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ORIGINAL ARTICLE

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Double optimal stopping times and dynamic pricing problem: description of the mathematical model

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Abstract In many industries, managers face the problem of selling a given stock of items by a deadline. We investigate the problem of dynamically pricing such inventories when demand is price sensitive and stochastic and the firm's objective is to maximize expected revenues. Examples that fit this framework include retailers selling fashion and seasonal goods and the travel and leisure industry, which markets space such as seats on airline flights, cabins on vacation cruises, hotels renting rooms before midnight and theaters selling seats before curtain time that become worthless if not sold by a specific time. Given a fixed number of seats, rooms, or coats, the objective for these industries is to maximize revenues in excess of salvage value. When demand is price sensitive and stochastic, pricing is an effective tool to maximize revenues. In this paper, we address the problem of deciding the optimal timing of a double price changes from a given initial price to given lower or higher prices. Under mild conditions, it is shown that it is optimal to decrease the initial price as soon as the time-to-go falls below a time threshold and increase the price if time-to-go is longer than adequate time threshold. These thresholds depend on the number of yet unsold items.

Keywords Point process · Yield management · Optimal stopping times · Perishable goods · Fashion apparel · Optimal dynamic pricing of inventories · Stochastic demand · Finite horizons · Intensity control

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1 Introduction

This paper studies a revenue management problem, in which a finite number of commodities are sold in finite time horizon. The problem is formulated as sales of the poissonian stream of goods on the predefined time, which can be chosen at some random moments based on the time-to-go and on the number of yet unsold items.

Many industries face the problem of selling a fixed stock of items over a finite horizon. These industries include airlines selling seats before planes depart, hotels renting rooms before midnight, theaters selling seats before curtain time and retailers selling seasonal goods such as air-conditioners or winter coats before the end of the season. Feng and Gallego (1995) considered the following optimization problem: for given a fixed number of seats, rooms, or coats to maximize revenues in excess of salvage value. They address the problem of deciding the optimal timing of *a single* price change from a given initial price to either a given lower or higher second price. They have shown that it is optimal to decrease (resp., to increase) the initial price as soon as the time-to-go falls below (resp., above) a time threshold that depends on the number of yet unsold items. Price adjustment should be done with care since pricing that aim to run out of stock are not necessarily optimal in maximizing expected revenues as has been pointed out by Gallego and van Ryzin (1994). Anjos et al. (2005) studied similar problem but they have considered rather time-dependent than stock-dependent demand and they presented descriptions of the time-dependent pricing functions under different assumptions for customer behavior. They assume that pricing function is continuous in time. This methodologies is much easier to determine the shape of the optimal price curves for different demand and reservation price distribution but it is difficult to practical realization.

Pricing decision is the minority of all important decisions, which can apparently influence a firm's profit-making within extremely short time. In an era of meagre profit, firms cannot stand any more injury caused of mistake at pricing. However, lots of managers still make pricing decision according to their experience or the action of other competitors without any mechanism of price-determining based on their firm's resource condition.

The subject of this research is to probe the abiding price-reducing strategy for fashion appearing firms. Fashion apparel is a kind of commodities with seasonality and popularity and is an example of all perishable goods. For all sorts of characteristic such as the need for long lead time before production, short time span for sale and the low salvage value after season, it makes firms reduce price to close out inventories by the end of seasons to evade value loss. When it comes to price-reducing, the fashion apparel is quite different from other commodities. It is a kind of commodity, which has speciality of phased and monotonicity on price reduction. Therefore, it lacks two kinds of elasticity, which are price-adjusting at any time and adjusting the price range at will. For the characteristic of close interdependence between product and time and the normal demand on price-reducing, fashion apparel firms need some decision tools which are more fast, correct, and practical than any other ones.

With two main parameters, which are *the levels of unsold inventory* and *the length of season remaining* this research constructs out an stochastic dynamic programming model to maximize profit by price change at two random moments, which are constructed based on observation of the market. This model can be proved to be able to extend to other similar industries with the same nature.

The yield management problems for perishable goods has been considered by many authors. Let us recall the works by Rothstein (1971), which developed an overbooking model for airlines and the model for a firm, which has inventories of a set of components that are used to produce a set of products considered by Gallego and van Ryzin (1997). In this last problem there is a finite horizon over which the firm can sell its products. Demand for each product is a stochastic point process with an intensity that is a function of the vector of prices for the products and the time at which these prices are offered. The problem is to price the finished products so as to maximize total expected revenue over the finite sales horizon. Aviv and Pazgal (2005) proposed model of optimal pricing of fashion-like seasonal goods. They took into consideration the effect of forward-looking consumer behavior and find that sellers incorrectly assume that customers are myopic. They discussed also some discounting strategies, which unlikely gave minimal improvement in revenue. One of the extensions yield management problem are models with overbooking. Karaesmen and van Ryzin (2004) considered model combining overbooking and seat allocation decisions on a single flight. They create efficient algorithm to determine the optimal policy parameters. Several models connected with optimal pricing are studied by Feng and Xiao. They widen problem proposed by Feng and Gallego (1995) and solved the case with possibility of the multiple markdowns or markups in Feng and Xiao (2000b). Later they considered model, which is similar to our problem in Feng and Xiao (2000a), but they assumed that each of prescribed prices can be active at any time. Another interesting model is problem of selling a finite number of substitutable commodities to two different market segments at respective prices Feng and Xiao (1999). The management closes the low price segment when the chance of selling all items at the high price is promising. These authors also studied seat inventory control in Feng and Xiao (2001a) and further, they join capacity allocation with pricing of perishable goods in article by Feng and Xiao (2001b).

The paper is organized as follows. In the next section the model of the retail with double price changes of goods is formulated and the various assumption are made. In Sect. 3 the technical, auxiliary investigation are performed. The main theorem are in Sect. 4 and it gives the optimal strategy for price changes when two changes are permitted. The numerical example is included in Sect. 4.1.

2 The double price changes problem

The double price changes problem is formulated as follows: we want to maximize the expected revenue from a sale of fixed number of commodities by given deadline when set of available prices is prescribed in advance and two changes to lower or higher price are allowed.

2.1 Notations and assumptions

We assume that the firm works in a market with imperfect competition, which affects that demand is price sensitive. Let $n \in \mathbf{N}$ be a number of items available for sales at time zero and let $t \in \mathbf{R}_+$ be the length of the sales horizon. Such model of the market has been considered by Feng and Gallego (1995). They control the demand by one change of price during the sales period. The set of available prices $\mathcal{P} = \{p_0, p_1, \ldots, p_6\}$ is obtained from past experience or consensus in the industry level. If observed demand for the commodities will be suitably high, it follows the price increasing for the first time from p_0 to p_2 eventually if the demand is low the price can be changed to p_1 ($p_1 < p_0 < p_2$). If the first change is to p_1 the second can be only to p_3 or p_4 ($p_3 < p_1 < p_4$). Similarly if the first change is to p_2 the second should be to p_5 or p_6 ($p_5 < p_2 < p_6$). Additionally, let us assume $p_4 < p_0 < p_5$. If observed demand is high enough (or suitable small) there is an opportunity to change directly to p_4 or p_6 (p_3 or p_5) without pricing at p_2 (or p_1). We assume that demand for the commodities is stochastic, price sensitive and modeled by homogeneous Poisson process.

