A FUZZY RELATIONAL EQUATION IN DYNAMIC FUZZY SYSTEMS

M. KURANO(Chiba University), M. YASUDA(Chiba University),
J. NAKAGAMI(Chiba University) & Y. YOSHIDA(Kitakyushu University)

Abstract : For a dynamic fuzzy system, the fundamental method is to analyze its recursive relation of the fuzzy states. It is similar that the Bellman equation is the important tool in the dynamic programming. Here we will consider the existence and the uniqueness of solution of a fuzzy relational equation. Two examples, which satisfies our conditions, are given to illustrate the results.

1 Introduction and notations

We use the notations in [4]. Let X be a compact metric space. We denote by 2^X the collection of all subsets of X, and denote by $\mathcal{C}(X)$ the collection of all closed subsets of X. Let ρ be the Hausdorff metric on 2^X. Then it is well-known ([3]) that $(\mathcal{C}(X), \rho)$ is a compact metric space. Let $\mathcal{F}(X)$ be the set of all fuzzy sets $\tilde{s} : X \to [0,1]$ which are upper semi-continuous and satisfy $\sup_{x \in X} \tilde{s}(x) = 1$. Let $\tilde{q} : X \times X \to [0,1]$ be a continuous fuzzy relation on X.

In this paper, we consider the existence and uniqueness of solution $\tilde{p} \in \mathcal{F}(X)$ in the following fuzzy relational equation (1.1) for a given continuous fuzzy relation $\tilde{q}$ on X (see [4]) :

$$\tilde{p}(y) = \sup_{x \in X} \{\tilde{p}(x) \wedge \tilde{q}(x,y)\}, \quad y \in X, \tag{1.1}$$

where $a \wedge b := \min\{a, b\}$ for real numbers a and b. We define a map $\tilde{q}_\alpha : 2^X \to 2^X$ $(\alpha \in [0,1])$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha \neq 0,\ D \in 2^X, D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{q}(x,y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0,\ D \in 2^X, D \neq \emptyset, \\ X & \text{for } 0 \leq \alpha \leq 1,\ D = \emptyset, \end{cases} \tag{1.2}$$

where cl denotes the closure of a set. Especially, we put $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$ for $x \in X$. We note that $\tilde{q}_\alpha : \mathcal{C}(X) \to \mathcal{C}(X)$.

Lemma 1.1 ([4, Lemma 2]). *For each $\alpha \in [0,1]$, the map $\tilde{q}_\alpha : \mathcal{C}(X) \to \mathcal{C}(X)$ is continuous with respect to ρ.*

For $\tilde{s} \in \mathcal{F}(X)$, the α-cut $\tilde{s}_\alpha$, $\alpha \in [0,1]$ is defined by

$$\tilde{s}_\alpha := \{x \in X \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \neq 0) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in X \mid \tilde{s}(x) > 0\}.$$

Lemma 1.2.

(i) *For $\tilde{s} \in \mathcal{F}(X)$, $\tilde{s}$ satisfies* (1.1) *if and only if*

$$\tilde{q}_\alpha(\tilde{s}_\alpha) = \tilde{s}_\alpha, \quad \alpha \in [0,1]. \tag{1.3}$$

(ii) *We suppose that a family of subsets $\{D_\alpha \mid \alpha \in [0,1]\} (\subset \mathcal{C}(X))$ satisfies the following conditions* (a), (b) *and* (c)*:*

(a) $D_\alpha \subset D_{\alpha'}$ *for* $0 \le \alpha' < \alpha \le 1$;

(b) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'} = D_\alpha$ *for* $\alpha \neq 0$;

(c) $\tilde{q}_\alpha(D_\alpha) = D_\alpha$ *for* $\alpha \in [0,1]$.

Then $\tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(x)\}$, $x \in X$, *satisfies* $\tilde{s} \in \mathcal{F}(X)$ *and* (1.1), *where* 1_D *denotes the characteristic function of a set* $D \in 2^X$.

