The Time Average Reward for Some Dynamic Fuzzy Systems

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Abstract

In this paper, by using a fuzzy relation, we define a dynamic fuzzy system with a bounded convex fuzzy reward on the positive orthant $\mathbf{R}_+^n$ of an n-dimensional Euclidean space. As a measure of the system's performance we introduce the time average fuzzy reward, which is characterized by the limiting fuzzy state under the contractive properties of the fuzzy relation. In one-dimensional case, the average fuzzy reward is expressed explicitly by the functional equations concerning the extreme points of its α-cuts. Also, a numerical example is given to illustrate the theoretical results.

Keywords: dynamic fuzzy system, time average fuzzy reward, contractive properties, fuzzy relational equation.

1. Introduction and notations

In the previous papers, Kurano etc. [4], [11], [13], [14], we have defined a dynamic fuzzy system using a fuzzy relation and proved a limit theorem for transition of fuzzy states under the contractive properties of the fuzzy relation. Here, the dynamic fuzzy system will be extended to the one with a bounded fuzzy reward on the positive orthant $\mathbf{R}_+^n$ of an n-dimensional Euclidean space and the time average fuzzy reward is introduced as a measure of the system's performance and characterized by the limiting fuzzy state or by various fuzzy relational equations. For sequential decision analyses in a fuzzy environment, refer to Bellman and Zadeh[1], Esogbue and Bellman[2], Kurano etc.[5].

Let X and Y be convex subsets of some Banach space. We denote by $\mathcal{C}(X)$ the collection of all compact convex subsets of X and ρ be the Hausdorff metric on $\mathcal{C}(X)$. Throughout this paper we denote a fuzzy set on X by its membership function $\tilde{s} : X \mapsto [0,1]$ and its α-cut by $\tilde{s}_\alpha$. For the details, refer to Zadeh[15], Novák[9] and the previous our papers.

A fuzzy set $\tilde{s}$ on X is called convex if

$$\tilde{s}(\lambda x + (1-\lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad \text{for any } x, y \in X \text{ and } \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$ for real numbers a and b. Also, a fuzzy relation $\tilde{p}$ defined on $X \times Y$ is called convex if

$$\tilde{p}(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \geq \tilde{p}(x_1, y_1) \wedge \tilde{p}(x_2, y_2)$$

for any $x_1, x_2 \in X, y_1, y_2 \in Y$, and $\lambda \in [0,1]$.

Let $\mathcal{F}(X)$ be the set of all convex fuzzy sets $\tilde{s}$ on X, which are upper semi-continuous and have a compact support. Clearly $\tilde{s} \in \mathcal{F}(X)$ implies $\tilde{s}_\alpha \in \mathcal{C}(X)$ for all $\alpha \in [0,1]$. The addition and the multiplicative operation of fuzzy sets are defined as follows (see Madan etc.[7]) : For any $\tilde{s}, \tilde{v} \in \mathcal{F}(\mathbf{R}_+^n)$ and $\lambda \in \mathbf{R}_+ := [0, \infty)$,

$$(\tilde{s} + \tilde{v})(x) := \sup_{y,z \in \mathbf{R}_+^n : y+z=x} \{\tilde{s}(y) \wedge \tilde{v}(z)\} \quad x \in \mathbf{R}_+^n \tag{1.1}$$

and

$$(\lambda \tilde{s})(x) := \tilde{s}(x/\lambda) \quad \text{if} \quad \lambda > 0, \quad x \in \mathbf{R}_+^n, \tag{1.2}$$

and $(\lambda \tilde{s})(x) := I_{\{0\}}(x)$ if $\lambda = 0$,where $I_A(\cdot)$ is the indicator function of a subset A of $\mathbf{R}_+^n$.

It is easily seen that, for $\alpha \in [0,1]$,

$$(\tilde{s} + \tilde{v})_\alpha = \tilde{s}_\alpha + \tilde{v}_\alpha \quad \text{and} \quad (\lambda \tilde{s})_\alpha = \lambda \tilde{s}_\alpha,$$

where $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any subsets A, B of $\mathbf{R}_+^n$.

Lemma 1.1.(Chen-wei Xu [3])

(i) *For any* $\tilde{s}, \tilde{v} \in \mathcal{F}(\mathbf{R}_+^n)$ *and* $\lambda \in [0, \infty)$,

$$\tilde{s} + \tilde{v} \in \mathcal{F}(\mathbf{R}_+^n) \quad \textit{and} \quad \lambda \tilde{v} \in \mathcal{F}(\mathbf{R}_+^n).$$

(ii) *Let* $\tilde{p}$ *be any lower semi-continuous convex fuzzy relation on* $X \times Y$*. Then*

$$\sup_{x \in X} \tilde{s}(x) \wedge \tilde{p}(x, \cdot) \in \mathcal{F}(Y) \quad \textit{for all } \tilde{s} \in \mathcal{F}(X).$$

Here, we give the notion of convergence for a sequence of fuzzy sets, which is used in Section 2.

