The Mean Value with Evaluation Measures and a Zero-Sum Stopping Game with Fuzzy Values

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Abstract. We firstly introduce an evaluation method of fuzzy numbers called the mean value with evaluation measures. Then, using this notion, a stopping game model with fuzzy random variables could be formulated. When a sequence of fuzzy-valued random variables(fuzzy RV) are observed, the important problem is how to treat and analyze the model. Previously the observed fuzzy RV's are evaluated by probabilistic expectation and scalarization functions, that is, λ-weighting functions and fuzzy measures. Here, as an alternative approach, the mean value with evaluation measure is discussed. Also the fuzzy RV's by the stopping times are defined and we apply it to a zero-sum stopping game with fuzzy values. The saddle points for this fuzzy stopping games are given under a regularity condition.

Keywords: stopping game, g-saddle point, fuzzy random variable, mean value, evaluation measure, λ-weighting function, dynamic programming.

1. Introduction and notations

A classical stopping game for sequences of random variables is known as Dynkin's stopping problem(Dynkin (1965)) and it is the two-person zero-sum game where each player aims the game so as to maximize/minimize his own expected payoff in the gambling. In this paper, we extend the game in the probabilistic and fuzzy configurations. A stopping game for sequence of fuzzy-valued random variables(fuzzy RV's) are considered, which are random variables taking values in fuzzy numbers. Fuzzy RV were first studied by Puri and Ralescu (1986) and have been discussed by many authors.

Our objective is to discuss the game when the fuzzy RV's are evaluated by probabilistic expectation and a new method called as the "Mean Value with Evaluation Measures". This paper treats a problem of fuzzy RV's and stopping times. Then the saddle point for fuzzy stopping games is given under a regularity condition.

In the rest of this section, we give the notations of fuzzy random variables. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. Let $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a} (\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\}$ $(\alpha \in (0, 1])$ and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. In this paper, we write the closed intervals by $\tilde{a}_\alpha := [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha]$ for $\alpha \in [0, 1]$. A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* if the maps $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are measurable for all $\alpha \in [0, 1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}^-_\alpha(\omega), \tilde{X}^+_\alpha(\omega)] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$.

In the rest of this section, we formulate a stopping game for sequences of fuzzy random variables. Let $\{\tilde{W}_n\}_{n=0}^\infty$, $\{\tilde{X}_n\}_{n=0}^\infty$ and $\{\tilde{Y}_n\}_{n=0}^\infty$ be three sequences of integrable bounded fuzzy random variables such that $E(\sup_n \tilde{X}^-_{n,0}) > -\infty$ and $E(\sup_n \tilde{Y}^+_{n,0}) < \infty$, where $\tilde{Y}^+_{n,0}(\omega)$ $(\tilde{X}^-_{n,0}(\omega))$ is the right-end (left-end) of the 0-cut of the fuzzy number $\tilde{Y}_n(\omega)$ ($\tilde{X}_n(\omega)$ resp.). We assume $\tilde{X}_n \preceq \tilde{W}_n \preceq \tilde{Y}_n$ almost surely for all $n = 0, 1, 2, \cdots$ and $P(\Lambda \cup \Gamma) = 1$, where $\Lambda := \{\omega \mid \liminf_{n\to\infty} \tilde{Y}^\pm_{n,\alpha}(\omega) \leq \limsup_{n\to\infty} \tilde{X}^\pm_{n,\alpha}(\omega)$ for all $\alpha \in [0, 1]\}$ and $\Gamma := \bigcup_{m=0}^\infty \{\omega \mid \tilde{Y}_n(\omega) = \tilde{X}_n(\omega)$ for all $n \geq m\}$, and $\preceq$ means the *fuzzy max order* ([3,4,6]), i.e. for fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, $\tilde{a} \preceq \tilde{b}$ means that $\tilde{a}^-_\alpha \leq \tilde{b}^-_\alpha$ and $\tilde{a}^+_\alpha \leq \tilde{b}^+_\alpha$ for all $\alpha \in [0, 1]$. Then there exist $\lim_{n\to\infty} \tilde{W}^\pm_{n,\alpha}$ almost surely for all $\alpha \in [0, 1]$. Therefore we can define a fuzzy random variable $\tilde{W}$ such that

$$\tilde{W}^\pm_{\infty,\alpha}(\omega) := \begin{cases} \lim\limits_{\alpha'\uparrow\alpha} \lim\limits_{n\to\infty} \tilde{W}^\pm_{n,\alpha'}(\omega) \text{ if } \alpha > 0 \\ \lim\limits_{\alpha'\downarrow 0} \lim\limits_{n\to\infty} \tilde{W}^\pm_{n,\alpha'}(\omega) \text{ if } \alpha = 0 \end{cases} \quad \text{for } \omega \in \Omega. \tag{1}$$

