A game variant of Stopping Problem on Jump Processes with a Monotone Rule

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Abstract

A continuous time version of the multi-variate stopping problem is considered. Associated with vector valued jump stochastic processes, stopping problems with a monotone logical rule are defined under the notion of Nash equilibrium point. The existence of an equilibrium strategy and its characterization by integral equations are obtained. Illustrative examples are provided.

Key words: *Jump process, Multi-objective, Stopping game, Nash equilibrium, Optimal stopping.*

1 Introduction

In the social life or the business world, group decision making is often done by summing up each opinion of individuals who wish to reflect their opinion to the whole group as much as possible. As one abstraction of such a situation, we shall try to propose a multi-valued stopping game by introducing a monotone logical function to sum up each opinion of individuals. The discrete time case had been already discussed by authors[4], [10]. Here we consider the continuous time case, which is formulated as a multi-objective extension of Karlin's model[3] and a rule's extension of Sakaguchi's model[7]. As a related result, Presman and Sonin[6] have obtained the multi-person best choice problem on the Poisson stream but their model of a decision to stop is different from ours.

The situation of our problem is as follows. The group of p players observe a p-dimensional stochastic process. Each player can declare to stop or continue the process at any time when the p-dimensional successive offers will have happened, and the individual declarations are summed up to make the group decision for the process by using a monotone logical rule . When the process is decided to stop by the group of p players, components of the stochastic process are given to each player as the reward, so that he wishes to make his expected gain as large as possible.

First, we introduce some definitions and notations to formulate our stopping problem in the section 2. Then, by preparing several lemmas, we show the existence of an equilibrium stopping strategy and obtain its characterization by an integral equation in the section 3. In the section 4, examples of the underlying model are given.

2 Formulation

We consider a p-dimensional vector valued stochastic process $\{X_t; t \geq 0\}$ with i-th component X_t^i, adapted to $\mathcal{F}_t$ on a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}_t$ is the σ-algebra generated by $\{X_s; 0 \leq s \leq t\}$. Let us assume that the process $\{X_t; t \geq 0\}$ is an independent jump process (see. for example, Feller[2]) , that is, there are two independent stochastic sequences $Z_n = (Z_n^1, Z_n^2, \cdots, Z_n^p)$ and $\tau_n, n = 0, 1, 2, \cdots$, which satisfy $X_t = Z_n$ if $\tau_n \leq t < \tau_{n+1}$ for any t, $t \geq 0$, under the following assumption.

Assumption 1.

(a) p-dimensional random vectors $Z_n = (Z_n^1, Z_n^2, \cdots, Z_n^p)$, $n = 0, 1, 2, \cdots$, are i.i.d. with a common distribution F on R^p, where $R = (-\infty, \infty)$.

(b) $\tau_0 = 0$ a.s. and $\tau_n - \tau_{n-1}, n = 1, 2, \cdots$ are i.i.d. with a common distribution G on R_+, where $R_+ = [0, \infty)$ and $G(0) = 0$.

(c) $\int_{R^p} |z| \, F(dz) < \infty$ and $\mu_G = \int_{R_+} t \, G(dt) < \infty$, where $| \cdot |$ is a norm on R^p.

In order to denote the declaration for each player $i, i = 1, \cdots, p$, when the process is $\{X_t; t \geq 0\}$, let $\sigma^i(t, x)$ be a $\{0, 1\}$-valued Borel measurable function on $R_+ \times R^p$ with $\sigma^i(0, x) = 1$. We call $\sigma^i = \sigma^i(\cdot, \cdot)$ an *individual strategy* for player i, and $\sigma = (\sigma^1, \cdots, \sigma^p)$ a *strategy*. The individual strategy $\sigma^i(t, x)$ may be interpreted as follows; when the amount x of the offer has happened and the time interval remaining until termination is t, $\sigma^i(t, x) = 1(0)$ means player i declares to stop (continue). In particular, $\sigma^i(0, x) = 1$ means that any player i must declare to stop when the time remaining until termination is 0.

