E(K) < \infty$*. Then, for any* $\epsilon > 0$*, the game* $\mathcal{G}_K\left(\widetilde{\Lambda}\right)$ *defined in Theorem 2 admits an* ϵ*-Nash equilibrium.*

Theorem 6 *Suppose that Assumptions* (A1) *is fulfilled. Then, for any* $\epsilon > 0$*, the game* $\mathcal{G}\left(\widetilde{S}_r^L\right)$ *admits an* ϵ*-Nash equilibrium.*

3 Markov stopping games

In this section we give examples of the considered games for which Assumption (A2) may be skipped and Nash equilibria exist. In these games sequences of rewards are functions of currently observed states of a homogeneous Markov chain with Borel state space and probability transitions that are combinations of finitely many measures on the state space with coefficients depending on the state. This class of transition probabilities is contained in the class of the ones considered by Nowak (2003) in his new model of discounted stochastic games. In this new stochastic game these coefficients depend on the state of the controlled Markov chain and on the players' actions. In stopping games, only at most two actions are possible and there is no players' influence on the Markov chain evolution. Nevertheless, stopping games may be presented as stochastic games with special type of transition probabilities. In what follows we present the Markov stopping game as a special case of a discounted stochastic game considered by Nowak (2003).

Let us introduce the following elements of Markov stopping games:

(i) E is a nonempty Borel state space, i.e. subset of a separable complete metric space, say Y, and there exists $y_0 \in Y \setminus E$.
(ii) $\xi = \{\xi_n\}_{n\geq 1}$ is a homogeneous Markov chain with state space E, determined on a probability space $(\Omega, \mathcal{F}, P)$,
(iii) $g_i : E \times \bigcap_{i=1}^{m}\{0, 1\} \to R$ is an utility function for Player i such that $g_i(\cdot, i_1, i_2, \ldots, i_m)$ is Borel measurable for each $i_j \in \{0,1\}$, $j=1, 2, \ldots, m$, where we interpret $i_j = 0$ (1) as the player j decision to continue (stop) the game. We assume that there exists some $K>0$ such that $|g_i(e, i_1, i_2, \ldots, i_m)| \leq K$ for each $e \in E$, $i_j \in \{0, 1\}$, and $j \in \{1, 2, \ldots, m\}$. Let $\beta \in (0, 1)$ be a discount factor such that, referring to our game model from Sect. 2, the reward for Player i at moment n is as follows

$$R^i_{n,D} = \beta^{n-1} g_i(\xi_n, i_1, i_2, \ldots, i_m), \quad \text{where } D = \{1 \leq j \leq m : i_j = 1\}. \tag{9}$$

(iv) p is a Borel measurable transition probability from E to E, i.e. if e is a state of the Markov chain at some moment n, then $p(B \mid e) = P(\xi_{n+1} \in B \mid \xi_n = e)$ for any Borel subset B of E.
(v) Let the set $\mathcal{S}$ of game strategies consist of players strategies which are stationary randomized stopping strategies, i.e. $\pi = (\pi_1, \pi_2, \ldots, \pi_m) \in \mathcal{S}$ iff $\pi_i : E \to [0, 1]$ are Borel measurable functions, $i = 1, 2, \ldots, m$. $\pi_i(e)$ is interpreted as conditional probability that Player i chooses stopping while observing the current state e and he has not decided to stop earlier. Referring to denotations of Sect. 2 we have $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ and for $i = 1, 2, \ldots, m$, $n \in \mathcal{N}$

$$p^i_n = \pi_i(\xi_n) \prod_{k=1}^{n-1} (1 - \pi_i(\xi_k)).$$

The mean reward function $V_i(\pi)(e)$ for Player i under the game strategy π is as follows

$$V_i(\pi)(e) = E\left(\sum_{n\in\mathcal{N}} \left(\sum_{D\in 2^{\mathcal{M}}\setminus\varnothing} R^i_{n,D} \prod_{l\in D} \pi_l(\xi_n)\right) \prod_{j\notin D} (1 - \pi_j(\xi_n))\varkappa_n | \xi_1 = e\right),$$

where $\varkappa_n = \prod_{k=1}^{n-1} \prod_{p=1}^{m} (1 - \pi_p(\xi_k))$.