Let us consider Poisson processes $\{N_i(s), \mathcal{F}_s^i\}_{s=0}^{\infty}$, $i \in \{0, \ldots, 6\}$, defined on probability space $(\Omega, \mathcal{F}, P)$. It is assumed that $N_i(s)$ are $\mathcal{F}_s^i$ measurable, where $\mathcal{F}_s^i \subset \mathcal{F}$ for all $s \in [0, t]$, with intensity λ_i. The intensities λ_i, $i \in \{0, \ldots, 6\}$, are determined from market research. They strictly correspond to demand at price p_i. It can be observed that if $p_i < p_k$ then $\lambda_i > \lambda_k$. Let us denote $r_i = \lambda_i p_i$, $i \in \{0, \ldots, 6\}$, the revenue rate at price p_i. We assume that $r_i > r_k$ if $p_i < p_k$ because in other case there is no economical reason to decrease the price. Without loss the generality we assume that the salvage value q of unsold items is zero because we can simply transform the problem to zero salvage case by defining new prices $p_i' = p_i - q$.

The opportunity to choose the direction of price change makes possible to respond to observed demand. Therefore, our model applies to pricing perishable goods, which must be sold by a deadline and observed demand vary with time for example. There are models with other assumptions about price changes. For example Feng and Gallego (1995) solved similar problem with increasing the price if demand is high enough. They have also investigated opportunity to choose the direction of price change when the one change is allowed only.

Let us denote by $J_i^k(n, t; 0)$ the maximum expected revenue from sale of n items during the time t if the price has been changed immediately from p_i to p_k. Owing to two possibilities to chose from the prices, we define maximum expected revenue in case of immediate change from p_i to p_{2i+1} or p_{2i+2} as

$$J_i(n, t, 0) = \max\left\{J_i^{2i+1}(n, t, 0), J_i^{2i+2}(n, t, 0)\right\}, \tag{1}$$

$i \in \{0, 1, 2\}$. Let us notice that the revenue obtained in the case of the price changes from p_i to p_{2i+1} or p_{2i+2} at time s is equal to

$$I_i(n, t, s) = p_i \min\{n, N_i(s)\} + \mathbf{E}\left[J_i(n - N_i(s), t - s, 0) | \mathcal{F}_s^i\right]. \tag{2}$$

2.2 The double change price model as optimal double stopping problem

Our problem is to find two optimal times of markups or markdowns, which maximizes expected revenue. From mathematical point of view such problem can be considered as the optimal double stopping problem. The double stopping problem with discrete time has been considered by Haggstrom (1967) and Stadje

(1985). It is hard to find in the literature formulation of double stopping problem for continuous time processes, adequate to our model. Davis (1993) has only considered single stopping problem for class of piecewise deterministic processes. Our poissonian sale's process is simple example from that class.

Let us define $\mathcal{F}_s = \mathcal{F}_s^0$. We denote $\mathcal{T}$ the set of stopping times with respect to $\{\mathcal{F}_s\}_{s\geq 0}$ and the set of strategies $\mathcal{S} = \{(\gamma_s)_{s\geq 0}\}$, $\mathcal{F}_s$-adapted processes such that for given $\tau \in \mathcal{T}$,

$$\gamma_s(\omega) = i \ \mathbb{I}_{\{\tau\leq s\}}(\omega), \quad i \in \{1, 2\}.$$

Further, let us define $\mathcal{F}_{st} = \sigma\{\mathcal{F}_s, \gamma_s, N_{\gamma_s}(u-s)\mathbb{I}_{\{s\leq u\}}(u) : u \leq t\}$ and $\mathcal{T}_s$ the set of stopping times σ, such that $\{\omega : \sigma > t\} \in \mathcal{F}_{st}$ for every $s \leq t$. For given $\tau \in \mathcal{T}$, strategy $(\gamma_s)_{s\geq 0} \in \mathcal{S}$ and $\sigma \in \mathcal{T}_\tau$, the set of strategies $\mathcal{S}_s = \{(\delta_t^{\gamma_s})_{t\geq s}\}$, such that $\delta_t^{\gamma_s}$ is $\mathcal{F}_{st}$-adapted for every $s \leq t$ is defined as follows

$$\delta_t^{\gamma_s}(\omega) = (2\gamma_{s\wedge t}(\omega) + i) \ \mathbb{I}_{\{\omega:s<\sigma(\omega)\leq t\}}(\omega), \quad i \in \{1, 2\}.$$

The two parameter decision process can be formulated

$$\delta_{st}(\omega) = \begin{cases} \gamma_s(\omega) & \text{for } s \geq t, \\ \delta_t^{\gamma_s}(\omega) & \text{for } s < t. \end{cases}$$

Let $J_k(n,t)$, $k \in \{0, \dots, 6\}$, be the maximum expected revenue (revenue function) if the sale is at price p_k and it remains n items and t units of time to deadline of sales period. In particular, if there are no possibility to change price we have

$$J_k(n,t) = p_k \mathbf{E} \min\{N_k(t), n\}, \quad k \in \{3, 4, 5, 6\}. \tag{3}$$

The compound structure of double stopping model follows that $J_i^k(n,t,0) = J_k(n,t)$, for all $i \in \{0, 1, 2\}$ and $k \in \{2i+1, 2i+2\}$.

Let us notice by $\mathcal{T}$ the class of stopping times with respect to $\mathcal{F}_s^0$, such that $0 \leq \tau \leq t$ and $N_0(\tau) \leq n$. Our goal is to find stopping time $\tau^* \in \mathcal{T}$, which maximizes the expected revenue and

$$J_0(n,t) = \mathbf{E} I_0(n,t,\tau^*) = \sup_{\tau\in\mathcal{T}} \mathbf{E} I_0(n,t,\tau). \tag{4}$$

From properties of conditional expectation and (2) we obtain

$$J_0(n,t) = p_0 \mathbf{E}\big[\min\{n, N_0(\tau^*)\} + J_0(n - N_0(\tau^*), t - \tau^*, 0)\big]. \tag{5}$$

Foregoing equation and (1) follows

$$\begin{aligned} J_0(n,t) = p_0 \mathbf{E}\big[& \min\{n, N_0(\tau^*)\} \\ & + \max\{J_1(n - N_0(\tau^*), t - \tau^*), J_2(n - N_0(\tau^*), t - \tau^*)\}\big]. \end{aligned}$$

It is easy to see that revenue function in double markup or markdown problem is compound of revenues from sale at price p_0 and maximum of the expected revenues from the single markup or markdown models with shorter horizon, fewer number of items and different initial price. The single markup or markdown problem has been solved by Feng and Gallego (1995). They have found two sequences of time thresholds x_n^i and z_n^i. In their paper the sequences of time thresholds has been determined such that it is optimal to change the price to lower if the number of unsold items is equal n and the time-to-go is shorter than x_n^i and similarly it is optimal switch the price to higher if the time-to-go is longer than z_n^i corresponding the number of unsold items. Aforementioned revenue functions $J_1(n,t)$ and $J_2(n,t)$ are determined in following theorem.

