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Egoroff's theorem on monotone non-additive measure spaces*

Jun Li†
Department of Applied Mathematics, Southeast University
Nanjing 210096, People's Republic of China
lijun@seu.edu.cn

Masami Yasuda
Department of Mathematics & Informatics, Faculty of Science
Chiba University, Chiba 263-8522, Japan
yasuda@math.s.chiba-u.ac.jp

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In this paper, the well-known Egoroff's theorem in classical measure theory is established on monotone non-additive measure spaces. Taylor's theorem, which concerns almost everywhere convergence of measurable function sequence in classical measure theory, is also generalized. The converse problem of the theorems are discussed, and a necessary and sufficient condition for the Egoroff's theorem is obtained on semicontinuous fuzzy measure space with S-compactness.

Keywords: Monotone measure; fuzzy measure; Egoroff's theorem

1. Introduction

Egoroff's theorem is one of the most important convergence theorems in classical measure theory. Wang [12] first generalized the well-known theorem to fuzzy measure spaces under the autocontinuity from above condition. The more researches on the theorem were made by Wang and Klir [13], Li *et al.*[5] and Liu [6]. Li [3] further prove that the Egoroff's theorem in classical measure theory remains valid on finite fuzzy measure spaces. In these discussions, the fuzzy measures are considered in the sense of Ralescu [9], that is, they are monotone set functions with the continuity both of below and of above.

In this paper, we shall investigate the convergence of measurable function sequence on monotone non-additive measure spaces. Here the non-additive measures considered are only nonnegative monotone set functions. Egoroff's theorem and Taylor's theorem [11] are generalized to monotone non-additive measure spaces by

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†Corresponding author.

using the strong order continuity and the property (S) of set functions. We also discuss the converse problem of the Egoroff's theorem by using strong order continuity of set function and obtain a necessary condition that Egoroff's theorem holds on monotone non-additive measure spaces. These are further improvements and generalizations of the related results in Li *et al.*[5]. Finally, we obtain an encouraging result: a necessary and sufficient condition that Egoroff's theorem remain true on lower semicontinuous fuzzy measure space with S-compactness is that the lower semicontinuous fuzzy measure be strongly order continuous.

2. Preliminaries

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X, $\mu : \mathcal{F} \to [0, +\infty]$ be a set function and let N denote the set of all positive integers. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

A set function $\mu : \mathcal{F} \to [0, +\infty]$ is called *monotone non-additive measure*, if it satisfies the following properties:

(1) $\mu(\emptyset) = 0$;

(2) $A \subset B$ implies $\mu(A) \leq \mu(B)$ (monotonicity).

If, moreover, μ satisfies:

(3) $A_1 \subset A_2 \subset \cdots$ implies

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \qquad \text{(continuity from below);}$$

(4) $A_1 \supset A_2 \supset \cdots$, and there exists n_0 with $\mu(A_{n_0}) < +\infty$ imply

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \qquad \text{(continuity from above),}$$

then μ is called fuzzy measure (Ralescu [9]).

μ is called a lower semicontinuous fuzzy measure, if it satisfies the conditions (1) – (3).

When μ is a monotone non-additive measure (resp. a fuzzy measure or a lower semicontinuous fuzzy measure), the triple $(X, \mathcal{F}, \mu)$ is called monotone non-additive measure space (resp. fuzzy measure space or lower semicontinuous fuzzy measure space).

The following Definition 1 and 2 were introduced by Li[4] and Sun[10], respectively.

Definition 1. μ is called strongly order continuous, if $\lim_{n\to+\infty}\mu(A_n) = 0$ whenever $\{A_n\}_n \subset \mathcal{F}$, $A_n \searrow B$ and $\mu(B) = 0$.

Definition 2. μ is called to have *property* (S), if for any $\{A_n\}_n$ with $\lim_{n\to+\infty}\mu(A_n) = 0$, there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\mu(\limsup A_{n_i}) = 0$.