We now describe our basic assumption similarly as Assumption A1 in Nowak (2003).

(A3) Assume that B is a countable subset of E and denote $C = E \setminus B$. There are atomless nonnegative measures μ_j on C, $j = 1, \ldots, k$, and nonnegative measures δ_t concentrated on subsets of B, $t = 1, \ldots, l$, and there are Borel measurable functions $c_j : E \to [0, 1]$, $b_t : E \to [0, 1]$ such that for all $e \in E$, we have

$$p(\cdot \mid e) = q(\cdot \mid e) + \delta(\cdot \mid e),$$

where δ is the “atomic part” of p of the form

$$\delta(\cdot|e) = \sum_{t=1}^{l} b_t(e)\delta_t(\cdot)$$

and the “atomless part” q of p is as follows

$$q(\cdot|e) = \sum_{j=1}^{k} c_j(e)\mu_j(\cdot).$$

Now, we can state our main result on the considered (discounted) Markov randomized stopping game written in a normal form as $\mathcal{G}(\mathcal{S}) = (\mathcal{S}, V_1, V_2, \ldots, V_m)$.

Theorem 7 *Every discounted Markov randomized stopping game* $\mathcal{G}(\mathcal{S})$ *satisfying* (i) *through* (v) *and* (A3) *has a Nash equilibrium.*

Proof We will present the game as a special type of a discounted stochastic game.

Let us consider m-person nonzero-sum stochastic game for which:

(i) $X = E \cup \{y_0\}$ is a nonempty Borel state space.
(ii) $A_i = \{0, 1\}$ is the set of actions of each Player i. Let $A = A_1 \times \cdots \times A_m$.
(iii) Let $A_i(x) = \{0, 1\}$ for $x \in E$ and $A_i(y_0) = \{1\}$ be sets of actions of Player i available at state x. Let

$$A(x) = A_1(x) \times A_2(x) \times \cdots \times A_m(x), \quad x \in X.$$

(iv) Let $\Delta = \{(x, a) : x \in X,\ a \in A(x)\}$. Δ is a Borel subset of $X \times A$. $r_i : \Delta \to R$ is a Borel measurable payoff function for Player i such that $r_i(y_0, a) = 0$, $r_i(x, a) = 0$ for $x \in E$, $a = (0, 0, \ldots, 0)$ and $r_i(x, a) = g_i(x, a)$ if $x \in E$ and $a = (a_1, \ldots, a_m) \in A(x)$ is such that

$$a_1 + a_2 + \cdots + a_m \geq 1.$$

(v) $\widehat{p}$ is a Borel measurable transition probabilty from $X \times A$ to X, called the law of motions among states, which is the transition probability of the stopped Markov chain with the absorbing state y_0, i.e. $\widehat{p}(\cdot \mid x, a) = p(\cdot \mid x)$ if $x \in E$, $a = (0, 0, \ldots, 0)$, $= \delta_0(\cdot)$ if $x \in E$, $a_1 + a_2 + \cdots + a_m \geq 1$, $= \delta_0(\cdot)$ if $x = y_0$, where $\delta_0(\cdot)$ is a measure concentrated at y_0.

Let $H^\infty = X \times A \times X \times \cdots$ be the space of all infinite histories of the game, endowed with the product σ-algebra. For $\pi = (\pi_1, \pi_2, \ldots, \pi_m) \in \mathcal{S}$ define $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \ldots, \widehat{\pi}_m)$ so that $\widehat{\pi}_i(x) = \pi_i(x)$ if $x \in E$, and $\widehat{\pi}_i(y_0) = 1$. $\widehat{\pi}_i(x)$ determines the probability distribution on $A_i(x)$ so that it is the probability of the action 1.

For any profile of strategies $\widehat{\pi}$ of the players and every initial state $X_1 = x \in E$, let a probability measure $P_x^{\widehat{\pi}}$ and a stochastic process $\{(X_n, Y_n)\}$ be defined on H^∞ in a canonical way, where the random variables X_n, Y_n describe the state and the actions chosen by the players, respectively, at the nth stage of the game. Now, we may define the β-discounted expected payoff function to Player i as follows

$$\gamma_i(\widehat{\pi})(x) = E_x^{\widehat{\pi}}\left(\sum_{n=1}^{\infty} \beta^{n-1} r_i(X_n, Y_n)\right),$$

where $E_x^{\widehat{\pi}}$ is the expectation operator with respect to the probability measure $P_x^{\widehat{\pi}}$.