Theorem 1 (Feng and Gallego 1995, Sect. 5) *When* $p_{2i+1} < p_i < p_{2i+2}$, $i \in \{1,2\}$, *there exist two strictly increasing sequences* x_n^i *and* z_n^i, $n \in \mathbf{N}$, *satisfying* $x_n^i < t_n^i < z_n^i$, *such that optimal revenue function is given by equation*

$$J_i(n,t) = \begin{cases} J_i(n,t,0) + F_i(n,t) & \textit{if } x_n^i \le t \le z_n^i, \\ J_i(n,t,0) & \textit{otherwise} \end{cases}$$

and

$$F_i(n,t) = \begin{cases} H_{2i+1}(n,t) & \textit{if } x_n^i \le t \le t_n^i, \\ H_{2i+2}(n,t) & \textit{if } t_n^i < t \le z_n^i, \\ 0 & \textit{otherwise,} \end{cases}$$

where $t_n^i = \inf\{t > 0 : J_{2i+1}(n,t) - J_{2i+2}(n,t) = 0\}$ *and* $H_{2i+1}(n,t)$ *is the solution to the differential equation*

$$\frac{\partial H_{2i+1}(n,t)}{\partial t} = -\lambda_i H_{2i+1}(n,t) + L_{2i+1}(n,t) \tag{6}$$

with boundary condition $H_{2i+1}(n, x_n^i) = 0$.

However, $H_{2i+2}(n,t)$ *is the solution to the differential equation*

$$\frac{\partial H_{2i+2}(n,t)}{\partial t} = -\lambda_i H_{2i+2}(n,t) + L_{2i+2}(n,t) \tag{7}$$

with boundary condition $H_{2i+2}(n, t_n^i) = H_{2i+1}(n, t_n^i)$,
where

$$\begin{aligned} L_k(n,t) &= G_k(n,t) + \lambda_i [J_i(n-1,t) - J_k(n-1,t)], \quad k \in \{2i+1, 2i+2\}, \\ G_k(n,t) &= r_i - r_k + p_k(\lambda_k - \lambda_i) P(N_k(t) \ge n). \end{aligned} \tag{8}$$

The thresholds are given by

$$x_n^i = \inf\{t \ge 0 : L_{2i+1}(n,t) = 0\}$$

and

$$z_n^i = \inf\left\{t \ge t_n^i : H_{2i+2}(n,t) \le 0\right\}.$$

The form of the function $G_k(n,t)$ will be explained in Lemma 2. On the basis of results described in Feng and Gallego (1995), we will find only stopping times for the first change to lower or higher price.

3 Auxiliary results

The main result is based on technical theorems, lemmas and remarks. They are collected in this section. Before showing the solution of our problem we make several preliminary remarks and cite facts, which are necessary to solve the model.

Lemma 1 (Feng and Xiao 2000b, Sect. 3) *Let $J_i(n,t)$ be an optimal revenue from the price change from p_i to p_{i+1}, given by equation*

$$J_i(n,t) = \sup_{\tau \in \mathcal{T}} E\big[p_i \min\{n, N_i(\tau)\} + J_i^{i+1}(n - N_i(\tau), t - \tau, 0)\big],$$

then $J_i(n,t)$ can be represented as

$$J_i(n,t) = J_i^{i+1}(n,t,0) + \tilde{F}_i(n,t), \tag{9}$$

where $\tilde{F}_i(n,t)$ is maximum premium resulting from optimal behavior.

Theorem 2 (Feng and Xiao 2000b, Sect. 3) *Let $J_i(n,t)$ be an optimal expected revenue from the price change from p_i to p_{i+1}. Suppose there exist a function $F_i(n,t)$ such that for each $n \in \mathbf{N}$, $F_i(n,t)$ is absolutely continuous, uniformly bounded and piecewise differentiable in $t \in \mathbf{R}_+$ with right continuous partial derivatives. If $F_i(n,t)$ satisfies*:

(i) $F_i(n,t) \geq 0$ *for all* $n \in \mathbf{N}$, $t \in \mathbf{R}_+$,
(ii) $F_i(0,t) = 0$ *and* $F_i(n,0) = 0$,
(iii) $F_i(n,t) = 0$ *implies*

$$\frac{\partial J_i(n,t)}{\partial t} \geq r_i - \lambda_i \left[J_i(n,t) - J_i(n-1,t)\right], \tag{10}$$

(iv) $F_i(n,t) > 0$ *implies*

$$\frac{\partial J_i(n,t)}{\partial t} = r_i - \lambda_i \left[J_i(n,t) - J_i(n-1,t)\right], \tag{11}$$

then $F_i(n,t) = \tilde{F}_i(n,t)$.

The proof of the theorem is based on general theory of optimal stopping (see Theorem T1 on page 203 in Brémaud 1981).

Let $N_i(s)$ denote a Poisson process with known, constant intensity λ_i. Then the following equation is satisfied:

$$\mathbf{E}\min\{n, N_i(s)\} = \sum_{k=1}^{n} P(N_i(s) \geq k). \tag{12}$$

Let us notice that

$$\mathbf{E}\min\{n, N_i(s)\} = \sum_{k=0}^{n-1} kP(N_i(s) = k) + nP(N_i(s) \geq n).$$

Thus

$$\mathbf{E}\min\{n, N_i(s)\} = (n-1)P(N_i(s) < n) - \sum_{k=0}^{n-1} P(N_i(s) < k) + nP(N_i(s) \geq n)$$
$$= -1 + P(N_i(s) \geq n) + \sum_{k=0}^{n-1} P(N_i(s) \geq k) = \sum_{k=1}^{n} P(N_i(s) \geq k).$$

Let us notice that

$$\frac{\partial}{\partial s} P(N_i(s) \geq n) = \lambda_i P(N_i(s) = n-1). \tag{13}$$

It is easy to see that

$$\frac{\partial}{\partial s} P(N_i(s) \geq n) = -\frac{\partial}{\partial s} \sum_{k=0}^{n-1} P(N_i(s) = k) = -\frac{\partial}{\partial s} \sum_{k=0}^{n-1} \frac{(\lambda_i s)^k}{k!} \mathrm{e}^{-\lambda_i s}$$
$$= \lambda_i P(N_i(s)=0) + \lambda_i \sum_{k=1}^{n-1} [P(N_i(s) = k) - P(N_i(s) = k-1)]$$
$$= \lambda_i P(N_i(s) = n-1).$$