Let $\mathbf{F}$ be the class of all finite real-valued measurable functions on measurable space $(X, \mathcal{F})$, and let $f, f_n \in \mathbf{F}$ $(n \in N)$. We say that $\{f_n\}_n$ *converges almost* *everywhere to* f *on* X, and denote it by $f_n \xrightarrow{a.e.} f$, if there is subset $E \subset X$ such that $\mu(E) = 0$ and f_n converges to f on $X - E$; $\{f_n\}_n$ *converges almost uniformly to* f *on* X, and denote it by $f_n \xrightarrow{a.u.} f$, if for any $\epsilon > 0$ there is a subset $E_\epsilon \in \mathcal{F}$ such that $\mu(X - E_\epsilon) < \epsilon$ and f_n converges to f uniformly on E_ϵ.

3. Egoroff's theorems

Now we generalize Egoroff's theorem and Taylor's theorem in classical measure theory to monotone non-additive measure space.

Theorem 1. *(Egoroff's theorem) Let* μ *be a monotone non-additive measure on* $\mathcal{F}$. *If* μ *is strongly order continuous and has property* (S), *then*

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u.} f.$$

Proof. Assume that μ is strongly order continuous and has property (S). Let E be the set of these points x in X at which $\{f_n(x)\}$ dose not converge to $f(x)$. Then $\mu(E) = 0$ and $\{f_n\}_n$ converges to f everywhere on $X - E$. If we denote

$$E_n^{(m)} = \bigcap_{i=n}^{+\infty} \left\{ x \in X : \ |f_i(x) - f(x)| < \frac{1}{m} \right\}$$

for any $m \geq 1$, then $E_n^{(m)}$ is increasing in n for each fixed m, and we get

$$X - E = \bigcap_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} E_n^{(m)}.$$

Since for any fixed $m \geq 1$, $X - E \ \subset \ \bigcup_{n=1}^{+\infty} E_n^{(m)}$, we have

$$X - E_n^{(m)} \ \searrow \ \bigcap_{n=1}^{+\infty} (X - E_n^{(m)}).$$

Noting that $\bigcap_{n=1}^{+\infty} (X - E_n^{(m)}) \subset E$ for any fixed $m \geq 1$, therefore $\mu(\bigcap_{n=1}^{+\infty} (X - E_n^{(m)})) = 0$ $(m = 1, 2, \ldots)$. By using the strong order continuity of μ, we have

$$\lim_{n \to +\infty} \mu(X - E_n^{(m)})) = 0. \quad \forall m \geq 1$$

Thus, there exists a subsequence $\{X - E_{n(m)}^{(m)}\}_m$ of $\{X - E_n^{(m)} : n, m \geq 1\}$ satisfying

$$\mu(X - E_{n(m)}^{(m)}) \leq \frac{1}{m}, \quad \forall m \geq 1$$

and therefore

$$\lim_{n \to +\infty} \mu(X - E_{n(m)}^{(m)}) = 0.$$

By applying the property (S) of μ to the sequence $\{X - E^{(m)}_{n(m)}\}_m$, then there exists a subsequence $\{X - E^{(m_i)}_{n(m_i)}\}_i$ of $\{X - E^{(m)}_{n(m)}\}_m$ such that

$$\mu\left(\overline{\lim_{i\to+\infty}}(X - E^{(m_i)}_{n(m_i)})\right) = 0.$$

and $m_1 < m_2 < \ldots$.

On the other hand, since

$$\left(\bigcup_{i=k}^{+\infty}(X - E^{(m_i)}_{n(m_i)})\right) \searrow \overline{\lim_{i\to+\infty}}(X - E^{(m_i)}_{n(m_i)})$$

therefore, by using the strong order continuity of μ, we have

$$\lim_{k\to+\infty}\mu\left(\bigcup_{i=k}^{+\infty}(X - E^{(m_i)}_{n(m_i)})\right) = 0.$$