Definition 1.1 (Kurano etc.[4], Nanda[8]). Let $\{\tilde{v}_t\}_{t=0}^{\infty}$ be a sequence fuzzy sets in $\mathcal{F}(X)$. Then we write $\tilde{v}_t \to \tilde{v} \in \mathcal{F}(X)$ as $t \to \infty$ if

$$\lim_{t\to\infty} \sup_{\alpha\in[0,1]} \rho(\tilde{v}_{t,\alpha}, \tilde{v}_\alpha) = 0, \tag{1.3}$$

where $\tilde{v}_{t,\alpha}$ and $\tilde{v}_\alpha$ are α-cuts of $\tilde{v}_t$ and $\tilde{v}$ respectively.

Note that for a sequence of sets $\{A_t\}_{t=1}^{\infty} \subset \mathcal{C}(X)$ and $A \in \mathcal{C}(X)$,

$$\lim_{t\to\infty} A_t = A$$

means that $\overline{\lim}_{t\to\infty} A_t = \underline{\lim}_{t\to\infty} A_t = A$, where

$$\overline{\lim_{t\to\infty}} A_t := \{z \in X \mid \underline{\lim_{t\to\infty}} d(z, A_t) = 0\},$$

$$\underline{\lim_{t\to\infty}} A_t := \{z \in X \mid \overline{\lim_{t\to\infty}} d(z, A_t) = 0\},$$

$d(z, D) := \min_{z'\in D} d(z, z')$ $\quad D \in \mathcal{C}(X)$ and d is a metric on X. It is known (Kuratowski[6]) that $\lim_{t\to\infty} \rho(A_t, A) = 0$ iff $\lim_{t\to\infty} A_t = A$, so that $\tilde{v}_t$ converges to $\tilde{v}$ as $t \to \infty$ in the sense of (1.3) means that $\lim \tilde{v}_{t,\alpha} = \tilde{v}_\alpha$ uniformly for $\alpha \in [0, 1]$.

Now, extending a discrete dynamic fuzzy system in Kurano etc. [4], [11], [13], [14], we consider the one with a fuzzy reward, which is characterized with the elements $(S, \tilde{q}, \tilde{r}, \tilde{s})$ as follows :

(i) The state space S is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state denoted as an element of $\mathcal{F}(S)$.

(ii) The law of the motion and the fuzzy reward for the system are denoted by the time invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0, 1]$ and $\tilde{r} : S \times [0, M]^n \mapsto [0, 1]$ respectively, where M is a fixed positive number and n is a positive integer. We assume that $\tilde{q} : S \times S \mapsto [0, 1]$ and $\tilde{r} : S \times [0, M]^n \mapsto [0, 1]$ are convex and continuous.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a fuzzy reward $R(\tilde{s})$ is incurred and we move to a new fuzzy state $Q(\tilde{s})$, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ and $R : \mathcal{F}(S) \mapsto \mathcal{F}([0, M]^n)$ are defined by

$$R(\tilde{s})(z) := \sup_{x\in S} \tilde{s}(x) \wedge \tilde{r}(x, z) \quad z \in [0, M]^n \tag{1.4}$$

and

$$Q(\tilde{s})(y) := \sup_{x\in S} \tilde{s}(x) \wedge \tilde{q}(x, y) \quad y \in S. \tag{1.5}$$

Note that by Lemma 1.1(ii) the maps R and Q are well-defined.

(iii) The initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$ is arbitrary.

For the dynamic fuzzy system $(S, \tilde{q}, \tilde{r}, \tilde{s})$, we can define a sequence of fuzzy rewards on $[0, M]^n$, $\{R(\tilde{s}_t)\}_{t=0}^{\infty}$, where

$$\tilde{s}_0 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \ \ (t \geq 0). \tag{1.6}$$

In Section 2, we define the time average fuzzy reward, which is characterized by the limiting fuzzy state under the contractive properties of the fuzzy relation $\tilde{q}$.

In Section 3, the one-dimensional case is treated and by introducing relative value functions the average fuzzy reward is expressed by the functional equations concerning the extreme points of its α-cuts.

Also, a numerical example is given to illustrate the theoretical results in this paper.

2. The average fuzzy reward

In this paper we specify the time average reward as a measure of the system's performance and discuss its characterization under the contractive assumption given in Kurano etc.[4].

We define the total T-time fuzzy reward $\tilde{R}_T(\tilde{s})$ by

$$\tilde{R}_T(\tilde{s}) := \sum_{t=0}^{T-1} R(\tilde{s}_t) \qquad T \geq 1, \tag{2.1}$$

where $\{\tilde{s}_t\}_{t=0}^{\infty}$ is given in (1.6).