From what has already been proved we obtain

$$\frac{\partial}{\partial s} \sum_{k=1}^{n} P(N_i(s) \geq k) = \lambda_i \left[1 - P(N_i(s) \geq n)\right] \tag{14}$$

and

$$\frac{\partial^2}{\partial s^2} \sum_{k=1}^{n} P(N_i(s) \geq k) = -\lambda_i^2 P(N_i(s) = n-1). \tag{15}$$

We define functions $G_{2i+1}(n,t)$ and $G_{2i+2}(n,t)$ for every $i \in \{0, 1, 2\}$ as

$$G_{2i+1}(n,t) = r_i - \lambda_i [J_{2i+1}(n,t) - J_{2i+1}(n-1,t)] - \frac{\partial J_{2i+1}(n,t)}{\partial t}, \tag{16}$$

$$G_{2i+2}(n,t) = r_i - \lambda_i [J_{2i+2}(n,t) - J_{2i+2}(n-1,t)] - \frac{\partial J_{2i+2}(n,t)}{\partial t}. \tag{17}$$

Lemma 2 *The functions $G_3(n,t)$ and $G_5(n,t)$ ($G_4(n,t)$ and $G_6(n,t)$) are continuous and strictly decreasing (increasing) in n and strictly increasing (decreasing) in t.*

Proof Applying (16), (3), (12) and (14), we have for $i \in \{1, 2\}$

$$G_{2i+1}(n,t) = r_i - \lambda_i p_{2i+1} P(N_{2i+1}(t) \geq n) - \frac{\partial}{\partial t} p_{2i+1} \sum_{k=1}^{n} P(N_{2i+1}(t) \geq k)$$
$$= r_i - r_{2i+1} + p_{2i+1}(\lambda_{2i+1} - \lambda_i) P(N_{2i+1}(t) \geq n). \tag{18}$$

By assumptions about intensities $\lambda_{2i+1} > \lambda_i$ so $G_{2i+1}(n,t)$ is continuous, strictly decreasing in n and increasing in t. Similarly,

$$G_{2i+2}(n,t) = r_i - r_{2i+2} + p_{2i+2}(\lambda_{2i+2} - \lambda_i)P(N_{2i+2}(t) \geq n) \tag{19}$$

and $\lambda_{2i+2} < \lambda_i$, hence $G_{2i+2}(n,t)$ is continuous, strictly increasing in n and decreasing in t. □

Let us mention two consequences of the Theorem 1.

Remark 1 The function $J_i(n,t)$, $i \in \{1,2\}$, has the form

$$J_i(n,t) = \begin{cases} J_{2i+1}(n,t) & \text{if } t < x_n^i, \\ J_{2i+1}(n,t) + H_{2i+1}(n,t) & \text{if } x_n^i \leq t \leq t_n^i, \\ J_{2i+2}(n,t) + H_{2i+2}(n,t) & \text{if } t_n^i < t \leq z_n^i, \\ J_{2i+2}(n,t) & \text{if } z_n^i < t. \end{cases} \tag{20}$$

Proof By assumptions about revenue rates for suitably small δt

$$\begin{aligned} J_{2i+1}(n,\delta t) - J_{2i+2}(n,\delta t) &= p_{2i+1}\mathbf{E}\min\{n, N_{2i+1}(\delta t)\} \\ &\quad - p_{2i+2}\mathbf{E}\min\{n, N_{2i+2}(\delta t)\} \\ &= (p_{2i+1}\lambda_{2i+1} - p_{2i+2}\lambda_{2i+2})\delta t \\ &= (r_{2i+1} - r_{2i+2})\delta t > 0 \end{aligned}$$

and when t is big enough we get $J_{2i+1}(n,t) - J_{2i+2}(n,t) = (p_{2i+1} - p_{2i+2})n < 0$.

Because $J_{2i+1}(n,t)$ and $J_{2i+2}(n,t)$ are continuous, increasing in t and strictly concave we can observe that $J_{2i+1}(n,t) \geq J_{2i+2}(n,t)$ for $t \in [0, t_n^i]$ and $J_{2i+1}(n,t) < J_{2i+2}(n,t)$ for $t \in (t_n^i, \infty)$, hence we have

$$\max\{J_{2i+1}(n,t), J_{2i+2}(n,t)\} = \begin{cases} J_{2i+1}(n,t) & \text{if } t \leq t_n^i, \\ J_{2i+2}(n,t) & \text{if } t_n^i < t, \end{cases}$$

which proves (20). □

Remark 2 The functions $J_i(n,t)$, $i \in \{1,2\}$, have the form

$$J_i(n,t) = \begin{cases} J_{2i+1}(n,t) & \text{if } t < x_n^i, \\ J_{2i+1}(n,t) + H_{2i+1}(n,t) & \text{if } x_n^i \leq t \leq z_n^i, \\ J_{2i+2}(n,t) & \text{if } z_n^i < t \end{cases} \tag{21}$$

or

$$J_i(n,t) = \begin{cases} J_{2i+1}(n,t) & \text{if } t < x_n^i, \\ J_{2i+2}(n,t) + H_{2i+2}(n,t) & \text{if } x_n^i \leq t \leq z_n^i, \\ J_{2i+2}(n,t) & \text{if } z_n^i < t. \end{cases} \tag{22}$$

It is a consequence of the fact that

$$H_{2i+2}(n,t) = H_{2i+1}(n,t) + J_{2i+1}(n,t) - J_{2i+2}(n,t) \quad \text{for } t \in [x_n^i, z_n^i], \tag{23}$$

which has been proved in Feng and Gallego (1995).

Lemma 3 *The functions $J_i(n,t)$, $i \in \{1,2\}$, are continuous for $t \in \mathbf{R}_+$ and differentiable for all $t \in \mathbf{R}_+ \backslash \{z_n^i\}$.*

Proof Let us use the form of $J_i(n,t)$, which is given by Eq. (21). Solving differential equation (6) with respect to boundary condition, we obtain

$$H_{2i+1}(n,t) = \int_{x_n^i}^{t} L_{2i+1}(n,u)\mathrm{e}^{-\lambda_i(t-u)}\mathrm{d}u.$$

Let us observe that $H_{2i+1}(n,t)$, $J_{2i+1}(n,t)$, $J_{2i+2}(n,t)$ are continuous and $H_{2i+1}(n,x_n^i) = 0$. By (23) it is obvious that $J_{2i+1}(n,z_n^i) + H_{2i+1}(n,z_n^i) = J_{2i+1}(n,z_n^i)$, therefore $J_i(n,t)$ is continuous for $t \in \mathbf{R}_+$.