For any $\epsilon > 0$, we take k_0 such that

$$\mu\left(\bigcup_{i=k_0}^{+\infty}(X - E^{(m_i)}_{n(m_i)})\right) < \epsilon,$$

that is,

$$\mu\left(X - \bigcap_{i=k_0}^{+\infty} E^{(m_i)}_{n(m_i)}\right) < \epsilon.$$

Put $E_\epsilon = \bigcap_{i=k_0}^{+\infty} E^{(m_i)}_{n(m_i)}$, then $\mu(X - E_\epsilon) < \epsilon$. Now, we just need to prove that $\{f_n\}$ converges to f on E_ϵ uniformly. Since

$$E_\epsilon = \bigcap_{i=k_0}^{+\infty}\bigcap_{j=n(m_i)}^{+\infty}\left\{x \in X : \ |f_j(x) - f(x)| < \frac{1}{m_i}\right\},$$

therefore, for any fixed $i \geq k_0$, $E_\epsilon \subset \bigcap_{j=n(m_i)}^{+\infty}\left\{x \in X : \ |f_j(x) - f(x)| < \frac{1}{m_i}\right\}$. For any given $\sigma > 0$, we take i_0 ($\geq k_0$) such that $\frac{1}{m_{io}} < \sigma$. Thus, as $j > n(m_{i_o})$, for any $x \in X_\epsilon$,

$$|f_j(x) - f(x)| < \frac{1}{m_{i_o}} < \sigma.$$

This shows that $\{f_n\}$ converges to f on X_ϵ uniformly. The proof of the theorem is thereby completed. □

In the following, we give a generalization of Taylor's theorem (Taylor[11]) concerning convergence almost everywhere in classical measure theory.

Theorem 2. *(Taylor's theorem) Let μ be a monotone non-additive measure on $\mathcal{F}$. If μ is strongly order continuous and has property (S), and $f_n \xrightarrow{a.e.} f$, then there exists a sequence $\{\delta_n\}$ of positive numbers, such that $\delta_n \searrow 0$ as $n \to \infty$, and*

$$\frac{|f_n(x) - f(x)|}{\delta_n} \xrightarrow{a.e.} 0 \quad (n \to \infty).$$

Proof. Using conclusion of Theorem 1 (Egoroff's theorem), it is similar to the proof of Theorem 1 in [11].

The following corollary gives an alternative form of Egoroff's theorem on monotone non-additive measure space.

Theorem 3. *Under the conditions given in Theorem 2, there exists a sequence $\{\delta_n\}$ of positive numbers, such that $\delta_n \searrow 0$ as $n \to \infty$,*

$$\frac{|f_n(x) - f(x)|}{\delta_n} \xrightarrow{a.u.} 0 \quad (n \to \infty).$$

Proof. This follows immediately on applying Theorem 1 to the measurable function sequence

$$\varphi_n(x) = \frac{|f_n(x) - f(x)|}{\delta_n}$$

where $\{\delta_n\}$ satisfy the conditions of Theorem 2. □

Theorem 3 is an apparently stronger form of Theorem 1.

Remark 1. In Theorem 1, 2 and 3, the continuity from below and the continuity from above of set functions are not required.

Example 1. Let $X = [0, 1]$, $\mathcal{F}$ denote σ-algebra on X, and let m be σ-additive measure on $\mathcal{F}$ and $\mu(X) = 1$. Let $\mu : \mathcal{F} \to [0, 1]$ be defined by

$$\mu(E) = \begin{cases} m(E) & \text{if } m(E) \le \frac{1}{2} \\ \frac{2}{3} & \text{if } m(E) > \frac{1}{2} \text{ and } E \ne X \\ 1 & \text{if } E = X. \end{cases}$$

It is not too difficult to verify that set function μ is monotone and strongly order continuous and has property (S). But μ is neither continuous from below nor continuous from above.

A set function $\mu : \mathcal{F} \to [0, +\infty]$ is called *order-continuous* if $\lim_{n\to+\infty}\mu(E_n) = 0$ whenever $E_n \searrow \emptyset$; *exhaustive* if $\lim\limits_{n\to+\infty} \mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_n$ [2,7,8]. μ is called to have *pseudometric generating property*, if for each $\epsilon > 0$ there is $\delta > 0$ such that for any $E, F \in \mathcal{F}$, $\mu(E) \vee \mu(F) < \delta$ implies $\mu(E \cup F) < \epsilon$ (Dobrakov and Farkova [1]).