The individual declarations are summed up by a logical rule. A logical rule is a map $\pi : \{0, 1\}^p \to \{0, 1\}$ and called *monotone* if $\pi(1, \cdots, 1) = 1$ and $\pi(\sigma^1, \cdots, \sigma^p) \leq \pi(\tilde{\sigma}^1, \cdots, \tilde{\sigma}^p)$ for $\sigma^i \leq \tilde{\sigma}^i (1 \leq i \leq p)$. A monotone logical rule includes a wide variety in a choice system such as a unanimity rule, an equal/unequal majority rule and a hierarchical rule, some of which are given in the last section. For example, if no less than $r (\leq p)$ members in a group of p players declare to stop the group decision is to stop the process (equal majority rule). That is, $\pi(\sigma^1, \cdots, \sigma^p) = 1(0)$ if $\sum_{i=1}^p \sigma^i \geq (<) r$. Refer also to our previous papers[4, 10].

For a strategy σ, a monotone logical rule π and a planning horizon T, a *stopping time* $t(T, \sigma, \pi)$ for the group of p players is defined by

$$t(T, \sigma, \pi) = \min\{\tau(\sigma, \pi), T\},$$

where $\tau(\sigma, \pi) =$ the first τ_k such that $\pi(\sigma(T - \tau_k, X_{\tau_k})) = 1$, where $\sigma(T - \tau_k, X_{\tau_k}) = (\sigma^1(T - \tau_k, X_{\tau_k}), \cdots, \sigma^p(T - \tau_k, X_{\tau_k}))$ for $k \geq 0$. Note that $t(T, \sigma, \pi)$ means the first time such that the declaration σ^i of each player i is summed up for the group of p players. to stop the process by the rule π. Since the monotone logical rule π is fixed, π is suppressed in $t(T, \sigma, \pi)$ hereonwards.

An *expected reward* of player i for a strategy σ is defined by

$$u^i(T, \sigma) = E[X^i_{t(T,\sigma)}], \quad T \geq 0.$$

Since the problem is fundamentally formulated as a non-cooperative game, a notion of Nash equilibrium point (see Nash[5]) can be utilized. A strategy ${}^*\sigma = ({}^*\sigma^1, \cdots, {}^*\sigma^p)$ is *equilibrium* if, for each i,

$$u^i(T, {}^*\sigma) \geq u^i(T, {}^*\sigma^{-i} \| \sigma^i)$$

for any individual strategy σ^i and any $T \geq 0$, where

$${}^*\sigma^{-i} \| \sigma^i = ({}^*\sigma^1, \cdots, {}^*\sigma^{i-1}, \sigma^i, {}^*\sigma^{i+1}, \cdots, {}^*\sigma^p).$$

In this paper, we will find an equilibrium strategy ${}^*\sigma$ and the corresponding stopping time $t(T, {}^*\sigma, \pi)$ given a monotone rule π.

3 Lemmas and Theorems

In this section, the existence of an equilibrium strategy and its characterization are obtained. First, we'll derive the integral equation for $u(T, \sigma) = (u^1(T, \sigma), \cdots, u^p(T, \sigma)), T \geq 0$, given a strategy σ. For each $n \geq 0$, let $\mathcal{G}_n$ be the σ-algebra generated by $\{(Z_k, \tau_k), k = 0, 1, \cdots, n-1, \tau_n\}$.

Then we have the following lemmas.

Lemma 1.

$$E[X^i_{t(T,\sigma)} I_{\{t(T,\sigma) \geq \tau_n\}} \mid \mathcal{G}_n] = u^i(T - \tau_n, \sigma) I_{\{t(T,\sigma) \geq \tau_n\}} \quad \text{a.e.},$$

where I_A is the indicator for a set A.

Proof. By Assumption 1, it holds that for $n \geq 0$

$$t(T, \sigma) = t(T - \tau_n, \sigma) + \tau_n \tag{3.1}$$

on $\{t(T, \sigma) \geq \tau_n\}$ and that

$$E[X^i_{t(T-\tau_n,\sigma)+\tau_n} \mid \mathcal{G}_n] = u^i(T - \tau_n, \sigma) \quad \text{a.e..} \tag{3.2}$$

So, the proof is completed by noting $\{t(T, \sigma) \geq \tau_n\} \in \mathcal{G}_n$. Q.E.D.