It is easy to check that $H_{2i+1}(n,t)$ and $J_{2i+1}(n,t)$ are differentiable for $t \in [x_n^i, z_n^i]$ and it follows that $J_i(n,t)$ is differentiable for $t \in (x_n^i, z_n^i)$. Using boundary condition we have

$$\begin{aligned}\left[\frac{\partial J_i(n,t)}{\partial t}\right]_{x_n^i} - \lim_{t \to x_n^i -} \frac{\partial J_i(n,t)}{\partial t} &= \left[\frac{\partial J_i(n,t)}{\partial t}\right]_{x_n^i} - \left[\frac{\partial J_{2i+1}(n,t)}{\partial t}\right]_{x_n^i} \\ &= L_{2i+1}(n,x_n^i) - \lambda_i H_{2i+1}(n,x_n^i) = 0.\end{aligned}$$

In the presence of above equation, $J_i(n,t)$ is differentiable at x_n^i as well. On the other side $J_i(n,t)$ has only one-sided derivatives at z_n^i. Let us notice, that on the basis of Theorem 1 we have $J_i(n-1,z_n^i) = J_{2i+2}(n-1,z_n^i)$. Using the form of $J_i(n,t)$, given by Eq. (22) we obtain

$$\begin{aligned}\left[\frac{\partial J_i(n,t)}{\partial t}\right]_{z_n^i} - \lim_{t \to z_n^i +} \frac{\partial J_i(n,t)}{\partial t} &= \frac{\partial}{\partial t}\left[J_{2i+2}(n,t) + H_{2i+2}(n,t) - J_{2i+2}(n,t)\right]_{z_n^i} \\ &= -\lambda_i H_{2i+2}(n,z_n^i) + L_{2i+2}(n,z_n^i) \\ &= G_{2i+2}(n,z_n^i) < 0,\end{aligned}$$

hence $J_i(n,t)$ is not differentiable at z_n^i. □

Lemma 4 *If $\rho(s)$ is monotone decreasing (increasing) function of $s \in \mathbf{R}$, then for $\lambda > 0$, $t \geq 0$, the function $f(t) = \int_{-\infty}^{t} \rho(s)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s$ is also monotone decreasing (increasing).*

Proof Because $\rho(s)$ is decreasing (increasing) we obtain

$$\lambda f(t) \geq \lambda\rho(t) \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\mathrm{d}s = \rho(t) \quad (\lambda f(t) \leq \rho(t)).$$

Let us notice that $f'(t) = -\lambda f(t) + \rho(t)$, hence $f'(t) \leq 0$ ($f'(t) \geq 0$). □

Let us define the marginal expected revenue as

$$\Delta_i(n,t) = J_i(n,t) - J_i(n-1,t). \tag{24}$$

Lemma 5 *$\Delta_i(n,t)$, $i \in \{1,2\}$, is increasing in t for $t \in [x_n^i, z_{n-1}^i]$ if and only if $\Delta_i(n,t)$ is decreasing in n and for $t \in (z_{n-1}^i, z_n^i]$ if and only if*

$$\lambda_i \Delta_i(n,t) \le r_i - r_{2i+2} + r_{2i+2} P(N_{2i+2}(t) \ge n-1). \tag{25}$$

Proof Using the fact that $H_{2i+1}(n,t) = J_i(n,t) - J_{2i+1}(n,t)$ for $t \le z_n^i$, we calculate

$$\begin{aligned}\frac{\partial \Delta_i(n,t)}{\partial t} &= r_{2i+1} P(N_{2i+1}(t) = n-1) - \lambda_i [H_{2i+1}(n,t) - H_{2i+1}(n-1,t)] \\ &\quad + L_{2i+1}(n,t) - L_{2i+1}(n-1,t) \\ &= r_{2i+1} P(N_{2i+1}(t) = n-1) + G_{2i+1}(n,t) - G_{2i+1}(n-1,t) \\ &\quad - \lambda_i [\Delta_i(n,t) - \Delta_i(n-1,t) + p_{2i+1} P(N_{2i+1}(t) = n-1)] \\ &= -\lambda_i [\Delta_i(n,t) - \Delta_i(n-1,t)]. \end{aligned} \tag{26}$$

However, if $t \in (z_{n-1}^i, z_n^i]$ we obtain

$$\begin{aligned}\frac{\partial \Delta_i(n,t)}{\partial t} &= \frac{\partial}{\partial t}[J_{2i+2}(n,t) + H_{2i+2}(n,t) - J_{2i+2}(n-1,t)] \\ &= r_{2i+2} P(N_{2i+2}(t) = n-1) + G_{2i+2}(n,t) - \lambda_i \Delta_i(n,t) \\ &\quad + \lambda_i [J_{2i+2}(n,t) - J_{2i+2}(n-1,t)] \\ &= -\lambda_i \Delta_i(n,t) + r_i - r_{2i+2} + r_{2i+2} P(N_{2i+2}(t) \ge n-1)\end{aligned}$$

and it cause that the lemma is true. □

Lemma 6 *If $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{x_n^i} \ge 0$ and $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{z_{n-1}^{i+}} \ge 0$, then $\Delta_i(n,t)$, $i \in \{1,2\}$, is increasing in t and decreasing in n for $t \in [x_n^i, z_n^i]$.*

Proof Theorem 2 implies that

$$\frac{\partial J_i(n,t)}{\partial t} = r_i - \lambda_i [J_i(n,t) - J_i(n-1,t)] \quad \text{for } t \in [x_n^i, z_n^i]. \tag{27}$$

Differentiating (27) with respect to t, yields

$$\frac{\partial^2 J_i(n,t)}{\partial t^2} = -\lambda_i \frac{\partial}{\partial t}[J_i(n,t) - J_i(n-1,t)] \quad \text{for } t \in [x_n^i, z_n^i]. \tag{28}$$

Multiplying (28) by $\mathrm{e}^{-\lambda_i(t-s)}$ and taking integral from x_n^i to t when $t \le z_{n-1}^i$, we get

$$\frac{\partial J_i(n,t)}{\partial t} = c\mathrm{e}^{-\lambda_i(t-x_n^i)} + \int_{x_n^i}^{t} \lambda_i \frac{\partial J_i(n-1,s)}{\partial s} \mathrm{e}^{-\lambda_i(t-s)} \mathrm{d}s, \tag{29}$$

where $c = [\frac{\partial}{\partial s} J_i(n,s)]_{x_n^i} = r_{2i+1} P(N_{2i+1}(x_n^i) \le n-1)$. Now let us define

$$\rho(t) = \begin{cases} 0 & \text{if } t < x_n^i, \\ \dfrac{\partial}{\partial t} J_i(n-1,t) - c & \text{if } x_n^i \le t \le z_{n-1}^i. \end{cases} \tag{30}$$

It is easy to calculate that

$$\frac{\partial}{\partial t} J_i(1,t) = \begin{cases} r_{2i+1}[1 - P(N_{2i+1}(t) \geq 1)] & \text{if } t < x_1^i, \\ r_i - \lambda_i J_i(1,t) & \text{if } x_1^i \leq t \leq z_1^i, \\ r_{2i+2}[1 - P(N_{2i+2}(t) \geq 1)] & \text{if } z_1^i < t. \end{cases} \quad (31)$$

Because $J_i(1,t)$ is increasing in t, we conclude that $\frac{\partial}{\partial t} J_i(1,t)$ is decreasing for $t \in \mathbf{R}_+$.

