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A Fuzzy Stopping Problem with the Concept of Perception

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Abstract. Stimulated by Zadeh's paper (*Journal of Statistical Planning and Inference*, 2002, 105, 233–264), we will try to consider a perceptive analysis of the optimal stopping problem. In this paper, the fuzzy perception value of the expectation of the optimal stopped reward is characterized and calculated by a new recursive equation. Also, a numerical example described by triangular fuzzy numbers is given.

Keywords: fuzzy perceptive stopping problem, fuzzy random variable, fuzzy perception reward, optimal stopping time

1. Introduction and Notation

The stopping problem in a stochastic model is to maximize $E(X_\delta)$ over all the stopping times δ for a given sequence of random variables $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, which was solved elegantly by many authors, for example (Chow, Robbins and Siegmund (1971)). However, in practice, we are often faced with the case that the value of random variables is partially observed by dimness of perception or measurement imprecision. For example, in the classical stopping problem of selling or buying an asset (Karlin (1962)), the price of the asset may not be observed exactly. Usually it is linguistically and roughly perceived through negotiations e.g. about \$10,000, the price considerably larger than \$10,000, etc. When it will take a long time to make an actual decision for the problem, we are still wrapped in a fog of dimness. But immediately before our decision making, the fog mist is cleared up and we can know the true value of the price so that the optimal procedure could be taken. Then, under dimness of perception or measurement imprecision, how can we estimate in advance the future reward obtained from the optimal procedure. A possible way of handling such a case is to use the fuzzy set (Zadeh (1965)), whose membership function can describe the perception value of price. Motivated by the example of the above, in this paper we try the perceptive analysis (Baswell and Taylor (1987), Zadeh

(2002)) of the stopping problem in which fuzzy perception is accommodated. In a concrete form, if, for each sequence of random variables $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, the perception level of $\boldsymbol{X}$ is given, the fuzzy perception value of the expectation $E(X_{\delta^*})$ of the optimal stopped reward is characterized and calculated by a new recursive equation, where δ^* is an optimal stopping time w. r. t. $\boldsymbol{X}$. The above problem of estimating the perception value will be called the perceptive stopping problem.

In remainder of this section, we will give some notation and the definition of a fuzzy perception function referring (Baswell and Taylor (1987)), by which the perceptive stopping problem is formulated in the sequel. For non-perception approaches to fuzzy stopping problems, refer to our previous works (Kurano et al (2002), Yoshida et al (2000)). Recently Zadeh wrote a summary paper of perception-based theory (Zadeh (2002)).

For any set A, the fuzzy set on A will be denoted by its membership function $\tilde{a}: A \to [0,1]$. The α-cut of $\tilde{a}$ is given by $\tilde{a}_\alpha := \{x \in A | \tilde{a}(x) \geq \alpha\} (\alpha \in (0,1])$ and $\tilde{a}_0 := \mathrm{cl}\{x \in A | \tilde{a}(x) > 0\}$, where $\mathrm{cl}\{B\}$ is the closure of a set B. For the theory of fuzzy sets, we refer to (Zadeh (1965)) and (Dubois and Prade (1980)).

Let $\mathbb{R}$ be the set of all real numbers and $\tilde{\mathbb{R}}$ the set of all fuzzy numbers, i.e., $\tilde{r} \in \tilde{\mathbb{R}}$ means that $\tilde{r}: \mathbb{R} \to [0,1]$ is normal, upper-semicontinuous and fuzzy convex and has a compact support. Let $\mathbb{C}$ be the set of all bounded and closed intervals of $\mathbb{R}$. Then,obviously for any $\tilde{r} \in \tilde{\mathbb{R}}$, it holds that $\tilde{r}_\alpha \in \mathbb{C} (\alpha \in [0,1])$. So, we write $\tilde{r}_\alpha = [\tilde{r}_\alpha^-, \tilde{r}_\alpha^+] (\alpha \in [0,1])$.