Let $Z = (Z^1, \cdots, Z^p)$ be a p-dimensional random variable whose distribution is F and $\mathcal{F}$ be the σ-algebra generated by Z. For any set $A \in \mathcal{F}$ and any $\alpha \in R$, let define operators $L^i, i = 1, \cdots, p$, by

$$L^i(A; \alpha) = E[Z^i I_A] + \alpha P(A^c).$$

Lemma 2. *For each $i, i = 1, \cdots, p$, $u^i(T) = u^i(T, \sigma)$ satisfies the following integral equation:*

$$u^i(T) = L^i(\{\pi(\sigma(T, Z)) = 1\}; G \circ u^i(T)) \tag{3.3}$$

where $\sigma(T, Z) = (\sigma^1(T, Z), \cdots, \sigma^p(T, Z))$ and $G \circ u^i(T) = \int_0^T u^i(T - s)G(ds)$.

Proof. By Assumption 1, we have

$$u^i(T) = E\left[Z_0^i I_{\{t(T,\sigma)=0\}}\right] + E\left[X^i_{t(T,\sigma)} I_{\{t(T,\sigma)\geq\tau_1\}}\right]. \tag{3.4}$$

¿From Lemma 1, it holds that

$$E[X^i_{t(T,\sigma)} I_{\{t(T,\sigma)\geq\tau_1\}} \mid \mathcal{G}_1] = u^i(T - \tau_1) I_{\{t(T,\sigma)\geq\tau_1\}} \quad \text{a.e..}$$

Thus, noting $\{t(T, \sigma) = 0\} = \{\pi(\sigma(T, Z_0)) = 1\}$ and $\{t(T, \sigma) \geq \tau_1\} = \{\pi(\sigma(T, Z_0)) = 1\}^c$, (3.3) follows from (3.4), replacing Z_0 by Z. Q.E.D.

To show the existence of an equilibrium strategy, we further need several lemmas.

Let S be the set of all $\{0, 1\}$-valued Borel measurable functions on $R_+ \times R^p$. For any number $\alpha \in R$ and $i, i = 1, \cdots, p$, define $\sigma^i[\alpha] \in S$ by $\sigma^i[\alpha] = 1$ if $Z^i \geq \alpha, = 0$ otherwise, which is called an individual strategy of a control-limit-type.

For any $(\sigma^1, \cdots, \sigma^p) \in S^p$, let $\pi(\sigma) = \pi(\sigma^1(T, Z), \cdots, \sigma^p(T, Z))$ for simplicity.

Lemma 3. *For any $\alpha \in R$ and $(\sigma^1, \cdots, \sigma^p) \in S^p$,*

$$L^i(\{\pi(\sigma) = 1\}; \alpha) \leq L^i(\{\pi(\sigma^{-i} \| \sigma^i[\alpha]) = 1\}; \alpha).$$

Proof. Since π is monotone, we have $\pi(\sigma^1, \cdots, \sigma^{i-1}, 1, \sigma^{i+1}, \cdots, \sigma^p) \geq \pi(\sigma^1, \cdots, \sigma^p) \geq \pi(\sigma^1, \cdots, \sigma^{i-1}, 0, \sigma^{i+1}, \cdots, \sigma^p)$ for all $\sigma \in S^p$. Thus, from the definition of $\sigma^{-i} \| \sigma^i[\alpha]$, it follows that

$$\{Z^i - \alpha \geq 0, \pi(\sigma^{-i} \| \sigma^i[\alpha]) = 1\} \supset \{Z^i - \alpha \geq 0, \pi(\sigma) = 1\} \text{ and}$$

$$\{Z^i - \alpha < 0, \pi(\sigma^{-i} \| \sigma^i[\alpha]) = 1\} \subset \{Z^i - \alpha < 0, \pi(\sigma) = 1\},$$

which implies

$$E[(Z^i - \alpha) I_{\{\pi(\alpha)=1\}} \leq E[(Z^i - \alpha) I_{\{\pi(\sigma^{-i} \| \sigma^i[\alpha])=1\}}.$$

So, the proof is completed by noting $L^i(A; \alpha) = E([Z^i - \alpha) I_A] + \alpha$ for all $A \in \mathcal{F}$. Q.E.D.