Assuming that $\frac{\partial}{\partial t} J_i(n-1,t)$ is decreasing for $t \in [x_n^i, z_{n-1}^i]$ and $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{x_n^i} \geq 0$ we deduce that $\rho(t)$ is also decreasing. Let us notice that

$$\frac{\partial J_i(n,t)}{\partial t} = c + \int_{-\infty}^{t} \rho(s) \mathrm{e}^{-\lambda(t-s)} \mathrm{d}s$$

and according to Lemma 4 $\frac{\partial}{\partial t} J_i(n,t)$ is decreasing for $t \in [x_n^i, z_{n-1}^i]$. In the same manner we can see that if $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{z_{n-1}^{i+}} \geq 0$, then $\frac{\partial}{\partial t} J_i(n,t)$ is decreasing for $t \in (z_{n-1}^i, z_n^i]$. From what has already been proved and Eq. (28) we conclude that $\frac{\partial}{\partial t}[J_i(n,t) - J_i(n-1,t)] \geq 0$ and $\Delta_i(n,t)$ is increasing in t. The decreasingness in n of $\Delta_i(n,t)$ follows easily from Theorem 2 and Lemma 5. □

Remark 3 The condition $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{x_n^i} \geq 0$, $i \in \{1,2\}$, is equivalent to

$$J_i(n-1, x_n^i) \geq \frac{1}{2}\left[J_i(n, x_n^i) + J_i(n-2, x_n^i)\right] \quad (32)$$

and the condition $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{z_{n-1}^{i+}} \geq 0$, $i \in \{1,2\}$, is equivalent to

$$\lambda_i\left[J_i(n, z_{n-1}^i) - J_{2i+2}(n-1, z_{n-1}^i)\right] \leq r_i - r_{2i+2} + r_{2i+2} P(N_{2i+2}(z_{n-1}^i) \geq n-1). \quad (33)$$

Equations (32) and (33) simply result from Lemma 5.

Lemma 7 *The function $G_1(n,t)$ $(G_2(n,t))$ is continuous except for $t = z_n^1$ $(t = z_n^2)$ and*

$$\lim_{t \to z_n^{1+}} G_1(n,t) - G_1(n, z_n^1) < 0 \left(\lim_{t \to z_n^{2+}} G_2(n,t) - G_2(n, z_n^2) < 0 \right).$$

Additionally, if the following conditions are fulfilled

(i) $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{x_n^i} \geq 0$ *and* $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{z_{n-1}^{i+}} \geq 0$ *for* $i = 1$ $(i = 2)$

(ii) $\Delta_2(n-1,t) \geq \max\{W_1(n,t), W_2(n,t)\}$ *for* $t \in [x_{n-1}^i, x_n^i \wedge z_{n-1}^i]$, *where*

$$W_1(n,t) = \frac{1}{\lambda_0 \lambda_2}\{r_5(\lambda_5 - \lambda_0) P(N_5(t) = n-1) + \lambda_0[r_2 - r_5 + r_5 P(N_5(t) \geq n-1)]\}, \quad (34)$$

$$W_2(n,t) = \frac{1}{\lambda_0 - \lambda_2}\{\lambda_0 \Delta_2(n,t) + r_5 - r_2 - r_5 P(N_5(t) \geq n)\}, \quad (35)$$

(iii) $r_6 P(N_6(t) \geq n-1) - r_5 P(N_5(t) \geq n-1) \geq \max\{Z_1(n,t), Z_2(n,t)\}$ *for* $t \in (x_n^i \wedge z_{n-1}^i, x_n^i)$*, where*

$$Z_1(n,t) = \frac{1}{\lambda_0}\{r_5(\lambda_5 - \lambda_0)P(N_5(t) = n-1) + \lambda_0(r_6 - r_5)\}, \quad (36)$$

$$Z_2(n,t) = r_6 - r_5 - r_5 P(N_5(t) = n-1) + \lambda_0[p_6 P(N_6(t) \geq n-1) - \Delta_2(n,t)], \quad (37)$$

then $G_1(n,t)(G_2(n,t))$ *is piecewise decreasing* (*increasing*) *in* n *and piecewise increasing* (*decreasing*) *in* t.

Proof Directly from definitions $G_k(n,t)$, $k \in \{1,2\}$, given by (16) and (17), these functions are not continuous at $t = z_n^k$, because from Lemma 3 $\frac{\partial}{\partial t} J_k(n,t)$ is not continuous at $t = z_n^k$ and

$$\lim_{t \to z_n^i{}^+} G_i(n,t) - G_i(n, z_n^i) = \lim_{t \to z_n^i{}^+} \frac{\partial J_i(n,t)}{\partial t} - \left[\frac{\partial J_i(n,t)}{\partial t}\right]_{z_n^i} = G_{2i+2}(n, z_n^i) < 0.$$

The properties included in Theorem 1, Remark 21, Remark 22, Eqs. (16), (17) applied to $i = 0$ make obvious that

$$G_k(n,t) = r_0 - r_{2k+1} + p_{2k+1}(\lambda_{2k+1} - \lambda_0)P(N_{2k+1}(t) \geq n) \quad \text{for } t \in [0, x_{n-1}^k) \quad (38)$$

$$G_k(n,t) = r_0 - r_{2k+2} + p_{2k+2}(\lambda_{2k+2} - \lambda_0)P(N_{2k+2}(t) \geq n) \quad \text{for } t \in (z_n^k, \infty). \quad (39)$$

Our assumptions that $\lambda_3 > \lambda_0$, $\lambda_4 > \lambda_0$ and $\lambda_5 < \lambda_0$, $\lambda_6 < \lambda_0$ make it obvious that $G_1(n,t)$ ($G_2(n,t)$) is increasing (decreasing) in t and decreasing (increasing) in n for $t \in [0, x_{n-1}^k) \cup (z_n^k, \infty)$. According to Theorem 1 we have for $t \in [x_n^i, z_n^i]$

$$G_k(n,t) = r_0 - r_k + (\lambda_k - \lambda_0)[J_k(n,t) - J_k(n-1,t)]. \quad (40)$$

By assumptions that $\lambda_1 > \lambda_0$, $\lambda_2 < \lambda_0$ and from Lemma 6, $G_1(n,t)$ and $G_2(n,t)$ fulfill the thesis of Lemma 7 on the suitable interval. The form of $G_k(n,t)$ for $t \in [x_{n-1}^i, x_n^i)$ is following

$$G_k(n,t) = r_0 - r_{2k+1} + p_{2k+1}(\lambda_{2k+1} - \lambda_0)P(N_{2k+1}(t) \geq n) + \lambda_0[J_{2k+2}(n-1,t) - J_{2k+1}(n-1,t)].$$

Feng and Gallego (1995) showed that $J_{2k+2}(n-1,t) - J_{2k+1}(n-1,t)$ is increasing function of t and decreasing function of n, so $G_1(n,t)$ fulfills the thesis of the lemma. Unfortunately $G_2(n,t)$ perhaps is increasing in t and decreasing in n. Our next goal is to determine the conditions, which guarantee suitable properties of