A partial order relation $\preccurlyeq$ on $\tilde{\mathbb{R}}$, called the fuzzy max order (Ramík and Řimánek (1985)), is defined as follows: For $\tilde{s}, \tilde{r} \in \tilde{\mathbb{R}}, \tilde{s} \preccurlyeq \tilde{r}$, if $\tilde{s}_\alpha^- \leq \tilde{r}_\alpha^-$ and $\tilde{s}_\alpha^+ \leq \tilde{r}_\alpha^+$ for all $\alpha \in [0,1]$ where $\tilde{s}_\alpha = [\tilde{s}_\alpha^-, \tilde{s}_\alpha^+]$ and $\tilde{r}_\alpha = [\tilde{r}_\alpha^-, \tilde{r}_\alpha^+]$. Here, we define $\widetilde{\max}\{\tilde{s}, \tilde{r}\} \in \tilde{\mathbb{R}}$ by

$$\widetilde{\max}\ \{\tilde{s}, \tilde{r}\}(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = x_1 \vee x_2}} \{\tilde{s}(x_1) \wedge \tilde{r}(x_2)\} \quad (y \in \mathbb{R}), \tag{1.1}$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$. Then, it is well-known (Ramík and Řimánek, (1985)) that $\tilde{s} \preccurlyeq \tilde{r}$ if and only if $\tilde{r} = \widetilde{\max}\ \{\tilde{s}, \tilde{r}\}$.

Let $(\Omega, \mathscr{M}, P)$ be a probability space. A map $\tilde{X}: \Omega \to \tilde{\mathbb{R}}$ is called a fuzzy perception function if for each $\alpha \in [0,1]$ the maps $\Omega \ni \omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\Omega \ni \omega \mapsto \tilde{X}_\alpha^+(\omega)$ are $\mathscr{M}$-measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbb{R} | \tilde{X}(\omega)(x) \geq \alpha\}$. Let $\mathscr{X}$ be the set of all integrable random variables on $(\Omega, \mathscr{M}, P)$. For any fuzzy perception function $\tilde{X}$, the expectation $E\tilde{X} \in \tilde{\mathbb{R}}$ is defined by

$$E\tilde{X}(x) = \sup_{\substack{X \in \mathscr{X} \\ EX = x}} \tilde{\mu}(\tilde{X})(X), \tag{1.2}$$

where $\tilde{\mu}(\tilde{X})$ is a fuzzy set of $\mathscr{X}$ and defined by

$$\tilde{\mu}(\tilde{X})(X) = \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \quad \text{for all } X \in \mathscr{X}. \tag{1.3}$$

Obviously, we have

$$E(\tilde{X})_\alpha = \left[\int \tilde{X}^-_\alpha(\omega)dP(\omega), \int \tilde{X}^+_\alpha(\omega)dP(\omega)\right], \quad (\alpha \in [0,1]). \tag{1.4}$$

Note that a fuzzy set $\tilde{\mu}(\tilde{X})$ on $\mathscr{X}$ is called fuzzy random variable induced $\tilde{X}$ (Baswell and Taylor (1987)). Regarding the another (equivalent) definition of fuzzy random variables, we refer to (Kwakernaak (1978)) and (Puri and Ralescu (1986)). In this paper, the definition of fuzzy random variables from a perceptive stand point by (Baswell and Taylor (1987)) is adopted for modeling a fuzzy perceptive stopping problem.

2. Stopped Fuzzy Perception Rewards

Let $\mathscr{X}^n$ be the set of all n-dimensional row vectors whose elements are in $\mathscr{X}$, i.e.,

$$\mathscr{X}^n = \{\boldsymbol{X} = (X_1, X_2, \ldots, X_n) | X_t \in \mathscr{X}, \quad t = 1, 2, \ldots, n\}.$$

A random variable $\sigma: \Omega \to \mathbb{N}_n := \{1, 2, \ldots, n\}$ is said to be a stopping time corresponding to $\boldsymbol{X} = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n$ if $\{\sigma = k\} \in \mathscr{B}(\boldsymbol{X}_k)(k = 1, 2, \ldots, n)$ where $\boldsymbol{X}_k = (X_1, X_2, \ldots, X_k)$ and $\mathscr{B}(\boldsymbol{X}_k)$ is the σ-field on Ω generated by the random vector $\boldsymbol{X}_k$. The set of such stopping times will be denoted by $\Sigma\{\boldsymbol{X}\}$.

The map δ on $\mathscr{X}^n$ with $\delta(\boldsymbol{X}) \in \Sigma\{\boldsymbol{X}\}$ for all $\boldsymbol{X} \in \mathscr{X}^n$ is called a stopping time function. A stopping time function δ is monotone if for any $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n) \in \mathscr{X}^n$ with $\boldsymbol{X} \leq \boldsymbol{Y}$, i.e., $X_t \leq Y_t (t = 1, 2, \ldots, n)$ *P-a.s.,* it holds that $E\boldsymbol{X}_\delta \leq E\boldsymbol{Y}_\delta$, where $\boldsymbol{X}_\delta := X_{\delta(\boldsymbol{X})}$ and $\boldsymbol{Y}_\delta := Y_{\delta(\boldsymbol{Y})}$.