Lemma 4. *For any $\alpha, \beta \in R$, with $\alpha \geq \beta$ and $(\sigma^1, \cdots, \sigma^p) \in S^p$,*

$$L^i(\{\pi(\sigma^{-i} \| \sigma^i[\alpha]) = 1\}; \alpha) \geq L^i(\{\pi(\sigma^{-i} \| \sigma^i[\beta]) = 1\}; \beta). \tag{3.5}$$

Proof. By Lemma 3, it holds that

$$L^i(\{\pi(\sigma^{-i} \| \sigma^i[\alpha]) = 1\}; \alpha) \geq L^i(\{\pi(\sigma^{-i} \| \sigma^i[\beta]) = 1\}; \alpha). \tag{3.6}$$

Since $\alpha \geq \beta$, then

$$L^i(\{\pi(\sigma^{-i}\|\sigma^i[\beta]) = 1\}; \alpha) \geq L^i(\{\pi(\sigma^{-i}\|\sigma^i[\beta]) = 1\}; \beta),$$

so that (3.5) follows from (3.6). Q.E.D.

Lemma 5. *For any fixed i and any strategies $\sigma = (\sigma^1, \cdots, \sigma^p) \in S^p$, let us consider the following integral equation with respect to $v(T) := v^i(T)$, for simplicity :*

$$v(T) = L^i(\{\pi(\sigma^{-i}\|\sigma^i[G \circ v(T)]) = 1\}; G \circ v(T)) \tag{3.7}$$

for $T \geq 0$. Then, we have :

(i) *The solution $v(T)$ exists uniquely in $L^1([0,\infty), dG)$, and*

(ii) *$v(T) \geq u^i(T;\sigma)$, for $T \geq 0$.*

Proof. First we shall show the uniqueness of the solution of (3.7). Let $\alpha = G \circ v(T)$ and $\alpha' = G \circ v'(T)$, where $v(T)$ and $v'(T)$ are two solutions of (3.7) in $L^1([0,\infty), dG)$. We generally assume $\alpha \geq \alpha'$.

Then, since

$$\pi(\sigma^{-i}\|\sigma^i[\alpha]) = \pi(\sigma^{-i}\|\sigma^i[\alpha']) \quad \text{on} \quad \{Z^i \leq \alpha'\} \cup \{Z^i > \alpha\},$$

we have :

$$\begin{aligned} & E\left[Z^i I_{\{\pi(\sigma^{-i}\|\sigma^i[\alpha])=1\}}\right] - E\left[Z^i I_{\{\pi(\sigma^{-i}\|\sigma^i[\alpha'])=1\}}\right] \\ &= E\left[Z^i I_{\{\alpha' < Z^i \leq \alpha, \pi(\sigma^{-i}\|\sigma^i[\alpha])=1\}}\right] - E\left[Z^i I_{\{\alpha' < Z^i \leq \alpha, \pi(\sigma^{-i}\|\sigma^i[\alpha'])=1\}}\right] \\ &\leq \alpha\, P\{\alpha' < Z^i \leq \alpha, \pi(\sigma^{-i}\|\sigma^i[\alpha]) = 1\} - \alpha' P\{\alpha' < Z^i \leq \alpha, \pi(\sigma^{-i}\|\sigma^i[\alpha']) = 1\} \\ &\leq \alpha\, P\{\pi(\sigma^{-i}\|\sigma^i[\alpha]) = 1\} - \alpha' P\{\pi(\sigma^{-i}\|\sigma^i[\alpha']) = 1\}. \end{aligned}$$

It follows from (3.7) that

$$v(T) - v'(T) \leq \alpha - \alpha'$$

Thus, we have from Lemma 4

$$0 \leq v(T) - v'(T) \leq G \circ v(T) - G \circ v'(T),$$

which implies

$$|v(T) - v'(T)| \leq \int_0^T |v(T-s) - v'(T-s)|\, G(ds) \quad \text{for all} \quad T \geq 0.$$

By the well-known Gronwall-Bellman's theorem (see, for example, Bellman [1]). we obtain the result

$$v(T) = v'(T) \quad \text{for all} \quad T \geq 0$$

in $L^1([0,\infty), dG)$.

Next, we shall show the existence of the solution of (3.7). For any strategy σ it holds from Lemma 2 that

$$(3.8) \qquad u^i(T;\sigma) = L^i(\{\pi(\sigma(T,Z)) = 1\}; G \circ u^i(T;\sigma)).$$