Proof. (i) is trivial. (ii) is from (i) and [4, Lemma 3]. □

2 The existence of solutions

For $\alpha \in [0,1]$ and $x \in X$, a sequence $\{\tilde{q}^k_\alpha(x)\}_{k=1,2,\cdots}$ is defined iteratively by

$$\tilde{q}^0_\alpha(x) := \{x\}, \quad \tilde{q}^1_\alpha(x) := \tilde{q}_\alpha(x) \quad \text{and} \quad \tilde{q}^{k+1}_\alpha(x) := \tilde{q}_\alpha(\tilde{q}^k_\alpha(x)) \quad \text{for } k = 1, 2, \cdots.$$

Then, let $G_\alpha(x) := \bigcup_{k=1}^\infty \tilde{q}^k_\alpha(x)$ and

$$F_\alpha(x) := \bigcup_{k=0}^\infty \tilde{q}^k_\alpha(x) = \{x\} \cup G_\alpha(x). \tag{2.1}$$

We now consider a class of invariant points for this iteration procedure, that is, $x \in G_\alpha(x)$. So put

$$R_\alpha := \{x \in X \mid x \in G_\alpha(x)\} \quad \text{for } \alpha \in [0,1]. \tag{2.2}$$

Each state of R_α is called as an "α-recurrent" state and it is studied by [7]. The following properties (i) and (ii) hold clearly:

(i) $\tilde{q}_\alpha(F_\alpha(x)) = G_\alpha(x)$ for $\alpha \in [0,1]$ and $x \in X$;

(ii) $R_\alpha \subset R_{\alpha'}$ for $0 \le \alpha' < \alpha \le 1$.

Lemma 2.1. *If* $z \in R_1$, *the following* (i) *and* (ii) *hold:*

(i) $\tilde{q}_\alpha(F_\alpha(z)) = F_\alpha(z)$ *for* $\alpha \in [0,1]$;

(ii) $F_\alpha(z) \subset F_{\alpha'}(z)$ *for* $0 \le \alpha' < \alpha \le 1$.

Proof. Since $z \in R_1 \subset R_\alpha$, we have

$$\tilde{q}_\alpha(F_\alpha(z)) = G_\alpha(z) = F_\alpha(z).$$

So, we obtain (i). (ii) is trivial. □

For $z \in R_1$, we define

$$\hat{F}_\alpha(z) := \bigcap_{\alpha' < \alpha} \mathrm{cl}\{F_{\alpha'}(z)\} \ (\alpha \neq 0) \quad \text{and} \quad \hat{F}_0(z) := \mathrm{cl}\{F_0(z)\}, \tag{2.3}$$

where $\mathrm{cl}\{F_{\alpha'}(z)\}$ denotes the closure of $F_{\alpha'}(z)$.

Lemma 2.2. *If* $z \in R_1$, *the following* (i), (ii) *and* (iii) *hold:*

(i) $\tilde{q}_\alpha(\hat{F}_\alpha(z)) = \hat{F}_\alpha(z)$ *for* $\alpha \in [0,1]$;

(ii) $\hat{F}_\alpha(z) \subset \hat{F}_{\alpha'}(z)$ *for* $0 \le \alpha' < \alpha \le 1$;

(iii) $\hat{F}_\alpha(z) = \lim_{\alpha' \uparrow \alpha} \hat{F}_{\alpha'}(z)$ *for* $\alpha \neq 0$.

Proof. (ii) is trivial from Lemma 2.1 and (iii) is also trivial from the definition. To prove (i), let $\alpha = 0$. From Lemma 2.1(i), we have $\tilde{q}_0(F_0(z)) = F_0(z)$. By the continuity of $\tilde{q}$, we can check $\tilde{q}_0(\mathrm{cl}\{F_0(z)\}) = \mathrm{cl}\{F_0(z)\}$ in a similar way to the proof of [4, Lemma 1]. Therefore, $\tilde{q}_0(\hat{F}_\alpha(z)) = \hat{F}_0(z)$. Let $\alpha > 0$ and $y \in \tilde{q}_\alpha(\hat{F}_\alpha(z))$. By Lemma 1.1, we have

$$y \in \bigcap_{\alpha' < \alpha} \tilde{q}_\alpha(\mathrm{cl}\{F_{\alpha'}(z)\}) = \bigcap_{n=1}^{\infty} \tilde{q}_\alpha(\mathrm{cl}\{F_{(\alpha-1/n)\vee 0}(z)\}).$$