Associated with the fuzzy relation $\tilde{q}$ and fuzzy reward $\tilde{r}$, are the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ $(\alpha \in [0, 1])$ and $R_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}([0, M]^n)$ $(\alpha \in [0, 1])$ defined as follows : For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \alpha > 0 \\ cl\{y \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \alpha = 0, \end{cases} \tag{2.2}$$

and

$$R_\alpha(D) := \begin{cases} \{z \in [0, M]^n \mid \tilde{r}(x, z) \geq \alpha \text{ for some } x \in D\} & \alpha > 0 \\ cl\{z \in [0, M]^n \mid \tilde{r}(x, z) > 0 \text{ for some } x \in D\} & \alpha = 0. \end{cases} \tag{2.3}$$

The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$ (identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \qquad t \geq 0.$$

We have the following lemma, which is easily verified by the ideas in the proof of Kurano etc.[4, Lemma 1].

Lemma 2.1.

(i) $\tilde{R}_T(\tilde{s}) \in \mathcal{F}([0, TM]^n)$ *for* $T \geq 1$.

(ii) $\tilde{s}_{t,\alpha} = Q^t_\alpha(\tilde{s}_\alpha)$ *for* $t \geq 0$, *where* $\tilde{s}_{t,\alpha} = (\tilde{s}_t)_\alpha$.

(iii) $(\tilde{R}_T(\tilde{s}))_\alpha = \sum_{t=0}^{T-1} R_\alpha(\tilde{s}_{t,\alpha})$ *for* $T \geq 1$.

From Lemma 2.1(ii),(iii), the α-cut of rewards, $\tilde{R}_T(\tilde{s})$, can be calculated only through $\tilde{s}_\alpha$. So we denote it as

$$\tilde{R}_{T,\alpha}(\tilde{s}_\alpha) := (\tilde{R}_T(\tilde{s}))_\alpha \quad \text{for } T \geq 0 \text{ and } \alpha \in [0,1].$$

From this α-cut set of $\tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$ we try to estimate the increasing amount of fuzzy reward per unit time.

For $K > 0$ and $\alpha \in [0,1]$, we define

$$G_{K,\alpha} := \left\{ r \in \mathbf{R}^n_+ \;\middle|\; \begin{array}{l} \text{there exists } \{z_T\}_{T=1}^\infty \text{ such that} \\ z_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|z_T - rT\| \leq K \text{ for all } T \geq 1 \end{array} \right\}. \tag{2.4}$$

The properties of $G_{K,\alpha}$ are formulated in the following lemma. The proof is omitted.

Lemma 2.2. *Let* $K > 0$. *Then :*

(i) $\{G_{K,\alpha} | \alpha \in [0,1]\} \subset \mathcal{C}(\mathbf{R}^n_+)$.

(ii) $G_{K,\alpha} \subset G_{K,\alpha'}$ *for* $0 \leq \alpha' \leq \alpha \leq 1$.

(iii) $\lim_{\alpha' \uparrow \alpha} G_{K,\alpha'} = G_{K,\alpha}$ *for* $\alpha \in (0,1]$, *i.e.,* $\lim_{\alpha' \uparrow \alpha} \delta(G_{K,\alpha'}, G_{K,\alpha}) = 0$.

From Kurano etc.[4, Lemma 3], we can define a fuzzy number

$$\tilde{g}(\tilde{s})(r) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{G_{K,\alpha}}(r)\} \quad r \in [0, M]^n \quad \text{for } \tilde{s} \in \mathcal{F}(S). \tag{2.5}$$

Then, $\tilde{g}(\tilde{s}) \in \mathcal{F}([0, M]^n)$ and $(\tilde{g}(\tilde{s}))_\alpha = G_{K,\alpha}$ for all $\alpha \in [0,1]$.

We call $\tilde{g}(\tilde{s})$ *an average fuzzy reward* for the dynamic fuzzy systems, which depends on the initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$ with suppression of K. In the remainder of this section, we will investigate the average fuzzy reward from the limiting behavior of the fuzzy states. The following lemma is useful in the sequel.