Obviously, the strong order-continuity of μ implies order-continuity, and for monotone set function, order-continuity implies exhaustivity [8].

Proposition 1. *Let μ be a fuzzy measure. If μ is order continuous and has pseudometric generating property, then it is strongly order continuous and has property (S).*

Proof. It can be easily obtained that μ is strongly order continuous, and it follows from Proposition 6 in [5] that μ has property (S).

Remark 2. Pseudometric generated measures goes back to Dobrakov and Drewnowski in seventies, and this was related to Frechet-Nikodym topology [1,2,8,14].

As special result of Theorem 1 and Proposition 1, we can obtained the following corollary immediately:

Corollary 1. *(Li et al.[3, Theorem 1]) Let μ be a fuzzy measure with order continuity and pseudometric generating property. Then, for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u.} f.$$

Remark 3. A monotone non-additive measure with strong order continuity and property (S) may not possess pseudometric generating property. Therefore Theorem 1 is an improvement and generalization of the related result in Li *et al.*[5].

Example 2. Let $X = \{a, b\}$ and $\mathcal{F} = \wp(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E \neq X \end{cases}$$

Then μ is a fuzzy measure and it is obvious that μ is strongly order continuous and has property (S). But μ has not pseudometric generating property. In fact, $\mu(\{a\}) = \mu(\{b\}) = 0$, but $\mu(\{a\} \cup \{b\}) = 1 \neq 0$.

In the following we discuss the converse problem of the Egoroff's theorem for monotone non-additive measure.

Theorem 4. *Let μ be a monotone non-additive measure. If for any $f, f_n \in \mathbf{F}$ $(n \in N)$, $f_n \xrightarrow{a.e.} f$ implies $f_n \xrightarrow{a.u} f$, then μ is strongly order continuous and hence order continuous.*

Proof. For any decreasing set sequence $\{E_n\}_n$ with $E_n \searrow E$ and $\mu(E) = 0$, we define a measurable function sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} 0 \text{ if } & x \notin E_n \\ 1 \text{ if } & x \in E_n \end{cases}$$

for any $n \geq 1$. It is easy to see that $f_n \xrightarrow{a.e.} 0$. If $f_n \xrightarrow{a.u} 0$, then from Theorem 6.11 in [13], we can get for any $\sigma > 0$, $\lim_{n\to\infty} \mu(\{x : |f_n(x)| \geq \sigma\}) = 0$. Therefore

$$\lim_{n\to+\infty} \mu(E_n) = \lim_{n\to+\infty} \mu(\{x : \; f_n(x) \geq \frac{1}{2}\}) = 0.$$

This shows μ is strongly order continuous and hence order continuous. □

As special result of Theorem 4, we have the following corollary:

Corollary 2. *(Li et al.[3, Theorem 8(3)]) Let μ be a fuzzy measure. If for any $f, f_n \in \mathbf{F}$ $(n \in N)$, $f_n \xrightarrow{a.e.} f$ implies $f_n \xrightarrow{a.u} f$, then μ is exhaustive.*

Remark 4. A order continuous (or exhaustive) fuzzy measure may not be strongly order continuous. Therefore Theorem 4 generalize the related result in Li *et al.*[5].

Example 3. Let $X = [0, +\infty)$, $\mathcal{F}$ be the class of all Lebesgue measurable sets on X, and m be Lebesgue's measure. Put

$$\mu(E) = \begin{cases} 0 & \text{if } 0 \notin E \\ m(E) & \text{if } 0 \in E \end{cases}$$

Then μ is an order-continuous (hence exhaustive) fuzzy measure and has property (S). However μ is not strongly order continuous. In fact, if we take $E_n = [0, \frac{1}{n}) \cup (n, +\infty)$, $n = 1, 2, \cdots$, then $E_n \searrow \{0\}$ and $\mu(\{0\}) = 0$. But $\mu(E_n) = \infty$, $n = 1, 2, \cdots$.

The following corollary is a direct result of Theorem 1 and 4:

Corollary 3. *Let μ be a monotone non-additive measure with property (S). Then, for any $f, f_n \in \mathbf{F}$ $(n \in N)$,*

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u} f$$

if and only if μ is strongly order continuous.