$\mathcal{M}_n$ denotes the smallest σ-field on Ω generated by $\{\tilde{W}^\pm_{k,\alpha}, \tilde{X}^\pm_{k,\alpha}, \tilde{Y}^\pm_{k,\alpha} \mid k = 0, 1, \cdots, n;\ \alpha \in [0, 1]\ \}$ for $n = 0, 1, \cdots$. A map $\tau : \Omega \mapsto \mathbb{N} \cup \{\infty\}$ is called a *stopping time* if $\{\omega \mid \tau(\omega) = n\} \in \mathcal{M}_n$ for all $n = 0, 1, 2, \cdots$. Let $\omega \in \Omega$. For player 1's stopping time τ and player 2's stopping time σ, we define a fuzzy random variable by

$$\tilde{Z}[\tau, \sigma](\omega) := \begin{cases} \tilde{X}_\tau(\omega) \text{ if } \tau(\omega) < \sigma(\omega) \\ \tilde{W}_\tau(\omega) \text{ if } \tau(\omega) = \sigma(\omega) \\ \tilde{Y}_\sigma(\omega) \text{ if } \tau(\omega) > \sigma(\omega). \end{cases} \tag{2}$$

In Section 3, we construct a random variable from the integrand of a λ-weighting function for α-cut of a fuzzy RV's with fuzzy measures, and an existence of saddle point for the problem is shown.

2. The mean of general fuzzy numbers

One of the aims in this talk is how to evaluate fuzzy numbers. For the purpose, we use fuzzy measures which are defined as follows.

Definition 1 (Wang and Klir (1993)). A map $M : \mathcal{B} \mapsto [0, 1]$ is called a *fuzzy measure* on $\mathcal{B}$ if M satisfies the following (M.i), (M.ii) and (M.iii) (or (M.i), (M.ii) and (M.iv)):

(M.i) $M(\emptyset) = 0$ and $M(\mathbb{R}) = 1$;
(M.ii) $M(I_1) \le M(I_2)$ holds for $I_1, I_2 \in \mathcal{B}$ satisfying $I_1 \subset I_2$;
(M.iii) $M(\bigcup_{n=0}^{\infty} I_n) = \lim_{n\to\infty} M(I_n)$ holds for $\{I_n\}_{n=0}^{\infty} \subset \mathcal{B}$ satisfying $I_n \subset I_{n+1}$ $(n = 0, 1, 2, \cdots)$;
(M.iv) $M(\bigcap_{n=0}^{\infty} I_n) = \lim_{n\to\infty} M(I_n)$ holds for $\{I_n\}_{n=0}^{\infty} \subset \mathcal{B}$ satisfying $I_n \supset I_{n+1}$ $(n = 0, 1, 2, \cdots)$.

In this paper, we use fuzzy measures M to evaluate the fuzziness of fuzzy numbers, and we call them *evaluation measures*. First we deal with fuzzy numbers $\tilde{a}$ whose membership functions are continuous, i.e. $\tilde{a} \in \mathcal{R}_c$, and next we discuss about general fuzzy numbers $\tilde{a} \in \mathcal{R}$. Let $m : \mathcal{I} \mapsto \mathbb{R}$ be a continuous function such that $m(I)$ is the middle point of a closed interval $I = [I^-, I^+] \in \mathcal{I}$:

$$m[I] := \frac{I^- + I^+}{2}. \tag{3}$$

Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. Using fuzzy measures M, we introduce mean values of a fuzzy number $\tilde{a} \in \mathcal{R}$ by(see Yoshida et al. (preprint))

$$\tilde{E}_{\tilde{a}}(\tilde{a}) = \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\, m(\tilde{a}_\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha. \tag{4}$$

where $\tilde{a}_\alpha$ is the α-cut of the fuzzy number $\tilde{a}$. Hence $M_{\tilde{a}}$ is a fuzzy measure depending on the fuzzy number $\tilde{a}$, and $M_{\tilde{a}}(\tilde{a}_\alpha)$ means the confidence degree that the fuzzy number $\tilde{a}$ takes values at the interval $\tilde{a}_\alpha$ (see Example 1).