Lemma 2.3. *Let $\{D_t\}_{t=1}^{\infty} \subset \mathcal{C}(S)$ and $D \in \mathcal{C}(S)$ such that $\lim_{t\to\infty} D_t = D$. Let $\alpha \in (0,1]$. For any ϵ $(\alpha > \epsilon > 0)$, there exists $T \geq 1$ such that*

$$R_{\alpha-\epsilon}(D) \supset R_{\alpha}(D_t) \quad \text{for all } t \geq T.$$

Proof. Suppose that for some ϵ $(\alpha > \epsilon > 0)$, there exist sequences $\{t_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ such that

$$t_k \to \infty \ (k \to \infty), \quad \text{and} \quad z_k \in R_{\alpha}(D_{t_k}) \setminus R_{\alpha-\epsilon}(D) \ (k = 1, 2, \cdots).$$

Then we have

$$\tilde{r}(x, z_k) < \alpha - \epsilon \quad \text{for all } x \in D, k = 1, 2, \cdots, \tag{2.10}$$

and there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that

$$x_k \in D_{t_k} \quad \text{and} \quad \tilde{r}(x_k, z_k) \geq \alpha \quad \text{for } k = 1, 2, \cdots. \tag{2.11}$$

From the compactness, we may assume that the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ are convergent. We put the limits $x^* = \lim_{k\to\infty} x_k$ and $z^* = \lim_{k\to\infty} z_k$. Then we have $x^* \in D$ since $\lim_{t\to\infty} D_t = D$. From (2.10) and (2.11), we obtain

$$\tilde{r}(x^*, z^*) \geq \alpha \quad \text{and} \quad \tilde{r}(x, z^*) \leq \alpha - \epsilon \quad \text{for all } x \in D.$$

It is a contradiction. Thus we get this lemma. □

In order to characterizing the average fuzzy reward $\tilde{g}(\tilde{s})$, we need the following two assumptions, the first one is a contractive property concerning the fuzzy relation $\tilde{q}$ which guarantee the existence of the limiting fuzzy state and the second is a Lipschitz condition related with the fuzzy reward $\tilde{r}$.

Assumption A. (Contraction and ergodic property)
There exists $t_0 \geq 1$ and β $(0 < \beta < 1)$ satisfying that

$$\rho(Q_{\alpha}^{t_0}(D_1), Q_{\alpha}^{t_0}(D_2)) \leq \beta\rho(D_1, D_2) \quad \text{for all } D_1, D_2 \in \mathcal{C}(S), \ \alpha \in [0,1].$$

Assumption B. (Lipschitz conditions)
There exists a constant $C > 0$ such that

$$\delta(R_{\alpha}(D_1), R_{\alpha}(D_2)) \leq C\rho(D_1, D_2) \quad \text{for all } D_1, D_2 \in \mathcal{C}(S), \ \alpha \in [0,1], \tag{2.12}$$

where δ is the Hausdorff metric on $\mathcal{C}([0, M]^n)$.

Lemma 2.4. (Kurano etc.[4, Theorem 1]) *Suppose that Assumption A holds.*

(i) *There exists a unique fuzzy state* $\tilde{p} \in \mathcal{F}(S)$, *which is independently of the initial fuzzy state* $\tilde{s}$, *satisfying*

$$\tilde{p}(y) = \max_{x \in S}\{\tilde{p}(x) \wedge \tilde{q}(x,y)\} \quad \text{for all } y \in S. \tag{2.13}$$

(ii) *For* $\alpha \in [0,1]$, *the* α*-cut* $\tilde{p}_\alpha$ *is a unique set of* $\mathcal{C}(S)$ *such that*

$$Q_\alpha(\tilde{p}_\alpha) = \tilde{p}_\alpha.$$

(iii) *Let* $\alpha \in [0,1]$. *It holds that*

$$\rho(Q_\alpha^t(D), \tilde{p}_\alpha) \leq \beta^{[t/t_0]} K_\alpha(D, \tilde{p}_\alpha) \quad \text{for all } D \in \mathcal{C}(S),\ t \geq 1,$$

where $K_\alpha(D,\tilde{p}_\alpha) := \sum_{l=0}^{t_0-1} \rho(Q_\alpha^l(D), \tilde{p}_\alpha)$ *and, for a real number* c, $[c]$ *is the largest integer equal to or less than* c.

Recently, Yoshida[12] has given the notion of α-recurrent set for the fuzzy relation and shown that the α-cut of the limiting fuzzy set $\tilde{p}$ in Lemma 2.4 is characterized as the maximum α-recurrent set.

Now, we can state one of main results, which shows that $\tilde{g}(\tilde{s})$ is represented using the limiting fuzzy state $\tilde{p}$.

Theorem 2.1. *Suppose that Assumptions A and B hold. For sufficient large all* K, *it holds that*

$$\tilde{g}(\tilde{s}) = R(\tilde{p}), \tag{2.14}$$

where $\tilde{p}$ *is the limiting fuzzy state given in Lemma 2.3. Further this is independent of the initial fuzzy state* $\tilde{s}$.