Now putting $\alpha_1^i = G \circ u^i(T,\sigma)$ and $\sigma_1^i = \sigma^i[\alpha_1^i]$, we define

$$u_1^i(T) = L^i(\{\pi(\sigma^{-i} \| \sigma_1^i) = 1\}; \alpha_1^i).$$

Then, we observe from Lemma 3 that

$$(3.9) \qquad u_1^i(T) \geq u_1^i(T;\sigma).$$

If we define, recursively, for each $n \geq 2$,

$$(3.10) \qquad u_n^i(T) = L^i(\{\pi(\sigma^{-i} \| \sigma_n^i) = 1\}; \alpha_n^i),$$

$$\alpha_n^i = G \circ u_{n-1}^i(T) \text{ and } \sigma_n^i = \sigma^i[\alpha_n^i],$$

we see that

$$(3.11) \qquad u_n^i(T) \geq u_{n-1}^i(T)$$

by applying Lemma 4. Hence by the monotone convergence theorem, when $n \to \infty$ in (3.10), it holds that the limit $u^i(T) := \lim_{n\to\infty} u_n^i(T)$ equals a solution $v(T) := v^i(T)$ of the equation (3.7) in $L^1([0,\infty), dG)$. Clearly, (ii) holds from (3.9) and (3.11). Q.E.D.

Condition 1. There are $v^i(T) \in L^1([0,\infty), dG), i = 1, \cdots, p$, which satisfy the following p simultaneous integral equations:

$$(3.12) \qquad v^i(T) = L^i(\{\pi(^*\sigma) = 1; G \circ v^i(T)), \quad i = 1, \cdots, p \, (T \geq 0)$$

where $^*\sigma = (^*\sigma^1, \cdots, ^*\sigma^p)$ and $^*\sigma^i = {}^*\sigma^i(T,Z) = \sigma^i[G \circ v^i(T)]$.

We are now ready to prove the main theorem.

Theorem 1. *Under Condition 1, it holds that*

(i) $u^i(T, {}^*\sigma) = v^i(T), \quad i = 1, \cdots, p,$ *for* $T \geq 0$.

(ii) $^*\sigma$ *is an equilibrium strategy.*

Proof . (i) By Lemma 2, $u^i(T; {}^*\sigma)$ satisfies the equation (3.12). Thus, from (i) of Lemma 5, the uniqueness of the solution of (3.12) implies (i) of Theorem 1. Also, (ii) follows from (ii) of Lemma 5. Q.E.D.

Remark 1. Theorem 1 says that under Condition I there exists an equilibrium strategy of control-limit-type, whose threshold for each player i is $\alpha^i = G \circ v^i(t)$ when the remaining time interval until termination is t.

Remark 2. In most cases the direct verification of Condition 1 seems to be difficult. However, if $G(T) < 1$ for all $T > 0$ as the case that $G(ds) = \lambda e^{-\lambda s} ds, \lambda > 0$ (exponential distribution), (3.12) has a unique solution $v(T) = (v^1(T), \cdots, v^p(T))$ in $L^\infty[0,\infty)^p$, where $L^\infty[0,\infty)$ denotes the set of all bounded Borel measurable function on $[0,\infty)$. This result is used as the examples in the next section. In fact, we define the map $U : L^\infty[0,\infty)^p \to L^\infty[0,\infty)^p$ by

$$Uu(T) = (L^1(\pi\{\sigma(u(T))) = 1\}; G \circ u^1(T), \cdots L^p(\pi\{\sigma(u(T))) = 1\}; G \circ u^p(T)),\ (t \geq 0)$$

where $u(T) = (u^1(T), \cdots, u^p(T)) \in L^\infty[0,\infty)^p$ and $\sigma(u(T)) = (\sigma^1[G \circ u^1(T)], \cdots, \sigma^p[G \circ u^p(T)])$. Then, by the same way as the proof of Lemma 5, we get

$$||Uu - Uu'||_T \leq G(T)||u - u'||_T \quad \text{for any } u, u' \in L^\infty[0,\infty)^p \text{ and } T \geq 0,$$

where $||u||_T = \max_{1\leq i\leq p} \sup_{0\leq t\leq T} |u^i(t)|$ for $u(T) = (u^1(T), \cdots, u^p(T)) \in L^\infty[0,\infty)^p$.

The above discussion shows that U is a contraction w.r.t. $||\cdot||_T$, so that U has a unique fixed point $v_T \in L^\infty[0,\infty)^p$. Since T is arbitrary, $v := \lim_{T\to\infty} v_T$ satisfies (3.12). Also, the uniqueness of the solution of (3.12) follows from Lemma 5.