Now, it is easy to see that (i)–(v) and (A3) assure that assumptions of Theorem 1 in Nowak (2003) are fulfilled. Hence, there exists a Nash equilibrium $\widehat{\pi}^*$ in the discounted stochastis game.

It is sufficient to note that

$$\gamma_i(\widehat{\pi})(x) = V_i(\pi)(x), \quad x \in E.$$

Thus $\pi^*(x) = \widehat{\pi}^*(x)$, $x \in E$, is a Nash equilibrium of the game $\mathcal{G}(\mathcal{S})$. □

It is possible to skip Assumption (A3) and still have existence result on Nash equilibria using Schauder Fixed Point Theorem but with additional constraints on stationary strategies π. This result will be presented in future.

4 The model of a market game

In many econometric papers [e.g. Duffie et al. (1994) or Karatzas et al. (1994) or Nowak and Szajowski (1999)] model of the economy is constructed via stochastic games or Markov decision processes. We will present model which is a kind of stochastic game with stopping.

We now describe the game model consisting of the following elements:

(i) $(E, \mathcal{B})$ is a measurable space, E is the state space of the controlled Markov process.
(ii) A^i is a compact metric space called an action space of the player $i, i \in \{1, \ldots, m\} = \mathcal{M}$. $A = A^1 \times \cdots \times A^m$ is the action space of all players.
(iii) $r_{\mathcal{C}}^i : E \to R$ is a measurable function, interpreted as a reward function for the player i, $i \in \mathcal{M}$, $\mathcal{C} \in 2^{\mathcal{M}}$, $\mathcal{C}$ denotes the collection of players who decide to quit the game.
(iv) $Q(\cdot \mid x, a)$ is a product measurable transition probability law from $E \times A$ into E.
(v) $A^0 = \{0, 1\}$, where 0 denotes the decision "to continue the game" and 1 is the decision "to quit (stop) the game".

Let $\Delta = A^0 \times A$ denote the set of possible realizations of players' decisions. Δ will be called the decision space. Suppose that at some stage k, $k \geq 1$, realizations of the state of the process and the players' decision are, respectively, $x \in E$ and $d = (d_0, d_1) \in \Delta$, where $d_0 = (d_0^1, \ldots, d_0^m) \in \Pi_{i=1}^m A^0$, $d_1 = (d_1^1, \ldots, d_1^m) \in A$. Let us define $\psi(d) = \sum_{i=1}^m d_0^i$. Then, the game is stopped iff $\psi(d) > 0$. Otherwise the process evolves to another state in E according to the transition probability law $Q(\cdot|x, d_1)$. The fact of stopping the game will be identified with attaining the absorbing state δ, say. Hence, we may extend the state space E by δ and accordingly the transition probability law. Thus, the new measurable state space is $(\widetilde{E}, \widetilde{\mathcal{B}})$, where $\widetilde{E} = E \cup \{\delta\}$, $\widetilde{B} = \sigma(\mathcal{B}, \{\delta\})$, and the new transition probability law $\widetilde{Q}(\cdot \mid x, d)$ is defined as follows. For $B \in \widetilde{\mathcal{B}}$, $x \in \widetilde{E}$, $d \in \Delta$ we have

$$\widetilde{Q}\,(B|x, d) = Q(B \setminus \{\delta\} \mid x, d_1) \quad \text{if } x \in E, \;\; \psi(d) = 0,$$

or

$$\widetilde{Q}\,(B|x, d) = 1 \quad \text{if } x \in E \quad \text{and} \quad \psi(d) > 0 \quad \text{and} \quad \delta \in B \quad \text{or if } x = \delta \in B,$$

or

$$\widetilde{Q}(B|x,d) = 0 \quad \text{if } x \in E,\ \ \psi(\dot{d}) > 0 \quad \text{and} \quad \delta \notin B.$$