From the continuity of $\tilde{q}$, for $n \geq 1$, there exists $x_n \in F_{(\alpha-1/n)\vee 0}(z)$ such that $\tilde{q}(x_n, y) \geq \alpha - 1/n$. By Lemma 2.1(i),

$$y \in \tilde{q}_{(\alpha-1/n)\vee 0}(F_{(\alpha-1/n)\vee 0}(z)) = F_{(\alpha-1/n)\vee 0}(z) \subset \mathrm{cl}\{F_{(\alpha-1/n)\vee 0}(z)\}$$

for all $n \geq 1$. So, $y \in \hat{F}_\alpha(z)$. Therefore, we obtain

$$\tilde{q}_\alpha(\hat{F}_\alpha(z)) \subset \hat{F}_\alpha(z).$$

While, from Lemma 2.1(i), we have

$$\mathrm{cl}\{F_{\alpha'}(z)\} \subset \tilde{q}_{\alpha'}(\mathrm{cl}\{F_{\alpha''}(z)\})$$

for $\alpha'' < \alpha' < \alpha$. Then

$$\hat{F}_\alpha(z) = \bigcap_{\alpha' < \alpha} \mathrm{cl}\{F_{\alpha'}(z)\} \subset \bigcap_{\alpha' < \alpha} \tilde{q}_{\alpha'}(\mathrm{cl}\{F_{\alpha''}(z)\}) = \tilde{q}_\alpha(\mathrm{cl}\{F_{\alpha''}(z)\})$$

for $\alpha'' < \alpha$. So, we get

$$\hat{F}_\alpha(z) \subset \bigcap_{\alpha'' < \alpha} \tilde{q}_\alpha(\mathrm{cl}\{F_{\alpha''}(z)\}) = \tilde{q}_\alpha\left(\bigcap_{\alpha'' < \alpha} \mathrm{cl}\{F_{\alpha''}(z)\}\right) = \tilde{q}_\alpha(\hat{F}_\alpha(z)).$$

Therefore, we can obtain (i). □

Let $z \in R_1$. Since $\{\hat{F}_\alpha(z) \mid \alpha \in [0,1]\}$ satisfies the conditions (a) – (c) of Lemma 1.2(ii), we obtain the following theorem.

Theorem 2.1.

(i) *If $R_1 \neq \emptyset$, then there exists a solution of* (1.1).

(ii) *Let $z \in \mathcal{R}_1$. Define a fuzzy state*

$$\tilde{s}^z(x) := \sup_{\alpha \in [0,1]} \left\{\alpha \wedge 1_{\hat{F}_\alpha(z)}(x)\right\}, \quad x \in X. \tag{2.4}$$

Then $\tilde{s}^z \in \mathcal{F}(X)$ satisfies (1.1).

Assume that $R_1 \neq \emptyset$. We introduce an equivalent relation $\sim$ on R_α as follows: For $z_1, z_2 \in R_\alpha$,

$$z_1 \sim z_2 \quad \text{means that} \quad z_1 \in F_\alpha(z_2) \text{ and } z_2 \in F_\alpha(z_1).$$

Then we could identify the states of R_α which is equivalent with respect to $\sim$, and so put

$$R_\alpha^\sim := R_\alpha / \sim .$$

Lemma 2.3. *For $z_1, z_2 \in R_1$,*

$$z_1 \sim z_2 \quad \textit{if and only if} \quad F_\alpha(z_1) = F_\alpha(z_2) \textit{ for all } \alpha \in [0,1].$$

Proof. Let $z_1 \sim z_2$. Then, we have $z_1 \in F_1(z_2) \subset F_\alpha(z_2)$ for any $\alpha \in [0,1]$. From the definition (2.1) of $F_\alpha(z_1)$, we obtain $F_\alpha(z_1) \subset F_\alpha(z_2)$. Since we have $F_\alpha(z_2) \subset F_\alpha(z_1)$ similarly, $F_\alpha(z_1) = F_\alpha(z_2)$ holds. The reverse proof is trivial. □

From Theorem 2.1 and Lemma 2.3, the number of solutions of (1.1) is greater than or equals to the number of "1-recurrent" sets. To consider the class of solution (1.1), let $P := \{\tilde{p} \in \mathcal{F}(X) \mid \tilde{p} \text{ is a solution of (1.1)}\}$. Then P has the following property:

Theorem 2.2. *Let $\tilde{p}^k \in P$ $(k = 1, 2, \cdots, l)$. Then:*

(i) *Put*

$$\tilde{p}(x) := \max_{k=1,2,\cdots,l} \tilde{p}^k(x) \quad \text{for } x \in X.$$

Then $\tilde{p} \in P$.

(ii) *Let $\{\alpha^k \in [0,1] \mid k = 1, 2, \cdots, l\}$ satisfy $\max_{k=1,2,\cdots,l} \alpha^k = 1$. Put*

$$\tilde{p}(x) := \max_{k=1,2,\cdots,l} \{\alpha^k \wedge \tilde{p}^k(x)\} \quad \text{for } x \in X.$$

Then $\tilde{p} \in P$.