Remark 3. If the observation cost is incurred at each arrival time of offers, a p-dimensional random vector (net profit) $Z_n = (Z_n^1, \cdots, Z_n^p)$ is defined by

$$Z_n = Y_n - (n+1)c,$$

where $Y_n = (Y_n^1, \cdots, Y_n^p)$ are i.i.d. with a common distribution F on R^p and $c = (c^1, \cdots, c^p)$ is a constant observation cost.

The corresponding p-simultaneous integral equations for (3.12) reduces

$$v^i(T) + c^i = L^i\left(\{\pi(^*\sigma) = 1\}; G \circ v^i(T)\right), \quad i = 1, \cdots, p\, (T \geq 0). \tag{3.13}$$

Then, we can prove in the identical fashion that for a solution $v^i(T) \in L^1([0,\infty), dG), i = 1, \cdots, p$, of (3.13) the same theorem as Theorem 1 holds.

Remark 4. When G is a degenerate distribution with total mass at unity, the integral equation (3.13) becomes

$$v^i(T) + c^i = L^i(\{\pi(^*\sigma) = 1\}; v^i(T-1)),\ i = 1, \cdots, p, \tag{3.14}$$

where $^*\sigma = (^*\sigma^1, \cdots, ^*\sigma^p)$ and $^*\sigma^i(T, Z) = {}^*\sigma^i[v^i(T-1)]$. Thus, if we define a sequence $\{v^i(n); n = 0, 1, \cdots, , i = 1, \cdots, p$, recursively, by

$$\begin{aligned} &v^i(0) = E[Z^i] \\ &v^i(n) = L^i(\{\pi(^*\sigma) = 1\}; v^i(n-1)),\ n = 1, 2, \cdots \end{aligned} \tag{3.15}$$

for each i, then we observe that

$$v^i(T) = v^i(n) \quad \text{if} \quad n \leq T < n+1 \quad \text{for some } n.$$

Assuming that $c^i > 0$ for all i, it follows that $v^i := \lim_{T\to\infty} v^i(T)$ exists. Now, as $T \to \infty$ in (3.14), we obtain

$$\begin{aligned} &E[(Z^i - v^i)^+ P(\{\pi(^*\sigma^{-i}\|1) = 1\}|Z^i)] \\ &\quad -E[(Z^i - v^i)^- P(\{\pi(^*\sigma^{-i}\|0 = 1\}|Z^i)] = c^i, \end{aligned} \tag{3.16}$$

$i = 1, 2, \cdots, p$, where $^*\sigma^{-i}\|k = (^*\sigma^1, \cdots, ^*\sigma^{i-1}, k, ^*\sigma^{i+1}, \cdots, ^*\sigma^p)$ for each $k = 0, 1$. We note that (3.16) is corresponding to (3.1) of [10].

4 Examples

In this section, we provide some examples involving the two-person stopping problem(p=2) with the unanimity and simple majority rule as the typical ones of a monotone logical rule.

Example 1. (The unanimity rule.) Let us consider a unanimity rule $\pi(\sigma^1, \sigma^2), \sigma^1, \sigma^2 \in \{0, 1\}$ defined by $\pi(\sigma^1, \sigma^2) = 1$ if $\sigma^1 = \sigma^2 = 1, = 0$ otherwise. Then the integral equation (3.12) of $v^i(T)$ becomes

$$(4.1) \qquad v^i(T) = \overline{v}^i_T + \iint_D (z^i - \overline{v}^i_T) F(dz^1, dz^2), \ i = 1, 2 \ (T \geq 0),$$

where $D = \{(z^1, z^2); z^1 \geq \overline{v}^1_T \text{ and } z^2 \geq \overline{v}^2_T\}$. The equilibrium strategy is of control-limit type and the threshold of player i is $\overline{v}^i_T = G \circ v^i(T) = \int_{(0,T]} v^i(T-s)\, G(ds), i = 1, 2.$

If Z^1 and Z^2 are i.i.d. with a common distribution $F(z)$, then $v(T) := v^1(T) = v^2(T)$ for all $T \geq 0$, (4.1) becomes

$$(4.2) \qquad v(T) = \overline{v}_T + (1 - F(\overline{v}_T)) \int_{\overline{v}_T}^{\infty} (1 - F(z))\, dz,$$

where $\overline{v}_T = \int_{(0,T]} v(T-s)\, G(ds)$.