$G_2(n,t)$. Let us calculate that

$$\begin{aligned}\frac{\partial G_2(n,t)}{\partial t} &= r_5(\lambda_5-\lambda_0)P(N_5(t)=n-1)\\ &\quad +\lambda_0[G_5(n-1,t)+\lambda_2 p_5 P(N_5(t)\geq n-1)-\lambda_2\Delta_2(n-1,t)]\\ &= r_5(\lambda_5-\lambda_0)P(N_5(t)=n-1)+\lambda_0[r_2-r_5+r_5P(N_5(t)\geq n-1)\\ &\quad -\lambda_2\Delta_2(n-1,t)]\end{aligned}$$

for $t\in[x_{n-1}^i, x_n^i\wedge z_{n-1}^i]$ and the derivative with respect to t is negative if and only if

$$\begin{aligned}\lambda_0\lambda_2\Delta_2(n-1,t) &\geq r_5(\lambda_5-\lambda_0)P(N_5(t)=n-1)\\ &\quad +\lambda_0[r_2-r_5+r_5P(N_5(t)\geq n-1)]\end{aligned}\tag{41}$$

for $t\in[x_{n-1}^i, x_n^i\wedge z_{n-1}^i]$. Similarly for $t\in(x_n^i\wedge z_{n-1}^i, x_n^i)$

$$\begin{aligned}\frac{\partial G_2(n,t)}{\partial t} &= \lambda_0[r_6-r_5+r_5P(N_5(t)\geq n-1)-r_6P(N_6(t)\geq n-1)]\\ &\quad +r_5(\lambda_5-\lambda_0)P(N_5(t)=n-1).\end{aligned}$$

So we conclude that by our assumptions $G_2(n,t)$ is decreasing in t. Let us observe that

$$G_2(n-1,t)=r_0-r_2+(\lambda_2-\lambda_0)[J_2(n-1,t)-J_2(n-2,t)]$$

for $t\in[x_{n-1}^i, x_n^i)$. We conclude that for $t\in[x_{n-1}^i, x_n^i\wedge z_{n-1}^i]$

$$\begin{aligned}G_2(n,t)-G_2(n-1,t) &= r_2-r_5+p_5(\lambda_5-\lambda_0)P(N_5(t)\geq n)\\ &\quad +\lambda_0[J_2(n-1,t)-J_5(n-1,t)]-(\lambda_2-\lambda_0)\Delta_2(n-1,t),\end{aligned}$$

however, for $t\in(x_n^i\wedge z_{n-1}^i, x_n^i)$

$$\begin{aligned}G_2(n,t)-G_2(n-1,t) &= r_6-r_5+p_5[\lambda_5P(N_5(t)\geq n)-\lambda_0P(N_5(t)\geq n)]\\ &\quad +\lambda_0[J_6(n-1,t)-J_5(n-1,t)]\\ &\quad +p_6[\lambda_0P(N_6(t)\geq n-1)-\lambda_6P(N_6(t)\geq n-1)]\end{aligned}$$

therefore the assumptions of Lemma 7 ensure that $G_2(n,t)$ is increasing in n. This completes the proof. □

4 Double reversible price change problem

Similarly as in single stopping problem solved by Feng and Gallego (1995), we can formulate theorem, which characterize the optimal policy. Unfortunately, owing to fact that functions $J_i(n,t)$ not always are concave we have to add two assumptions, which guarantee concavity for $t\in[z_{n-1}^i, z_n^i]$. The values of time thresholds x_n^0 and z_n^0 are calculated in following theorem.

Theorem 3 *If* $p_1 < p_0 < p_2$ *and*

(i) $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{x_n^i} \geq 0$ *and* $\frac{\partial}{\partial t}[\Delta_i(n,t)]_{z_{n-1}^{i+}} \geq 0$ *for* $i \in \{1,2\}$,

(ii) $\Delta_2(n-1,t) \geq \max\{W_1(n,t), W_2(n,t)\}$ *for* $t \in [x_{n-1}^i, x_n^i \wedge z_{n-1}^i]$,

(iii) $r_6 P(N_6(t) \geq n-1) - r_5 P(N_5(t) \geq n-1) \geq \max\{Z_1(n,t), Z_2(n,t)\}$ *for* $t \in (x_n^i \wedge z_{n-1}^i, x_n^i)$,

then the optimal revenue function $J_0(n,t)$ *in double reversible price change problem is determined as follows.*

For fixed n, let

$$\begin{aligned} t_n^0 &= \inf\{t > 0 : J_1(n,t) - J_2(n,t) = 0\}, \\ L_1(n,t) &= G_1(n,t) + \lambda_0[J_0(n-1,t) - J_1(n-1,t)], \\ L_2(n,t) &= G_2(n,t) + \lambda_0[J_0(n-1,t) - J_2(n-1,t)], \end{aligned}$$

then

$$J_0(n,t) = \begin{cases} \max\{J_1(n,t), J_2(n,t)\} + F_0(n,t) & \text{if } x_n^0 \leq t \leq z_n^0, \\ \max\{J_1(n,t), J_2(n,t)\} & \text{otherwise,} \end{cases}$$

where

$$F_0(n,t) = \begin{cases} H_1(n,t) & \text{if } x_n^0 \leq t \leq t_n^0, \\ H_2(n,t) & \text{if } t_n^0 < t \leq z_n^0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_2(n,t) = H_1(n,t_n^0)\mathrm{e}^{-\lambda_0(t-t_n^0)} + \int_{t_n^0}^{t} L_2(n,u)\mathrm{e}^{-\lambda_0(t-u)}\mathrm{d}u.$$

(i) *If* $L_1(n,z_n^1)L_1(n,z_n^1+) > 0$, *then*

$$H_1(n,t) = \int_{x_n^0}^{t} L_1(n,u)\mathrm{e}^{-\lambda_0(t-u)}\mathrm{d}u \tag{42}$$

and the thresholds are given by

$$x_n^0 = \inf\{t \geq 0 : L_1(n,t) = 0\}, \tag{43}$$

$$z_n^0 = \inf\{t \geq t_n^0 : H_2(n,t) \leq 0\}. \tag{44}$$

(ii) *If* $L_1(n,z_n^1)L_1(n,z_n^1+) \leq 0$, *then let*

$$x_n^{01} = \inf\left\{0 \leq t \leq z_n^1 : L_1(n,t) = 0\right\}, \quad x_n^{02} = \inf\left\{z_n^1 < t : L_1(n,t) = 0\right\},$$

$$A = \inf\left\{z_n^1 < t \leq x_n^{02} : \int_{x_n^{01}}^{t} L_1(n,u)\mathrm{e}^{-\lambda_0(t-u)}\mathrm{d}u < 0\right\}.$$

If $A = \emptyset$ *then* $H_1(n,t)$ *is given by* (42), x_n^0 *by* (43) *and* z_n^0 *by* (44). *If the set* A *is not empty then let* $y_n^0 = \inf A$. *The function* $H_1(n,t)$ *can be noticed as*

$$H_1(n,t) = \begin{cases} \int\limits_{x_n^{01}}^{t} L_1(n,u)\mathrm{e}^{-\lambda_0(t-u)}\mathrm{d}u & \text{if } t \le y_n^0, \\ 0 & \text{if } y_n^0 < t \le x_n^{02}, \\ \int\limits_{x_n^{02}}^{t} L_1(n,u)\mathrm{e}^{-\lambda_0(t-u)}\mathrm{d}u & \text{if } x_n^{02} < t \end{cases}$$

and the threshold x_n^0 *is given by*

$$x_n^0 = x_n^{01}\mathbb{I}_{\{t \le z_n^1\}} + x_n^{02}\mathbb{I}_{\{t > z_n^1,\ x_n^{02} \le t_n^0\}}$$

and z_n^0 *is given by* (44).