For any $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n) \in \mathscr{X}^n$ and $\beta \in [0,1]$, let

$$\begin{aligned}\boldsymbol{Z} :=& \beta\boldsymbol{X} + (1-\beta)\boldsymbol{Y} \\ =& (\beta X_1 + (1-\beta)Y_1, \ldots, \beta X_n + (1-\beta)Y_n) \in \mathscr{X}^n.\end{aligned}$$

Then δ is called convex if $E\boldsymbol{Z}_\delta \leq \beta E\boldsymbol{X}_\delta + (1-\beta)E\boldsymbol{Y}_\delta$ for all $\beta \in [0,1]$, where $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$ and $\boldsymbol{Z}_\delta := Z_{\delta(\boldsymbol{Z})}$. The set of all monotone and convex stopping time functions will be denoted by Δ.

Let $\widetilde{\boldsymbol{X}} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ be a sequence of fuzzy perception functions. For any $\delta \in \Delta$, the δ-stopped fuzzy perception reward $\widetilde{\boldsymbol{X}}_\delta$ is defined by

$$\tilde{\boldsymbol{X}}_\delta(\omega)(x) := \sup_{\substack{\boldsymbol{X}_\delta(\omega)=x \\ \boldsymbol{X}=(X_1,\ldots,X_n)\in\mathscr{X}^n}} \left\{\tilde{X}_1(\omega)(X_1(\omega)) \wedge \cdots \wedge \tilde{X}_n(\omega)(X_n(\omega))\right\}. \tag{2.1}$$

Note that $\tilde{\boldsymbol{X}}_\delta(\omega)(x)$ may be a fuzzy set on $\mathbb{R}$ but not necessarily a fuzzy perception function.

Similarly as (1.2), we define the expected value of $\tilde{\boldsymbol{X}}_\delta(\omega)(x)$ by

$$E\widetilde{\boldsymbol{X}}_\delta(x) := \sup_{\substack{E(X)=x \\ X\in\mathscr{X}}} \inf_{\omega\in\Omega}\left\{\tilde{\boldsymbol{X}}_\delta(\omega)(X(\omega))\right\}. \tag{2.2}$$

For each $\alpha \in [0,1]$, we use notations that $\tilde{\boldsymbol{X}}_\alpha^- := (\tilde{X}_{1,\alpha}^-, \ldots, \tilde{X}_{n,\alpha}^-) \in \mathscr{X}^n$ and $\tilde{\boldsymbol{X}}_\alpha^+ := (\tilde{X}_{1,\alpha}^+, \ldots, \tilde{X}_{n,\alpha}^+) \in \mathscr{X}^n$ in component wise, where the α-cut of $\tilde{X}_k$ is described by $\tilde{X}_{k,\alpha} = [\tilde{X}_{k,\alpha}^-, \tilde{X}_{k,\alpha}^+]$ respectively.

Theorem 2.1 *For any $\delta \in \Delta$, it holds that*

(i) $E\widetilde{\boldsymbol{X}}_\delta \in \tilde{\mathbb{R}}$ and
(ii) $(E\widetilde{\boldsymbol{X}}_\delta)_\alpha = [E((\widetilde{\boldsymbol{X}}_\alpha^-)_\delta), E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta)]$ *for* $\alpha \in [0,1]$.

For the proof of Theorem 2.1, we need several preliminary lemmas. Here, we put, for each $\alpha \in [0,1]$,

$$Z^\alpha(\beta) := \beta\tilde{\boldsymbol{X}}_\alpha^+ + (1-\beta)\tilde{\boldsymbol{X}}_\alpha^- \quad (\beta \in [0,1]). \tag{2.3}$$

Lemma 2.1 For any $\delta \in \Delta, E(Z^\alpha(\beta)_\delta)$ is continuous with respect to $\beta \in [0,1]$.