Proof. A rough sketch of the proof is as follows and the details are omitted. First we show that

$$(\tilde{g}(\tilde{s}))_\alpha = G_{K,\alpha} \subset R_\alpha(\tilde{p}_\alpha) = (R(\tilde{p}))_\alpha. \tag{2.15}$$

Suppose that there exists $r \in G_{K,\alpha} \backslash R_\alpha(\tilde{p}_\alpha)$. Then $r \notin R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ for some $\epsilon > 0$. Since $R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ is closed and convex, there exists a unique $z_0 \in R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ such that

$$0 < \gamma := \|z_0 - r\| \leq \|z - r\| \quad \text{for all } z \in R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha). \tag{2.17}$$

By Lemma 2.3, there exists $T^* > 0$ such that

$$R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha) \supset R_\alpha(\tilde{s}_{t,\alpha}) \quad \text{for all } t \geq T^*. \tag{2.18}$$

From $r \in G_{K,\alpha}$, there exists $\{r_T\}_{T=0}^\infty$ such that

$$r_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|r_T - rT\| \leq K \text{ for all } T \geq 1. \tag{2.19}$$

On the other hand, from Lemma 2.1(iii), there exists a sequence $\{r_{T,t}\}$ such that

$$r_{T,t} \in R_\alpha(\tilde{s}_{t,\alpha}) \ (t = 0, 1, 2, \cdots, T-1) \text{ and } r_T = \sum_{t=0}^{T-1} r_{T,t} \quad (T \geq 1). \tag{2.20}$$

Noting the supporting hyperplane of $R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ at z_0, we have

$$\langle z_0 - r, r_{T,t} - r \rangle \geq \|z_0 - r\|^2 = \gamma^2 \quad \text{for all } t, T \ (T > t \geq T^*)$$

and

$$\left\langle z_0 - r, \sum_{t=T^*}^{T-1} (r_{T,t} - r) \right\rangle \geq (T - T^*)\gamma^2 \quad \text{for all } T > T^*.$$

By Cauchy-Schwartz inequality,

$$\left\| \sum_{t=T^*}^{T-1} (r_{T,t} - r) \right\| \geq (T - T^*)\gamma \quad \text{for all } T > T^*. \tag{2.21}$$

After some calculations we see that

$$\|r_T - rT\| = \left\| \sum_{t=0}^{T-1} (r_{T,t} - r) \right\| \to \infty \quad (T \to \infty).$$

So this contradicts (2.19) and we obtain (2.15).

Next we prove

$$R_\alpha(\tilde{p}_\alpha) \subset G_{K,\alpha} \quad \text{for sufficient large all } K. \tag{2.22}$$

From Assumption B, we have

$$\delta(R_\alpha(\tilde{s}_{t,\alpha}), R_\alpha(\tilde{p}_\alpha)) \leq C\rho(\tilde{s}_{t,\alpha}, \tilde{p}_\alpha) \quad \text{for } t \geq 0. \tag{2.23}$$

Also, from Lemmas 2.1(ii) and 2.4(iii),

$$\rho(\tilde{s}_{t,\alpha}, \tilde{p}_\alpha) \leq \beta^{[t/t_0]} K_\alpha(\tilde{s}_\alpha, \tilde{p}_\alpha) \quad \text{for } t \geq 0. \tag{2.24}$$

Since S is compact, there exists a constant $C^* > 0$ such that

$$\delta(R_\alpha(\tilde{s}_{t,\alpha}), R_\alpha(\tilde{p}_\alpha)) \leq C^* \beta^t \quad \text{for } t \geq 0$$

by using (2.23),(2.24). Therefore, for any $r \in R_\alpha(\tilde{p}_\alpha)$, there exists $\{r_t\}_{t=0}^\infty$ such that

$$r_t \in R_\alpha(\tilde{s}_{t,\alpha}) \text{ and } \|r_t - r\| \leq C^* \beta^t \tag{2.25}$$

for $t \geq 0$. Then

$$\left\| \sum_{t=0}^{T-1} r_t - rT \right\| = \left\| \sum_{t=0}^{T-1} (r_t - r) \right\| \leq \sum_{t=0}^{T-1} \|r_t - r\| \leq \sum_{t=0}^{T-1} C^* \beta^t \leq C^*/(1-\beta)$$

for all $T \geq 1$. Thus we get $r \in G_{K,\alpha}$ for all $K \geq C^*/(1-\beta)$. Therefore (2.22) holds for all $K \geq C^*/(1-\beta)$. Together with (2.15), we get (2.14) for sufficient large all K. It is trivial that (2.14) is independent of the initial fuzzy state $\tilde{s}$ from Lemma 2.4(i). □

From now on we take $K \geq C^*/(1-\beta)$. The following corollary shows that $\tilde{g}(\tilde{s})$ is given as the limit of $\{R(\tilde{s}_t)\}_{t=0}^{\infty}$ by the method of Cesaro averaging. The proof is omitted.