Let $h_n = (x_1, d_1, \ldots, x_{n-1}, d_{n-1}, x_n) \in H_n = E \times \Delta \times \Pi_{i=2}^{n-1}(\widetilde{E} \times \Delta) \times \widetilde{E}, n \geq 2$, be the history of the game till the moment n and x_n is the current state of the process at n. Let $h_1 = x_1 \in E = H_1$. Let us define $\psi_n(h_n) = \sum_{k=1}^{n-1} \psi(d_k)$. Suppose that $\psi_n(h_n) = 0$ which means that till the moment n the game has not been stopped. Let the policy of Player i at the nth stage of the game be $\gamma_n^i(h_n) \in \mathcal{P}(A^0) \otimes \mathcal{P}(A^i)$, where $\mathcal{P}(A^0)$ and $\mathcal{P}(A^i)$ are the sets of probability distributions over A^0 and A^i, respectively. Thus $\gamma_n^i(h_n)$ may be identified with the pair $(p_n^i(h_n), \mu_n^i(h_n))$, where $p_n^i(h_n) \in [0, 1]$ is the probability that the player i stops the game at n given the history h_n, $\mu_n^i(h_n)$ is the probability distribution over the action space A^i given the history h_n. Hence, the game strategy at the nth stage $\widetilde{\gamma}_n(\cdot)$ and the total game strategy $\widetilde{\gamma}(\cdot)$ are such that $\widetilde{\gamma}_n(\cdot) = (\gamma_n^1(\cdot), \ldots, \gamma_n^m(\cdot))$ and $\widetilde{\gamma}(\cdot) = (\widetilde{\gamma}_1(\cdot), \widetilde{\gamma}_2(\cdot), \ldots)$, respectively. In the case when $\gamma_n^i(h_n) = \gamma_n^i(x_n)$, for every $n \in \mathcal{N}$ and every $i \in \mathcal{M}$, $\widetilde{\gamma}$ will be called Markov game strategy. Moreover, $\widetilde{\gamma}$ is called stationary Markov game strategy if $\gamma_n^i(x_n) = \gamma^i(x_n)$, for any n and i. Let Γ denote the set of all stationary Markov game strategies, $\gamma = (\gamma^1, \ldots, \gamma^m) \in \Gamma$ and $x \in E$ be an initial state. Let $\{X_n\}_{n\in\mathcal{N}}$, $\{D_n\}_{n\in\mathcal{N}}$ denote the induced Markov chain and decision process, respectively. Let us denote, for $x \in \widetilde{E}$, $d = (d_0, d_1) \in \Delta$, $i \in \mathcal{M}$, the reward for the player i as $r^i(x, d)$. Hence, $r^i(x, d) = r_{\mathcal{C}}^i(x)$ if $x \in E$, $\psi(\dot{d}) > 0$ and $\mathcal{C} = \{i \in \mathcal{M} : d_{0,i} = 1, d_0 = (d_{0,1}, \ldots, d_{0,m})\}$, otherwise it is equal to 0. Now, let the discounted mean reward for the player i be as follows

$$H^i(\gamma, x) = E_x^\gamma\left(\sum_{n\in\mathcal{N}} \beta^{n-1} r^i(X_n, D_n)\right), \tag{10}$$

where $\beta \in (0, 1)$ is a discount factor.

One may obtain existence of Nash equilibria of the above game using Theorem 1 in Nowak (2003) which demands the assumption similar to (A3) on the decomposition of the transition probability.

5 Proofs of main results

In this section we will prove theorems formulated in Sect. 2. First, let us recall a general result of game theory which will be used in the proofs, namely the following theorem (Nash 1951; Fan 1966).

Theorem 8 *Suppose that for an m-person nonzero-sum game* $\mathcal{G} = (S_1 \times \cdots \times S_m, H_1, \ldots, H_m)$ *the following assumptions are fulfilled:*

(i) $S_1, \ldots, S_m$ *are nonempty compact convex subsets of a linear separated topological space* X.
(ii) *Functions* $H_i : S_1 \times \cdots \times S_m \to R$, $i = 1, \ldots, m$, *are continuous.*
(iii) $H_i(s^{-i}, \cdot)$ *are quasi-concave, for any* $s \in S_1 \times \cdots \times S_m$, $i = 1, \ldots, m$.