Proof: For any β, β' with $0 \le \beta < \beta' < 1$,

$$Z^\alpha(\beta') = \frac{\beta'-\beta}{1-\beta}\widetilde{\boldsymbol{X}}_\alpha^+ + \left(1 - \frac{\beta'-\beta}{1-\beta}\right)Z^\alpha(\beta).$$

So, from the monotonicity and convexity of $\delta \in \Delta$, we have for $0 \le \beta < \beta' < 1$,

$$\begin{aligned} E(Z^\alpha(\beta')_\delta) &\le E(Z^\alpha(\beta')_\delta) \\ &\le \frac{\beta'-\beta}{1-\beta}E\left((\tilde{\boldsymbol{X}}_\alpha^+)_\delta\right) + \left(1-\frac{\beta'-\beta}{1-\beta}\right)E(Z^\alpha(\beta)_\delta), \end{aligned}$$

which implies that $\lim_{\beta'\downarrow\beta} E(Z^\alpha(\beta')_\delta) = E(Z^\alpha(\beta)_\delta)$.

Similarly, we have for $0 \le \beta'' < \beta < 1$,

$$E(Z^\alpha(\beta)_\delta) \le \frac{\beta-\beta''}{1-\beta''}E\left((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta\right) + \left(1-\frac{\beta-\beta''}{1-\beta''}\right)E(Z^\alpha(\beta'')_\delta).$$

Thus, it holds that

$$0 \leq E(Z^{\alpha}(\beta)_{\delta}) - E(Z^{\alpha}(\beta'')_{\delta}) \leq \frac{\beta - \beta''}{1 - \beta''}\big(E\big((\tilde{X}^{+}_{\alpha})_{\delta}\big) - E\big(Z^{\alpha}(\beta'')_{\delta}\big)\big)$$
$$\leq \frac{\beta - \beta''}{1 - \beta''}\big(E\big((\tilde{X}^{+}_{\alpha})_{\delta}\big) - E\big((\tilde{X}^{-}_{\alpha})_{\delta}\big)\big).$$

Thus we get $\lim_{\beta'' \uparrow \beta} E(Z^{\alpha}(\beta'')_{\delta}) = E(Z^{\alpha}(\beta)_{\delta})$.

The following lemma follows easily from (2.1) and (2.2).

Lemma 2.2 For any $\delta \in \Delta$ and $\alpha \in [0,1]$, it holds that

$$(E\tilde{X}_{\delta})_{\alpha} = \big\{EX_{\delta} \mid X = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n,$$
$$X_t(\omega) \in \big[\tilde{X}^{-}_{t,\alpha}(\omega), \tilde{X}^{+}_{t,\alpha}(\omega)\big] \quad \text{for } t = 1, 2, \ldots, n\big\}.$$

The proof of Theorem 2.1 Since (ii) means (i), it suffices to show that (ii) holds. By Lemma 2.2 and monotonicity of δ, the inclusion $\subset$ of (ii) is immediate. Also, the inclusion $\supset$ follow from the observation that $Z^{\alpha}(1) = \widetilde{X}^{+}_{\alpha}, Z^{\alpha}(0) = \widetilde{X}^{-}_{\alpha}$ and Lemma 2.1. □

By Theorem 2.1, we observe that $E\widetilde{X}_{\delta} \in \tilde{\mathbb{R}}$ for all $\delta \in \Delta$. Here we can specify the perceptive fuzzy stopping problem investigated in the next section: The problem is to maximize $E\widetilde{X}_{\delta}$ for all $\delta \in \Delta$ with respect to the fuzzy max order $\preccurlyeq$ on $\tilde{\mathbb{R}}$.

3. Optimal Fuzzy Perception Values and Recursive Equations

In this section, for any given sequence of fuzzy perception functions $\widetilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$, we find the optimal stopping time function δ^* and to characterize the optimal fuzzy perception value $E\widetilde{X}_{\delta^*}$.

For each sequence of random variables $X = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n$, we denote by $\delta^*(X)$ the optimal stopping time for X (Chow, Robbins and Siegmund (1971)), which is thought as a stopping time function.

Lemma 3.1 $\delta^* \in \Delta$.

Proof: For $X = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n$, involving X, we define the sequence $\{\gamma^n_k = \gamma^n_k)(X)\}^n_{k=1}$ by

$$\begin{aligned} &\gamma^n_n(X) = X_n, \\ &\gamma^n_k(X) = \max\left\{X_k, E\big[\gamma^n_{k+1} | \mathscr{B}(X_k)\big]\right\} \quad (k = n-1, \ldots, 1), \end{aligned} \tag{3.1}$$

where $X_k = (X_1, X_2, \ldots, X_k)$. Then, by the usual theory of optimal stopping problems (Chow, Robbins and Siegmund (1971)), we have $E(X_{\delta^*}) = E\gamma^n_1(X)$.