The proof is analogous to proof of Theorem 3 in Feng and Gallego (1995). The proof consist in showing that function $F_0(n,t)$ fulfills the assumptions of Theorem 2. Monotone structure of solution result from monotone properties of $G_1(n,t)$ and $G_2(n,t)$. It is optimal to change the price for the first time to p_1 if the time-to-go is less than the corresponding threshold x_n^0 and there is n unsold items yet and it is optimal to change the price from starting price p_0 to p_2 if the time-to-go is more than z_n^0 and there are n unsold items. Let us pay attention that in some cases we obtain two different solutions for $t \in [0, z_n^i]$ and $t \in (z_n^i, \infty]$. It is a consequence of discontinuity of the function $G_1(n,t)$ at $t = z_n^1$. After first change we proceed according to policy described by Feng and Gallego (1995) in Theorem 1.

4.1 Numerical examples

The developer has three houses to sell. The current price of one house stand at $p_0 = 20$. Market analyze results that demand intensity at this price is equal $\lambda_0 = 2$. The developer can change the price for the first time to $p_1 = 16$ or $p_2 = 22$. The suitable intensities are equal $\lambda_1 = 3$ and $\lambda_2 = 1.5$. Then the seller is considering price changes to lower or higher one more time. If he sells the houses by p_1 he can choose one from prices $p_3 = 14$ or $p_4 = 18$ with suitable intensities $\lambda_3 = 3.5$ and $\lambda_4 = 2.5$. If he changed the price to p_2 for the first time then he can choose one price from $p_5 = 21$ or $p_6 = 23$ with suitable demand intensities $\lambda_5 = 1.75$ and $\lambda_6 = 1$. The prices and intensities satisfies our model assumptions that $r_i < r_j$ when $p_i > p_j$. The assumptions of Theorem 3 are fulfilled for our data. Now using this theorem and Theorem 1 we will solve the double stopping problem and obtain maximal expected revenue and optimal policy. We compute six time threshold sequences: concerning first change $\{x_n^0, z_n^0\}$ and concerning second change $\{x_n^1, z_n^1, x_n^2, z_n^2\}$. The values are given in Tables 1 and 2.

Let us notice that we obtained two solutions for $n = 1$ because $L_1(n, z_n^1)L_1(n, z_n^1+) < 0$. Figure 1 shows the graph of $L_1(1,t)$ in time-to-go. It is easy to construct optimal policy using foregoing tables. For example if the developer has

Table 1 The values of optimal time thresholds for the first price change

n	x_n^0	z_n^0
1	$0.230\,\mathbb{I}_{\{t\leq 0.304\}} + 0.324\,\mathbb{I}_{\{t>0.304\}}$	$0.982\,\mathbb{I}_{\{t\leq 0.304\}} + 0.989\,\mathbb{I}_{\{t>0.304\}}$
2	0.557	1.850
3	0.876	2.629

Table 2 The values of optimal time thresholds for the second price change

n	x_n^1	z_n^1	x_n^2	z_n^2
1	0.044	0.304	0.716	2.901
2	0.153	0.732	1.268	4.497
3	0.297	1.148	1.831	5.924

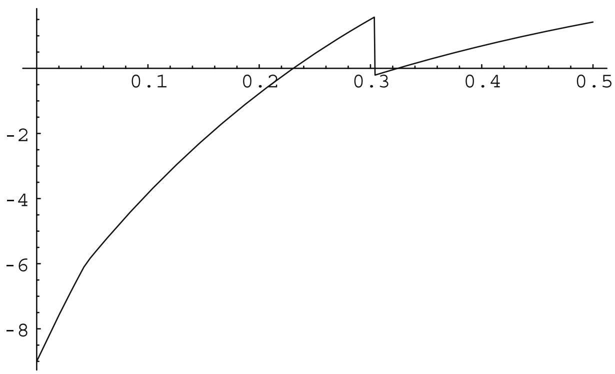

Fig. 1 $L_1(1, t)$ as a function of t

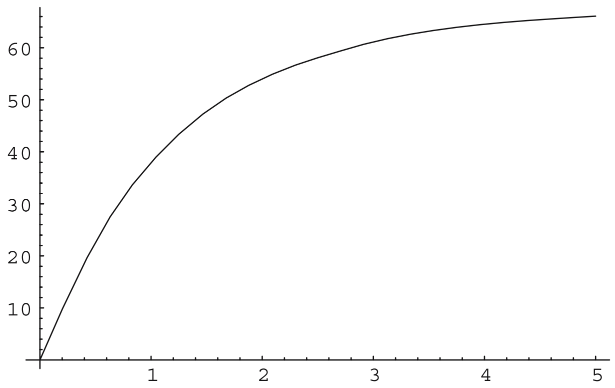

Fig. 2 $J_0(3, t)$ as a function of t

still three houses to sell, he should decrease the price for the first time if time to go is less than threshold $x_3^0 = 0.876$ or increase if the time to go is more than $z_3^0 = 2.629$. For the second time he should proceed similarly taking into consideration thresholds suitable to remaining items. Figure 2 shows the graph of optimal expected revenue function in time-to-go.

5 Final remarks

The double stopping problem has been used as a tool in mathematical economics very rarely. Recently, the authors has investigated the application of such approach to the risk process (see Karpowicz and Szajowski 2006). An insurance company, endowed with an initial capital $a > 0$, receives insurance premiums and pays out successive claims. The losses occur according to renewal process. At any moment the company may broaden or narrow down the offer, which entail the change of the parameters. This change concerns the rate of income, the intensity of renewal process and the distribution of claims. After the change the management wants to know the moment of the maximal value of the capital assets. The goal is to find two optimal stopping times: the best moment of change the parameters and the moment of maximal value of the capital assets. The double stopping problem is usually the simplest form of the impulse control (see Chap. 5 of Øksendal and Sulem 2005). The review of such models in financial mathematics, a cash management problem, optimal control of an exchange rate and portfolio optimization under transaction costs has been given by Korn (1999). These models are based on the fundamental process $X(t)$, which is given as the solution of some stochastic differential equation. The controller is allowed to choose intervention times θ_i, where he can shift the process $X(t)$ to another value. The yield management problems for perishable goods, which we consider in this paper, are another example of application of the multiple optimal stopping problem or the impulse control models.

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