Then, the game $\mathcal{G}$ *has a Nash equilibrium strategy.*

Let X denote the space of random sequences $\{\xi_n\}_{n\in\overline{\mathcal{N}}}$ such that $\xi_n \in L^2(\Omega, \mathcal{F}_n, P)$, $n \in \mathcal{N}$, and $E\big(\sum_{n\in\overline{\mathcal{N}}} \xi_n^2\big) < \infty$. X equipped with the inner product $\langle \xi, \eta\rangle = \sum_{n\in\overline{\mathcal{N}}} \xi_n \eta_n$ is a Hilbert space. Let us note that sets of randomized strategies are embedded in X, namely we have $S_r^L \subset S \subset X$. First, we will state results needed in the proofs of Theorems 1–3.

Lemma 1 *Let Assumption* (*A*2) *be satisfied. Then,*

(i) *S is weakly compact in X,*
(ii) *S_r^L is compact in X.*

Proof (i) Directly by definition, S is convex and bounded subset of the Hilbert space X . We will show that S is closed subset of X, which together with the former assures (i). Thus, let $p^k = \{p_n^k\}_{n\in\overline{\mathcal{N}}} \in S$, $k = 1, 2, \dots$, and $p = \{p_n\}_{n\in\overline{\mathcal{N}}} \in X$ be such $\|p^k - p\|^2 = E\left(\sum_{n\in\overline{\mathcal{N}}}(p_n^k - p_n)^2\right) \to 0$ as $k \to \infty$. Hence, there exists subsequence of naturals $\{k_j\}$ such that, for any n, we have $p_n^{k_j} \to p_n$ as $j \to \infty$, a.s. Therefore, p_n is $\mathcal{F}_n$-measurable, $0 \le p_n \le 1$, a.s. To prove that $p \in S$ it remains to show that $\sum_{n\in\overline{\mathcal{N}}} p_n = 1$, a.s. For any j and M we may write $p_1^{k_j} + \dots + p_M^{k_j} + \sum_{M<n\in\overline{\mathcal{N}}} p_n^{k_j} = 1$, which gives us,

$$p_1 + \dots + p_M + \lim_{j\to\infty} \sum_{M<n\in\overline{\mathcal{N}}} p_n^{k_j} = 1, \text{ a.s.} \tag{11}$$

Denote the limit above as γ_M. We have $\gamma_M \ge \gamma_{M+1} \ge 0$. Therefore $\gamma_M \to \gamma \ge 0$. Note that γ cannot be >0 for $\sum_{M<n\in\overline{\mathcal{N}}} p_n^k \to 0$ as $M \to \infty$, for any k. Hence, taking M arbitraly large in (11) we get $\sum_{n\in\overline{\mathcal{N}}} p_n = 1$.

(ii) is proved in Lemma 1 in Ferenstein (2005). □

Proof of Theorem 1 We will show that Assumptions (A1), (A2) assure that for the game $\mathcal{G}(\widetilde{S})$ the assumptions of Theorem 8 are satisfied. First, note that players' strategy sets $S_i = S$. S is an nonempty convex subset of the Hilbert space X and according to Lemma 1 it is weakly compact in X. Moreover, let us note that for any $i \in \mathcal{M}$ and $\widetilde{s} \in \widetilde{S}$, $H_i(\widetilde{s}^{-i}, \cdot)$ is quasi-concave since H_i is linear with respect to the player's i strategy set S. Hence, it remains to show that the functions $H_i : \widetilde{S} = \Pi_{i=1}^m S \to R$, $i \in \mathcal{M}$, are weakly continuous. Let $\{X_n\}_{n\in\mathcal{N}}$ be a sequence of random variables $\{\mathcal{F}_n\}_{n\in\mathcal{N}}$ adapted such that $E(\sup|X_n|) < \infty$, $\lim_{n\to\infty} X_n = 0$, a.s. Let us note that any payoff H_i is the sum over $D \in 2^{\mathcal{M}} \setminus \varnothing$ of functions $J(\widetilde{s})$, $\widetilde{s} \in \widetilde{S}$, written in the following forms

$$J(\widetilde{s}) = E\left(\sum_{n\in\mathcal{N}} X_n \prod_{i\in D} p_n^i \prod_{j\notin D}\left(1 - p_1^j - \dots - p_n^j\right)\right). \tag{12}$$