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n) \in \mathscr{X}^n$ with $X_t \leq Y_t (t = 1, 2, \ldots, n)$ P-a.s.. Then, by induction on k, we can easily prove that $\gamma_k^n(\boldsymbol{X}) \leq \gamma_k^n(\boldsymbol{Y})$ for $k = n, n-1, \ldots, 1$. Thus, we get

$$E(\boldsymbol{X}_{\delta^*}) = E(\gamma_1^n(X)) \leq E(\gamma_1^n(\boldsymbol{Y})) = E(\boldsymbol{Y}_{\delta^*}),$$

which shows the monotonicity of δ^*. For $\boldsymbol{Z} = \beta \boldsymbol{X} + (1-\beta)\boldsymbol{Y}$ $(\beta \in [0,1])$, we have

$$\begin{aligned} E[Z_{\delta*(\boldsymbol{Z})}] &= \beta E\big[X_{\delta*(\boldsymbol{Z})}\big] + (1-\beta) E\big[Y_{\delta*(\boldsymbol{Z})}\big] \\ &\leq \beta E\big[X_{\delta*(\boldsymbol{X})}\big] + (1-\beta) E\big[Y_{\delta*(\boldsymbol{Y})}\big], \end{aligned}$$

where $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$. This shows the convexity of δ^*. □

By Lemma 3.1, we observe that δ^* is an optimal stopping time function. For simplicity, we assume the sequence of perception functions $\widetilde{\boldsymbol{X}} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ is independent with each $\tilde{X}_t (t = 1, 2, \ldots, n)$. Then, in the following theorem it will be shown that the optimal fuzzy perception value $E\widetilde{\boldsymbol{X}}_{\delta^*}$ is given by the backward recursive equation:

$$\begin{aligned} \tilde{\gamma}_n^n &= E\tilde{X}_n, \\ \tilde{\gamma}_n^k &= E\widetilde{\max}\big\{\tilde{X}_k, \tilde{\gamma}_{k+1}^n\big\} \quad (k = n-1, \ldots, 2, 1). \end{aligned} \tag{3.2}$$

Since the α-cut of $\tilde{\gamma}_n^k$ in (3.2) can be denoted by

$$\tilde{\gamma}_{k,\alpha}^n = [\tilde{\gamma}_{k,\alpha}^{n,-}, \tilde{\gamma}_{k,\alpha}^{n,+}] \quad (k = 1, 2, \ldots, n),$$

then, the α-cut expression of (3.2) is as follows: For $\alpha \in [0,1]$,

$$\begin{aligned} \tilde{\gamma}_{n,\alpha}^{n,\pm} &= E\tilde{X}_{n,\alpha}^{\pm} \\ \gamma_{k,\alpha}^{n,\pm} &= E\max\Big\{\tilde{X}_{k,\alpha}^{\pm} \tilde{\gamma}_{(k+1),\alpha}^{n,\pm}\Big\} \quad (k = n-1, \ldots, 2, 1). \end{aligned} \tag{3.3}$$

Theorem 2 *It holds that* $E\widetilde{\boldsymbol{X}_{\delta^*}} = \tilde{\gamma}_1^n$.

Proof: By (3.2) and (3.3), we have that, for $\alpha \in [0,1]$,

$$\begin{aligned} \tilde{\gamma}_{k,\alpha}^n &= \Big[E\max\{\tilde{X}_{k,\alpha}^-, \gamma_{(k+1),\alpha}^{n,-}\}, E\max\{\tilde{X}_{k,\alpha}^+, \gamma_{(k+1),\alpha}^{n,+}\}\Big] \\ &= \Big[E\max\{\tilde{X}_{k,\alpha}^-, \tilde{\gamma}_{(k+1)}^n(\tilde{\boldsymbol{X}}_\alpha^-)\}, E\max\{\tilde{X}_{k,\alpha}^+, \tilde{\gamma}_{(k+1)}^n(\tilde{\boldsymbol{X}}_\alpha^+)\}\Big] \end{aligned}$$

where $\gamma_{(k+1)}^n(\tilde{\boldsymbol{X}}_\alpha^-)$ and $\gamma_{(k+1)}^n(\tilde{\boldsymbol{X}}_\alpha^+)$ are defined in (3.1). Applying Theorem 2.1, we get $(E\tilde{\boldsymbol{X}}_{\delta^*})_\alpha = (\tilde{\gamma}_1^n)_\alpha$. Thus, $E\tilde{\boldsymbol{X}}_{\delta^*} = \tilde{\gamma}_1^n$, as required. □