Corollary 2.1. *Under the same condition as Theorem 2.1, it holds that*

$$\lim_{T\to\infty} \frac{1}{T} \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) = (\tilde{g}(\tilde{s}))_\alpha \quad \textit{for all } \alpha \in [0,1]. \tag{2.26}$$

3. One-Dimensional Case

In this section we consider the case of $n = 1$, i.e. $\tilde{r} \in \mathcal{F}(S \times [0, M])$, and characterize an average fuzzy reward $\tilde{g}(\tilde{s})$ by the functional equations concerning with the extremal points of its α-cuts. Throughout this section it is assumed that Assumptions A and B hold.

Since $\mathcal{C}([0, M])$ is the set of all closed intervals, we can write the map R_α : $\mathcal{C}(S) \mapsto \mathcal{C}([0, M])$ by the following notation :

$$R_\alpha(D) := [\min R_\alpha(D), \max R_\alpha(D)] \quad \text{for all } D \in \mathcal{C}(S). \tag{3.1}$$

Let

$$\tilde{R}_{T,\alpha}(D) := \sum_{t=0}^{T-1} R_\alpha(Q_\alpha^t(D)) \quad \text{for } D \in \mathcal{C}(S).$$

Then, by Lemma 2.1(iii), it holds that

$$\min \tilde{R}_{T,\alpha}(D) = \sum_{t=0}^{T-1} \min R_\alpha(Q_\alpha^t(D)) \tag{3.2}$$

and

$$\max \tilde{R}_{T,\alpha}(D) = \sum_{t=0}^{T-1} \max R_\alpha(Q_\alpha^t(D)), \tag{3.3}$$

where

$$\tilde{R}_{T,\alpha}(D) = [\min \tilde{R}_{T,\alpha}(D), \max \tilde{R}_{T,\alpha}(D)].$$

From Lemma 2.4(iii) and Assumption B we observe that $R_\alpha(Q_\alpha^t(D))$ converges to $R_\alpha(\tilde{p}_\alpha)$ exponentially first as $t \to \infty$. Thus, by (3.2) and (3.2),

$$\underline{h}_\alpha(D) := \lim_{T\to\infty} (\min \tilde{R}_{T,\alpha}(D) - T \times \min R_\alpha(\tilde{p}_\alpha)) \tag{3.4}$$

and

$$\overline{h}_\alpha(D) := \lim_{T\to\infty} (\max \tilde{R}_{T,\alpha}(D) - T \times \max R_\alpha(\tilde{p}_\alpha)) \tag{3.5}$$

converge for all $D \in \mathcal{C}(S)$. The function $\underline{h}_\alpha$ ($\overline{h}_\alpha$ resp.) is called *a lower (upper) relative value function*, whose basic ideas are appearing in the theory of Markov decision processes (c.f. [10]). By Theorem 2.1, we have

$$\tilde{g}(\tilde{p})_\alpha = [\min R_\alpha(\tilde{p}_\alpha), \max R_\alpha(\tilde{p}_\alpha)], \tag{3.6}$$

where the extremal points are characterized in the following theorem.

Theorem 3.1. *Let $\alpha \in [0,1]$. Then the following (i) and (ii) hold.*

(i) *Let $\underline{h}_\alpha$ and $\overline{h}_\alpha$ be defined by (3.4) and (3.5). Then, the following equations hold:*

$$\underline{h}_\alpha(D) + \min R_\alpha(\tilde{p}_\alpha) = \min R_\alpha(D) + \underline{h}_\alpha(Q_\alpha(D)) \tag{3.7}$$

and

$$\overline{h}_\alpha(D) + \max R_\alpha(\tilde{p}_\alpha) = \max R_\alpha(D) + \overline{h}_\alpha(Q_\alpha(D)) \tag{3.8}$$

for all $D \in \mathcal{C}(S)$.

(ii) *Conversely, if there exist bounded functions $\underline{h}_\alpha$ and $\overline{h}_\alpha$ on $\mathcal{C}(S)$ and constants $\underline{K}_\alpha$ and $\overline{K}_\alpha$ satisfying that*

$$\underline{h}_\alpha(D) + \underline{K}_\alpha = \min R_\alpha(D) + \underline{h}_\alpha(Q_\alpha(D)) \tag{3.9}$$

and

$$\overline{h}_\alpha(D) + \overline{K}_\alpha = \max R_\alpha(D) + \overline{h}_\alpha(Q_\alpha(D)) \tag{3.10}$$

for all $D \in \mathcal{C}(S)$, then $\tilde{g}(\tilde{s})_\alpha = [\underline{K}_\alpha, \overline{K}_\alpha]$.