It is sufficient to prove that $J : \widetilde{S} \to R$ is weakly continuous. Let $\widetilde{s}^k = (s_1^k, \dots, s_m^k) \in \widetilde{S}$, $s_i^k = \{p_n^{i,k}\}_{n\in\mathcal{N}}$, $k = 1, 2, \dots$, be a sequence of game strategies converging weakly to $\widetilde{s} = (s_1, \dots, s_m) \in \widetilde{S}$, $s_i = \{p_n^i\}_{n\in\mathcal{N}}$. We will show that $J(\widetilde{s}^k) \to J(\widetilde{s})$

as $k \to \infty$. One may assume that $D = \{1, \ldots, l\}, l \in \mathcal{M}$. Then, for any natural M, $J(\widetilde{s}^k)$ may be written as follows

$$J(\widetilde{s}^k) = E\left(\sum_{n=1}^{M} X_n \prod_{i=1}^{l} p_n^{i,k} \prod_{j=l+1}^{m} \left(1 - p_1^{j,k} - \cdots - p_n^{j,k}\right)\right)$$
$$+ E\left(\sum_{n>M} X_n \prod_{i=1}^{l} p_n^{i,k} \prod_{j=l+1}^{m} \left(1 - p_1^{j,k} - \cdots - p_n^{j,k}\right)\right)$$
$$= A_M(\widetilde{s}^k) + B_M(\widetilde{s}^k), \tag{13}$$

where $A_M(\widetilde{s}^k)$ and$B_M(\widetilde{s}^k)$ denote the corresponding expectations above.

Note that for any $h \in X$ and $i \in \mathcal{M}$ we have

$$\left\langle h, s_i^k - s_i \right\rangle \to 0 \quad \text{as } k \to \infty, \tag{14}$$

since X is Hilbert space. Moreover, from Assumption (A2), for any k and n there exist sequences of reals $\alpha_{j,n}^{i,k}$, $\alpha_{j,n}^{i}$, $j = 1, \ldots, k_n$, such that we have, a.s.,

$$p_n^{i,k} = \sum_{j=1}^{k_n} \alpha_{j,n}^{i,k} I_{B_j^n} \quad \text{and} \quad p_n^{i} = \sum_{j=1}^{k_n} \alpha_{j,n}^{i} I_{B_j^n}. \tag{15}$$

Now (14) and (15) assure that for any n and $j \le k_n$ we have $\alpha_{j,n}^{i,k} - \alpha_{j,n}^{i} \to 0$ as $k \to \infty$ and we may write, for any M,

$$A_M(\widetilde{s}^k) = E\left(\sum_{n=1}^{M} X_n \sum_{j=1}^{k_n} \prod_{i=1}^{l} \alpha_{j,n}^{i,k} I_{B_j^n} \prod_{j=l+1}^{m} \left(1 - p_1^{j,k} - \cdots - p_n^{j,k}\right)\right), \tag{16}$$

$$B_M(\widetilde{s}^k) = E\left(\sum_{n>M} X_n \prod_{i=1}^{l} p_n^{i,k} \prod_{j=l+1}^{m} \left(1 - p_1^{j,k} - \cdots - p_n^{j,k}\right)\right), \tag{17}$$

$$|B_M(\widetilde{s}^k)| \le E\left(\sup_{n>M} |X_n|\right). \tag{18}$$

Now, from (14) to (16) and the assumption that the sequence $\{X_n\}_{n \in N}$ satisfies the integrability condition analogous to $(A1)$, for any M we have

$$\lim_{k\to\infty} A_M(\widetilde{s}^k) = A_M(\widetilde{s}). \tag{19}$$

Moreover, from (7) and assumptions on $\{X_n\}_{n \in \mathcal{N}}$ the sequence $\{B_M(\widetilde{s}^k)\}_{M \in \mathcal{N}}$ converges uniformly to 0 as $M \to \infty$. Hence, (13) and (19) give us weak continuity of $J(\cdot)$ which completes the proof. □