Proof. (ii) Taking the α-cut of $\tilde{p} \in \mathcal{F}(X)$, we have

$$\tilde{p}_\alpha := \bigcup_{k:\alpha^k \geq \alpha} \tilde{p}^k_\alpha.$$

Then,

$$\tilde{q}_\alpha(\tilde{p}_\alpha) = \tilde{q}_\alpha\left(\bigcup_{k:\alpha^k \geq \alpha} \tilde{p}^k_\alpha\right) = \bigcup_{k:\alpha^k \geq \alpha} \tilde{q}_\alpha(\tilde{p}^k_\alpha) = \bigcup_{k:\alpha^k \geq \alpha} \tilde{p}^k_\alpha = \tilde{p}_\alpha.$$

Therefore, we obtain (ii) from Lemma 1.2(i). (i) is proved similarly. □

3 The uniqueness of solutions

In this section, we discuss the uniqueness of solutions of the equation (1.1) under convexity and compactness. Let B be a convex subset of an n-dimensional Euclidean space $\mathcal{R}^n$ and $C_c(B)$ the class of all closed and convex subsets of B. Throughout this section, we assume that the state space X is a convex and compact subset of $\mathcal{R}^n$. The fuzzy set $\tilde{s} \in \mathcal{F}(X)$ is called convex if its α-cut $\tilde{s}_\alpha$ is convex for each $\alpha \in [0,1]$. Let $\mathcal{F}_c(X) := \{\tilde{s} \in \mathcal{F} : \tilde{s} \text{ is convex}\}$.

By applying Kakutani's fixed point theorem ([2]), we have the following.

Lemma 3.1. *Let $\alpha \in [0,1]$ and $\tilde{q}_\alpha(x)$ is convex for each $x \in X$. Then, for any $A \in C_c(X)$ with $A = \tilde{q}_\alpha(A)$, there exists an $x \in X$ such that $\tilde{q}(x,x) \geq \alpha$.*

Proof. The map $\tilde{q}_\alpha : A \to C_c(A)$ with $\tilde{q}_\alpha(x) \in C_c(A)$ for all $x \in A$ is continuous from Lemma 1.1, so Kakutani's fixed point theorem guarantees the existence of an element $x \in A$ such that $x \in \tilde{q}_\alpha(x)$, which implies $\tilde{q}(x,x) \geq \alpha$. This completes the proof. □

We assume that $\tilde{q}_\alpha(x)$ is convex for each $x \in X$. As a consequence, we have a property of the solutions of (1.1).

Proposition 3.1. *Let $p \in \mathcal{F}_c(X)$ be a solution of (1.1). Then, for each $\alpha \in [0,1]$, there exists an $x \in p_\alpha$ with $\tilde{q}(x,x) \geq \alpha$.*

Proof. By Lemma 1.2, $\tilde{p}_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha)$ for each $\alpha \in [0,1]$. Thus, Lemma 3.1 clearly proves the desired result. □

Now, we give sufficient conditions for the uniqueness of solutions of (1.1). Let $U_\alpha := \{x \in X \mid \tilde{q}(x,x) \geq \alpha\}$ for $\alpha \in [0,1]$.

Assumption A. The following A1 – A3 hold.

A1. The set U_1 is a one-point set, say u. That is, $U_1 = \{u\}$.

A2. $U_\alpha \subset F_\alpha(u)$ for each $\alpha \in [0,1]$, where u is given by A1 and $F_\alpha(u)$ is defined by (2.1).

A3. Let $\alpha \in [0,1]$ and $A \in C_c(X)$. If $A = \tilde{q}_\alpha(A)$, then

$$A = \bigcup_{x \in U_\alpha \cap A} F_\alpha(x)$$

Theorem 3.1. *Under Assumption A, the equation* (1.1) *has a unique solution in* $\mathcal{F}_c(X)$.