Proof. (i) By the definition of (3.4), it implies

$$\begin{aligned}\underline{h}_\alpha(D) &= \lim_{T\to\infty} \textstyle\sum_{t=0}^{T-1} (\min R_\alpha(Q_\alpha^t(D)) - \min R_\alpha(\tilde{p}_\alpha)) \\ &= \min R_\alpha(D) - \min R_\alpha(\tilde{p}_\alpha) \\ &\quad + \textstyle\sum_{t=1}^{\infty} (\min R_\alpha(Q_\alpha^{t-1}(Q_\alpha(D))) - \min R_\alpha(\tilde{p}_\alpha)) \\ &= \min R_\alpha(D) - \min R_\alpha(\tilde{p}_\alpha) + \underline{h}_\alpha(Q_\alpha(D)),\end{aligned}$$

which leads to (3.7). Also, (3.8) can be shown analogously to (3.7).

(ii) Let $\underline{h}_\alpha(D)$ and $\underline{K}_\alpha$ be as in (3.9). Then, it holds that for each t ($t \geq 0$),

$$\underline{h}_\alpha(Q_\alpha^t(D)) + \underline{K}_\alpha = \min R_\alpha(Q_\alpha^t(D)) + \underline{h}_\alpha(Q_\alpha^{t+1}(D)). \tag{3.11}$$

By summing (3.11) for $t = 0, 1, \cdots, T-1$, we get

$$\underline{h}_\alpha(D) + T \times \underline{K}_\alpha = \sum_{t=0}^{T-1} \min R_\alpha(Q_\alpha^t(D)) + \underline{h}_\alpha(Q_\alpha^T(D)).$$

So

$$\underline{K}_\alpha = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \min R_\alpha(Q_\alpha^t(D)) \quad \text{for } D \in \mathcal{C}(S).$$

Thus, from Theorem 2.1 and Corollary 2.1,

$$\underline{K}_\alpha = \min R_\alpha(\tilde{p}_\alpha).$$

We also obtain $\overline{K}_\alpha = \max R_\alpha(\tilde{p}_\alpha)$ similarly. Therefore we get $\tilde{g}(\tilde{s})_\alpha = [\underline{K}_\alpha, \overline{K}_\alpha]$ by (3.6). $\square$

Here we give a numerical example to illustrate the theoretical results in this section. Let $S := [0,1]$, $M := 1$. Take the fuzzy relation and the fuzzy reward by

$$\tilde{q}(x,y) = (1 - 3\,|y-x|\,/2) \vee 0, \quad x, y \in [0,1], \tag{3.12}$$

and

$$\tilde{r}(x,z) = (1 - 6\,|x-z|) \vee 0, \quad x, z \in [0,1]. \tag{3.13}$$

We observe that $\tilde{q}$ and $\tilde{r}$ satisfy Assumptions A for $t_0 = 1$, and B respectively. Let $\alpha \in [0,1]$. From (2.2) and (2.3),

$$Q_\alpha(\{x\}) = [(x/2 - (1-\alpha)/3) \vee 0, (x/2 + (1-\alpha)/3)]$$

for $x \in [0,1]$. So, for $0 \le a \le b \le 1$,

$$Q_\alpha([a,b]) = \bigcup_{x\in[a,b]} Q_\alpha(\{x\}) = [T_1(a), T_2(b)] \tag{3.14}$$

where maps $T_i; i = 1,2$ on $[0,1]$ are given by $T_1(x) := (x/2 - (1-\alpha)/3) \vee 0$, $T_2(x) := x/2 + (1-\alpha)/3$. Similarly we have

$$R_\alpha([a,b]) = [(a - (1-\alpha)/6) \vee 0, (b + (1-\alpha)/6) \wedge 1]. \tag{3.15}$$

A unique fixed point $\tilde{p}_\alpha$ of the map $Q_\alpha : \mathcal{C}([0,1]) \mapsto \mathcal{C}([0,1])$ is given as $\tilde{p}_\alpha = [\min \tilde{p}_\alpha, \max \tilde{p}_\alpha] = [0, 2(1-\alpha)/3]$, by solving $T_1(\min \tilde{p}_\alpha) = \min \tilde{p}_\alpha$ and $T_2(\max \tilde{p}_\alpha) = \max \tilde{p}_\alpha$ from (3.14). Therefore, from (3.15) and Theorem 2.1, we get $\tilde{g}(\tilde{s})_\alpha = R_\alpha([\min \tilde{p}_\alpha, \max \tilde{p}_\alpha]) = [0, 5(1-\alpha)/6]$. By (2.9), the average fuzzy reward is

$$\tilde{g}(\tilde{s})(x) = \begin{cases} 1 - 6x/5 & 0 \le x \le 5/6 \\ 0 & 5/6 < x \le 1. \end{cases} \tag{3.18}$$