Now suppose $G(ds) = \lambda e^{-\lambda s} ds, \lambda > 0$, that is, the time interval between successive offers is exponentially distributed. Then since $\frac{d\overline{v}_T}{dT} = \lambda(v(T) - \overline{v}_T)$, we have the following differential equation from (4.2) .

$$(4.3) \qquad \frac{d\overline{v}_T}{dT} = \lambda(1 - F(\overline{v}_T)) \int_{\overline{v}_T}^{\infty} (1 - F(z))\, dz,$$

which corresponds to (10) of Sakaguchi[7].

Example 2. (The simple majority rule.)

A simple majority rule $\pi(\sigma^1, \sigma^2), \sigma^1, \sigma^2 \in \{0, 1\}$ for $p = 2$ is defined by $\pi(\sigma^1, \sigma^2) = 1$ if $\sigma^1 + \sigma^2 \geq 1, = 0$ otherwise. If Z^1 and Z^2 are non-negative and i.i.d. with a common distribution $F(z)$, then $v(T) := v^1(T) = v^2(T)$, and (3.13) becomes

$$(4.4) \qquad v(T) = \overline{v}_T + \iint_D (z^1 - \overline{v}_T) F(dz^1) F(dz^2),$$

where $D = \{(z^1, z^2); z^1 \geq \overline{v}_T \text{ or } z^2 \geq \overline{v}_T\}$. The equilibrium strategy is of control-limit type and the threshold of player i is $\overline{v}_T = G \circ v(T) = \int_{(0,T]} v(T-s)\, G(ds)$. Then, we have from Assumption 1 that $\mu_F = \int_R x F(dx) < \infty$,

$$v(T) = \mu_F - F(\overline{v}_T) \int_{[0,\overline{v}_T]} z\, F(dz) + \overline{v}_T \{F(\overline{v}_T)\}^2.$$

Since $\int_{[0,y]} z\, F(dz) = -y[1-F(y)] + \int_0^y (1-F(z))\, dz$ for all $y > 0$, we obtain that

$$v(T) = \mu_F + F(\overline{v}_T) \int_0^{\overline{v}_T} F(z)\, dz,$$

so that

$$\overline{v}_T = \int_{(0,T]} G(ds) \left\{ \mu_F + F(\overline{v}_{T-s}) \int_0^{\overline{v}_{T-s}} F(z)\, dz \right\}. \tag{4.5}$$

Next, suppose $G(ds) = \lambda e^{-\lambda s}\, ds$. Then, by elementary calculus, (4.5) becomes

$$\overline{v}_T \cdot \exp\{\lambda T\} = \lambda \int_0^T e^{\lambda s}\, ds \left\{ \mu_F + F(\overline{v}_s) \int_0^{\overline{v}_s} F(z)\, dz \right\}. \tag{4.6}$$

By taking the derivative of both sides of (4.6) with respect to T, we have the following differential equation:

$$\frac{d\overline{v}_T}{dT} = \lambda \left\{ \mu_F - \overline{v}_T + F(\overline{v}_T) \int_0^{\overline{v}_T} F(z)\, dz \right\}. \tag{4.7}$$

Thus, by rewriting equation (4.7),

$$\left\{ \mu_F - \overline{v} + F(\overline{v}) \int_0^{\overline{v}} F(z)\, dz \right\}^{-1} d\overline{v} = \lambda\, dt,$$

from which we obtain the inverse function of $\overline{v}_T$, $T(\overline{v})$, given by

$$T(\overline{v}) = \lambda^{-1} \int_0^{\overline{v}} M^{-1}(\xi)\, d\xi, \tag{4.8}$$

where $M(\xi) = \mu_F - \xi + F(\xi) \int_0^{\xi} F(z)\, dz$.

For a numerical example, supposing $F(z) = z$, $0 \le z \le 1$, from (4.8) we get

$$T(\overline{v}) = 2\lambda^{-1} \int_0^{\overline{v}} 1/\{(\xi - 1)(\xi^2 + \xi - 1)\}\, d\xi,$$

so that

$$\lim_{T\to\infty} \overline{v}_T = (\sqrt{5}-1)/2 (\approx 0.6180) \quad \text{and} \quad \lim_{T\to\infty} v(T) = (\sqrt{5}-1)/2,$$

which are the threshold of the control-limit strategy and the expected reward for the infinite horizon problem (refer Table 3.1 of [10] on a discrete time case) .

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