Proof. Let $\tilde{p}, \tilde{p}' \in \mathcal{F}_c(X)$ be solutions of (1.1). By Lemma 3.1, $\tilde{p}_1 \cap U_1 \neq$ and $\tilde{p}'_1 \cap U_1 \neq$. Since U_1 is a one-point set, $u \in \tilde{p}_1$ and $u \in \tilde{p}'_1$. Thus, by A3, $\tilde{p}_1 = F_1(u)$ and $\tilde{p}'_1 = F_1(u)$, which implies $\tilde{p}_1 = \tilde{p}'_1$. We now show that $\tilde{p}_\alpha = \hat{F}_\alpha(u)$ for $0 < \alpha \leq 1$. Since $u \in \tilde{p}_{\alpha'} = \tilde{q}_{\alpha'}(\tilde{p}_{\alpha'})$, it holds that $F_{\alpha'}(u) \subset \tilde{p}_{\alpha'}$. Therefore, since $\tilde{p}_\alpha$ is closed,

$$\hat{F}_\alpha(u) = \bigcup_{\alpha' < \alpha} \mathrm{cl}\{F_{\alpha'}(u)\} \subset \bigcup_{\alpha' < \alpha} \tilde{p}_{\alpha'}(u) = \tilde{p}_\alpha.$$

On the other hand, we have

$$\begin{aligned} \tilde{p}_\alpha &\subset \textstyle\bigcup_{x \in U_\alpha \cap \tilde{p}_\alpha} \mathrm{cl}\{F_\alpha(x)\}, \quad \text{(from A3)} \\ &\subset \textstyle\bigcup_{x \in F_\alpha(u) \cap \tilde{p}_\alpha} \mathrm{cl}\{F_\alpha(x)\}, \quad \text{(from A2)} \\ &\subset \textstyle\bigcup_{x \in \hat{F}_\alpha(u)} \mathrm{cl}\{F_\alpha(x)\}. \end{aligned}$$

From that $x \in \hat{F}_\alpha(u)$ means $\hat{F}_\alpha(x) \subset \hat{F}_\alpha(u)$, it holds that

$$\tilde{p}_\alpha \subset \mathrm{cl}\{F_\alpha(u)\} \subset \hat{F}_\alpha(u).$$

The above shows $\tilde{p}_\alpha = \hat{F}(u)$. Similarly $\tilde{p}'_\alpha = \hat{F}_\alpha(u)$. Thus, $\tilde{p}_\alpha = \tilde{p}'_\alpha$. This completes the proof. □

4 Numerical examples

Here two numerical examples of Section 2 and 3 are given to comprehend computational aspect of this paper.

Example 1. Let $X = [0,1]$. For any $g : [0,1] \to [0,1]$, let

$$\tilde{q}(x,y) := (1 - \mid y - g(x) \mid) \vee 0, \quad x, y \in [0,1].$$

We assume that $g(\cdot)$ is strictly increasing and continuous and that there exists a unique $x_0 \in [0,1]$ with $x_0 = g(x_0)$. Under the above condition, $R_1 = \{x_0\}$ and for each $\alpha \in [0,1)$,

$$U_\alpha = [\underline{x}_\alpha, \overline{x}_\alpha], \tag{4.1}$$

when $\underline{x}_\alpha, \overline{x}_\alpha$ is a unique solution of $x = g(x) - (1-\alpha)$, $x = g(x) + (1-\alpha)$ respectively and $\underline{x}_\alpha = 0, \overline{x}_\alpha = 1$ if the solution does not exist in [0,1].

Clearly, U_α is a unique solution of the equation $A = \tilde{p}_\alpha(A)$ in $C_c([0,1])$, so that Assumption A in Section 3 holds in this case. Thus, by Theorem 3.1,

$$\tilde{s}(x) = \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{U_\alpha}(x)\} \tag{4.2}$$

is a unique convex solution of (1.1). For a concrete example such as $g(x) = (2x^2+1)/4$, then it is seen that $R_1 = \{(2-\sqrt{2})/2\}$ and

$$\begin{aligned} \underline{x}_\alpha &= \left(1 - \sqrt{5/2 - 2\alpha}\right) \vee 0, \\ \overline{x}_\alpha &= \begin{cases} 1, & 3/4 < \alpha \\ 1 - \sqrt{2\alpha - 3/2}, & 3/4 \leq \alpha \leq 1. \end{cases} \end{aligned}$$

By (4.1) and (4.2), the unique solution is as follows (Fig.1):

$$\tilde{s}(x) = \begin{cases} -x^2/2 + x + 3/4, & 0 \le x \le 1 - \sqrt{2}/2 \\ x^2/2 - x + 5/4, & 1 - \sqrt{2}/2 < x \le 1. \end{cases}$$