Proof of Theorem 2 Suppose (A1), (A2) and that K is a random horizon satisfying the assumptions of Theorem 2. We will show that for the game $\mathcal{G}_K\left(\widetilde{S}\right)$, equivalent to $\mathcal{G}_K\left(\widetilde{\Lambda}\right)$, one may apply Theorem 8. Following the proof of Theorem 1 we state that the assumptions (i) and (iii) are fulfilled with every player's strategy set

S which is weakly compact in X. We will show that player's payoffs are weakly compact. Let $\widetilde{\tau} = (\tau_1, \ldots, \tau_m) \in \widetilde{\Lambda}$, and $\widetilde{s} = (s_1, \ldots, s_m) \in \widetilde{S}$ be corresponding random stopping game strategy, i.e. $s_i = \{p_n^i\}_{n\in\mathcal{N}}$, $p_n^i = P(\tau_i = n \mid \mathcal{F}_n)$, a.s., for $i \in \mathcal{M}$, $n \in \mathcal{N}$. Now, the player's i payoff may be rewritten as follows

$$V_K^i(\widetilde{\tau}) = E\left(R^i_{t(\widetilde{\tau}),D(\widetilde{\tau})} I_{\{t(\widetilde{\tau})\le K\}}\right)$$

where we have denoted $\overline{G}_n = P(K > n)$, $n = 1, 2, \ldots$

Now, it is easy to see that H_i is the finite sum of functions of the following forms

$$U(\widetilde{s}) = E\left(\sum_{n\in\mathcal{N}} X_n \overline{G}_{n-1} \prod_{i\in D} p_n^i \prod_{j\notin D}\left(1 - p_1^j - \cdots - p_n^j\right)\right), \quad \widetilde{s} \in \widetilde{S}, \quad (20)$$

where according to Assumption (A1) a sequence $\{X_n\}_{n\in\mathcal{N}}$ is $\{\mathcal{F}_n\}_{n\in\mathcal{N}}$ adapted and $E(\sup|X_n|) < \infty$. Moreover, $X_n\overline{G}_{n-1} \to 0$ as $n \to \infty$, a.s., since $E(K) < \infty$. Thus the function $U(\cdot)$ is weakly continuous since it has the same properties as $J(\cdot)$ in (12) where we insert $X_n\overline{G}_{n-1}$ instead of X_n, $n = 1, 2, \ldots$. Thus $H_i(\cdot)$, $i \in \mathcal{M}$, are weakly continuous. Hence, the assumption (ii) of Theorem 8 is satisfied for the game $\mathcal{G}_K(\widetilde{S})$. The proof is completed. □

Remark 1 Let us note that in the light of the formula (20) the random horizon of the randomized stopping game $\mathcal{G}_K(\widetilde{\Lambda})$ is equivalent to the game $\mathcal{G}(\widetilde{S})$ in which the sequence of players' rewards is discounted with $\overline{G}_{n-1}$, $n = 1, 2, \ldots$

Proof of Theorem 3 Assume that (A1), (A2) are fulfilled and that $\widetilde{S}_r^L$ is the game strategy set, $r \in X$, $L \in \mathcal{N}$. Thanks to Lemma 1(ii) and Theorem 8 we need to show that $H_i(\cdot)$, $i \in \mathcal{M}$, are continuous. It is easy to see that $H_i(\widetilde{s})$ may be written as the finite sum of functions written as follows

$$J(\widetilde{s}) = A_M(\widetilde{s}) + B_M(\widetilde{s}),$$

with

$$A_M(\widetilde{s}) = E\left(\sum_{n=1}^{M} X_n \prod_{i=1}^{l} p_n^i \prod_{j=l+1}^{m}\left(1 - p_1^j - \cdots - p_n^j\right)\right),$$

$$B_M(\widetilde{s}) = E\left(\sum_{n>M} X_n \prod_{i=1}^{l} p_n^i \prod_{j=l+1}^{m}\left(1 - p_1^j - \cdots - p_n^j\right)\right),$$

where sequence $\{X_n\}_{n\in\mathcal{N}}$ satisfies integrability assumption corresponding to $(A1)$. Now, continuity of $J(\cdot)$ is obvious since A_M is continuous for any M and $B_M(\widetilde{s}) \to 0$ as $M \to \infty$, uniformly in S_r^L. □

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