Finally we calculate the lower and the upper relative value functions $\underline{h}_\alpha$ and $\overline{h}_\alpha$. We put

$$Q^t_\alpha([a,b]) = [T^t_1(a), T^t_2(b)] \quad \text{for } 0 \le a \le b \le 1 \text{ and } t \ge 0, \tag{3.19}$$

where maps $T^t_i; i = 1, 2, (t \ge 0)$ on $[0,1]$ are

$$T^0_i(x) = x, \quad T^{t+1}_i(x) = T_i T^t_i(x).$$

Then we can easily check

$$T^t_1(x) = \left(2^{-t}x - 2(1-\alpha)(1-2^{-t})/3\right) \vee 0 \quad \text{for } x \in [0,1]. \tag{3.20}$$

Similarly

$$T^t_2(x) := \left(2^{-t}x + 2(1-\alpha)(1-2^{-t})/3\right) \quad \text{for } x \in [0,1]. \tag{3.21}$$

Let $0 \le a \le b \le 1$. From (3.2) and (3.3), we get

$$\min \tilde{R}_{T,\alpha}([a,b]) = \sum_{t=0}^{T-1} \left\{ \left(T^t_1(a) - (1-\alpha)/6\right) \vee 0 \right\}$$

and

$$\max \tilde{R}_{T,\alpha}([a,b]) = \sum_{t=0}^{T-1} \left\{ \left(T^t_2(b) + (1-\alpha)/6\right) \wedge 1 \right\}.$$

From the definition of $\underline{h}_\alpha$ and $\overline{h}_\alpha$ and (3.17), the lower relative value function is

$$\begin{aligned} \underline{h}_\alpha(a) &:= \underline{h}_\alpha([a,b]) \\ &= \begin{cases} 2(1-2^{-t^*})(a + 2(1-\alpha)/3) - 5t^*(1-\alpha)/6 & \alpha < 1 \\ 2a & \alpha = 1, \end{cases} \end{aligned}$$

where t^* is the smallest non-negative integer such that

$$2^{-t^*}(a + 2(1-\alpha)/3) - 5(1-\alpha)/6 < 0.$$

And the upper relative value function is

$$\begin{aligned} \overline{h}_\alpha(b) &:= \overline{h}_\alpha([a,b]) \\ &= \begin{cases} 2b - 4(1-\alpha)/3 & \text{if } 0 \le b < (5+\alpha)/6 \\ b + (3\alpha - 1)/2 & \text{if } (5+\alpha)/6 \le b \le 1. \end{cases} \end{aligned}$$

When $\alpha = 1/2$, the lower and the upper relative value functions are

$$\underline{h}_\alpha(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/12 \\ x - 1/12 & \text{if } 1/12 \le x < 1/2 \\ 3x/2 - 1/3 & \text{if } 1/2 \le x \le 1 \end{cases}$$

and

$$\overline{h}_\alpha(x) = \begin{cases} 2x - 2/3 & \text{if } 0 \le x < 11/12 \\ x + 1/4 & \text{if } 11/12 \le x \le 1 \end{cases}$$

We also find that, when $\alpha = 1$,

$$\underline{h}_\alpha(x) = \overline{h}_\alpha(x) = 2x \quad \text{for } 0 \le x \le 1.$$

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Manag. Sci., Ser.B*, **17**(1970), 141-164.

[2] A.O.Esogbue and R.E.Bellman, Fuzzy dynamic programming and its extensions, *TIMS/Studies in Manag. Sci.*, **20** *,147-167, North-Holland, Amsterdam, 1984* .

[3] Chen-wei Xu, On convexity of fuzzy sets and fuzzy relations, *Infor. Sci.*, **59** (1992) 92-102.

[4] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[5] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Markov–Type Fuzzy Decision Processes with a Discounted Reward on a Closed Interval *To appear in European Journal of Operational Research.*

[6] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[7] L.Madan, L.Puri and Dan A.Palescu, The concept of normality for fuzzy random variables, *Ann. Prob.* **13** (1985) 1373-1379.

[8] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989) 123-126.

[9] V.Novák, *Fuzzy sets and their Applications* (Adom Hilger, Bristol-Boston, 1989).

[10] S.M.Ross, *Applied probability models with optimization applications* (Holden-Day, San Francisco, 1970).

[11] M.Yasuda, A Linear Structure and Convexity for Relations in Dynamic Fuzzy Systems, *Compu.Math.Applic.*, Vol.27, 23 – 27 (1994).

[12] Y.Yoshida, The recurrence of dynamic fuzzy systems, Submitted.

[13] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure: The contractive case. *Fuzzy Sets and Systems*,**60** (1993) 283-294.

[14] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure: The unbounded case. *Fuzzy Sets and Systems*,**66** (1994) 83-95.

[15] L.A.Zadeh, Fuzzy sets, *Inform. Control* **8** (1965) 338-353.