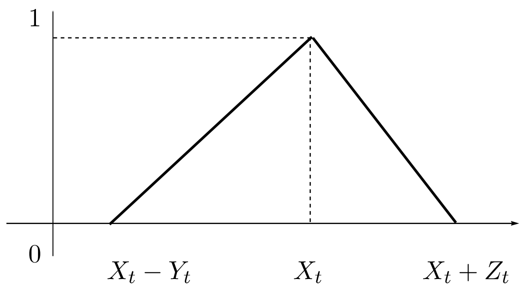

Figure 1. The fuzzy perception $\tilde{X}_t = (Y_t, X_t, Z_t)$.

As a numerical example, we will compute the optimal fuzzy perception value for the perception stopping problem described by simple triangular fuzzy numbers. The triangular fuzzy number (a, m, b) with $a > 0$ and $b > 0$ is given by

$$(a, m, b)(x) = \begin{cases} \max\{(x - m + a)/a, 0\} & \text{if } \ x \leq m \\ \max\{(x - m - b)/b, 0\} & \text{if } \ x > m. \end{cases}$$

Obviously, the α-cut of (a, m, b) is

$$(a, m, b)_\alpha = [m - a(1 - \alpha), m + b(1 - \alpha)] \quad \alpha \in [0, 1].$$

Let $\tilde{\boldsymbol{X}} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ be independent and identically distributed sequence of fuzzy perception functions with $\tilde{X}_t = (Y_t, X_t, Z_t)$ $(t = 1, 2, \ldots, n)$. (See Figure 1). We

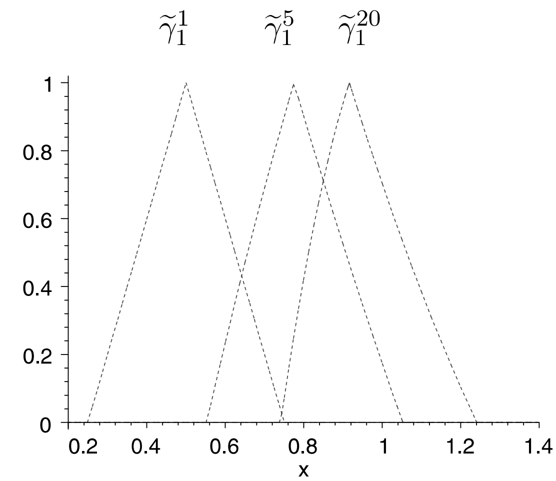

Figure 2. The graph of $\tilde{\gamma}_1^n (n = 1, 5, 20)$.

assume that $X_t \sim U[0,1]$ and $Y_t, Z_t \sim U[0,1/2]$ $(t = 1,2,\ldots,n)$, where $X \sim U[a,b]$ $(a < b)$ means that the distribution of X is a uniform distribution on $[a,b]$.

The optimal fuzzy perception value $E\tilde{\mathbf{X}}_{\delta^*} = \tilde{\gamma}_1^n$ is computed recursively by (3.3), which is given as follows.

$$\begin{aligned}
\tilde{\gamma}_{n,\alpha}^{n,-} &= (1+\alpha)/2, \tilde{\gamma}_{n,\alpha}^{n,+} = (3-\alpha)/2 \\
\tilde{\gamma}_{k,\alpha}^{n,-} &= E\max\{X_k - (1-\alpha)Y_k, \tilde{\gamma}_{(k+1),\alpha}^{n,-}\} \\
\tilde{\gamma}_{k,\alpha}^{n,+} &= E\max\{X_k + (1-\alpha)Z_k, \tilde{\gamma}_{(k+1),\alpha}^{n,+}\} \quad (\alpha \in [0,1], k = n-1, n-2, \ldots, 1).
\end{aligned}$$

The graph of $\tilde{\gamma}_1^n (n = 1,5,20)$ evaluated by Maple 7 is shown in Figure 2, and we observe that $\tilde{\gamma}_1^{20}$ is concave on its left-side slope and convex on its right-side slope.

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