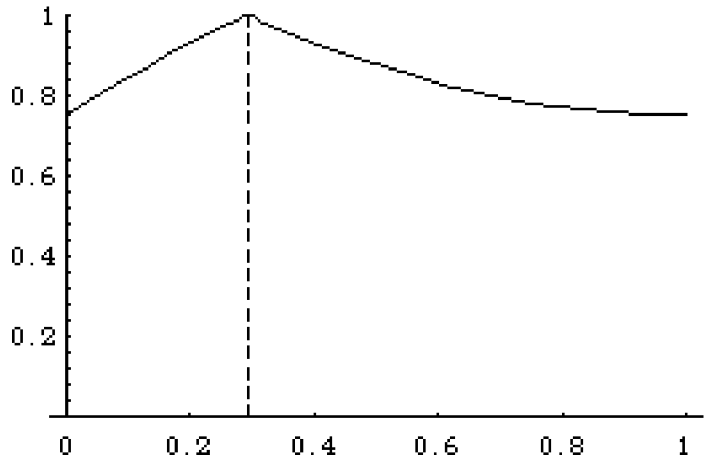

Fig.1 The unique solution $\tilde{s}$.

Example 2. This example has two peaks for the fuzzy relation. Let $X = [0, 1]$ and

$$\tilde{q}(x, y) = (1- \mid y - (x^2 + 1)/4 \mid) \vee (1- \mid y - (x^2 + 2)/4 \mid).$$

for $x, y \in [0, 1]$. Then, $R_1 = \{a, b\}$, where $a = 2 - \sqrt{3}, b = 2 - \sqrt{2}$. By simple calculation, we get

$$\hat{F}_\alpha(a) = [\underline{x}^a_\alpha, \overline{x}^a_\alpha] \quad \text{and} \quad \hat{F}_\alpha(b) = [\underline{x}^b_\alpha, \overline{x}^b_\alpha]$$

for $\alpha \in [0, 1]$, where

$$\underline{x}^a_\alpha = \begin{cases} 0, & 0 \le \alpha \le 3/4 \\ 2 - \sqrt{7 - 4\alpha}, & 3/4 < \alpha \le 1, \end{cases}$$

$$\overline{x}^a_\alpha = \begin{cases} 1, & 0 \le \alpha \le 7/8 \\ 2 - \sqrt{4\alpha - 1}, & 7/8 < \alpha \le 1, \end{cases}$$

$$\underline{x}^b_\alpha = \begin{cases} 0, & 0 \le \alpha \le 7/8 \\ 2 - \sqrt{6 - 4\alpha}, & 7/8 < \alpha \le 1, \end{cases}$$

$$\overline{x}^b_\alpha = \begin{cases} 1, & 0 \le \alpha \le 3/4 \\ 2 - \sqrt{4\alpha - 2}, & 3/4 < \alpha \le 1, \end{cases}$$

By Theorem 2.1, the solutions of (1.1) are given as follows(Fig.2):

$$\tilde{s}^a(x) = \begin{cases} -x^2 + 2x + 3/4, & 0 \le x \le 2 - \sqrt{3} \\ x^2 - 4x + 5/4, & 2 - \sqrt{3} \le x \le 2 - \sqrt{10}/2 \\ 7/8, & 2 - \sqrt{10}/2 \le x \le 1, \end{cases}$$

$$\tilde{s}^b(x) = \begin{cases} 7/8, & 0 \le x \le 2 - \sqrt{10}/2 \\ -x^2 + 4x + 2/4, & 2 - \sqrt{10}/2 \le x \le 2 - \sqrt{2} \\ x^2 - 4x + 6/4, & 2 - \sqrt{2} \le x \le 1. \end{cases}$$

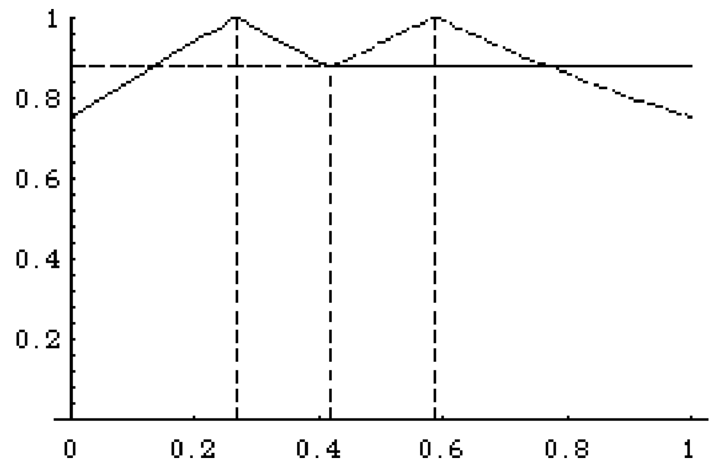

Fig.2 The unique solutions $\tilde{s}^a$ and $\tilde{s}^b$.

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