A measurable space $(X, \mathcal{F})$ is called S-compact [13], if for any sequence of sets in $\mathcal{F}$ there exists some convergent subsequence. Any countable measurable space is S-compact [13].

Proposition 2. *Let μ be a monotone non-additive measure on S-compact space $(X, \mathcal{F})$. If μ is continuous from below, then it has property (S).*

Proof. Suppose $\{A_n\}_n \subset \mathcal{F}$ and $\lim_{n\to+\infty}\mu(A_n) = 0$. Since $(X, \mathcal{F})$ S-compact, there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\limsup A_{k_i} = \liminf A_{k_i}$, that is, $\bigcap_{m=1}^{+\infty} \bigcup_{i=m}^{+\infty} A_{k_i} = \bigcup_{m=1}^{+\infty} \bigcap_{i=m}^{+\infty} A_{k_i}$. Therefore, from continuity from below,

$$\begin{aligned} \mu(\limsup A_{k_i}) &= \mu(\bigcap_{m=1}^{+\infty} \bigcup_{i=m}^{+\infty} A_{k_i}) = \mu(\bigcup_{m=1}^{+\infty} \bigcap_{i=m}^{+\infty} A_{k_i}) \\ &= \lim_{m\to+\infty} \mu(\bigcap_{i=m}^{+\infty} A_{k_i}) \le \lim_{m\to+\infty} \mu(A_{k_m}) \\ &= 0, \end{aligned}$$

and thus we have $\mu(\limsup A_{k_i}) = 0$. This shows that μ has property (S). □

Combining Theorem 1, Theorem 4 and Proposition 2, we can obtain the following result:

Theorem 5. *Let $(X, \mathcal{F})$ be S-compact space (especially, X is countable) and μ be a lower semicontinuous fuzzy measure. Then, for any $f, f_n \in \mathbf{F}$ $(n \in N)$,*

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u.} f$$

if and only if μ is strongly order continuous.

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References

1. I. Dobrakov, J. Farkova, "On submeasures II", *Math. Slovaca* 30(1980) 65−81.
2. L. Drewnowski, "On the continuity of certain non-additive set function", *Colloquium Math.* 38(1978) 243−253.
3. J. Li, "On Egoroff's theorems on fuzzy measure space", *Fuzzy Sets and Systems* 135(2003) 367−375.
4. J. Li, "Order continuity of monotone set function and convergence of measurable functions sequence", *Applied Mathematics and Computation* 135(2003) 211−218.
5. J. Li, M. Yasuda, Q. Jiang, H. Suzuki, Z. Wang, G.J. Klir, "Convergence of sequence of measurable functions on fuzzy measure space", *Fuzzy Sets and Systems* 87(1997) 317−323.
6. Y. Liu, B. Liu, "The relationship between structural characteristics of fuzzy measure and convergences of sequences of measurable functions", *Fuzzy Sets and Systems* **120**(2001) 511−516.
7. T. Murofushi, M. Sugeno, "A theory of fuzzy measures: representations, the Choquet integral and null sets", *J. Math. Anal. Appl.* 159(1991) 532−549.
8. E. Pap, *Null-additive Set Functions*, (Kluwer, Dordrecht, 1995).
9. D. Ralescu, G. Adams, "The fuzzy integral", *J. Math. Anal. Appl.* 75(1980) 562−570.
10. Q. Sun, Property (s) of fuzzy measure and Riesz's theorem, *Fuzzy Sets and Systems* 62(1994) 117−119.
11. S. J. Taylor, "An alternative form of Egoroff's theorem", *Fundamenta Mathematicae* **48** (1960) 169−174.
12. Z. Wang, "The autocontinuity of set function and the fuzzy integral", *J. Math. Anal. Appl.* 99(1984) 195−218.
13. Z. Wang, G. J. Klir, *Fuzzy Measure Theory*, (Plenum, New York, 1992).
14. H. Weber, FN-Topologies and Group-Valued measures, Handbook of Measure Theory (Ed. E. Pap), Elsevier, Amsterdam, 2002, 703-743.