Example 1. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. An evaluation measure $M_{\tilde{a}}$ is called the *possibility evaluation measure*, the *necessity evaluation measure* and the *credibility evaluation measure* induced from the fuzzy number $\tilde{a}$ if it is given by the following (5) – (7) respectively:

$$M_{\tilde{a}}^P(I) := \sup_{x\in I} \tilde{a}(x), \qquad I \in \mathcal{B}; \tag{5}$$

$$M_{\tilde{a}}^N(I) := 1 - \sup_{x\notin I} \tilde{a}(x), \qquad I \in \mathcal{B}; \tag{6}$$

$$M_{\tilde{a}}^C(I) := \frac{1}{2}\left(M_{\tilde{a}}^P(I) + M_{\tilde{a}}^N(I)\right), \qquad I \in \mathcal{B}. \tag{7}$$

We note that $M_{\tilde{a}}^P$, $M_{\tilde{a}}^N$ and $M_{\tilde{a}}^C$ satisfy Definition 1(M.i) – (M.iv) since $\tilde{a}$ is continuous and has a compact support. From (5) – (7), we have $M_{\tilde{a}}^P(\tilde{a}_\alpha) = 1$, $M_{\tilde{a}}^N(\tilde{a}_\alpha) = 1-\alpha$ and $M_{\tilde{a}}^C(\tilde{a}_\alpha) = 1-\alpha/2$, and the corresponding mean values $\tilde{E}_{\tilde{a}}(\tilde{a})$ are reduced to

$$\tilde{E}^P(\tilde{a}) := \int_0^1 m(\tilde{a}_\alpha)\, \mathrm{d}\alpha; \tag{8}$$

$$\tilde{E}^N(\tilde{a}) := \int_0^1 2(1-\alpha)\, m(\tilde{a}_\alpha)\, \mathrm{d}\alpha; \tag{9}$$

$$\tilde{E}^C(\tilde{a}) := \int_0^1 \frac{4}{3}(1-\frac{\alpha}{2})\, m(\tilde{a}_\alpha)\, \mathrm{d}\alpha. \tag{10}$$

They are called a *possibility mean*, a *necessity mean* and a *credibility mean* of the fuzzy number $\tilde{a}$ respectively.

From now on, we suppose the following Assumption M which guarantees the regularity of the mean values $\tilde{E}(\tilde{a})$.

Assumption M. There exists a nonincreasing function $\rho : [0,1] \mapsto [0,1]$ such that

$$M_{\tilde{a}}(\tilde{a}_\alpha) = \rho(\alpha), \quad \alpha \in [0,1] \quad \text{for all } \tilde{a} \in \mathcal{R}_c.$$

In Example 1, the possibility evaluation measure $M_{\tilde{a}}^P$, the necessity evaluation measure $M_{\tilde{a}}^N$ and the credibility evaluation measure $M_{\tilde{a}}^C$ have the following corresponding nonincreasing functions ρ in Assumption M:

$\rho^P(\alpha) := 1$ $(\alpha \in [0,1])$ in case of the possibility evaluation measure;
$\rho^N(\alpha) := 1 - \alpha$ $(\alpha \in [0,1])$ in case of the necessity evaluation measure;
$\rho^C(\alpha) := 1 - \alpha/2$ $(\alpha \in [0,1])$ in case of the credibility evaluation measure.

We discuss the mean of fuzzy numbers $\tilde{a}$. Let us consider a uniform distribution on a closed interval $I = [I^-, I^+](\in \mathcal{I})$. Its mean $m[I]$ are given as follows:

$$m[I] = \frac{\int_I x \, \mathrm{d}x}{\int_I \mathrm{d}x} = \frac{I^- + I^+}{2}. \tag{11}$$

The value (11) has been already discussed in (3), and the mean of a fuzzy number $\tilde{a} \in \mathcal{R}_c$ is given by (4):

$$\tilde{E}_{\tilde{a}}(\tilde{a}) = \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\, m[\tilde{a}_\alpha] \, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha) \, \mathrm{d}\alpha. \tag{12}$$

Next, we extend it to the mean of general fuzzy numbers $\tilde{a} \in \mathcal{R}$ whose membership functions are upper-semicontinuous but are not necessarily continuous. Let $\tilde{a} \in \mathcal{R}$. We define the mean of the general fuzzy number $\tilde{a} \in \mathcal{R}$ by

$$\tilde{E}(\tilde{a}) := \lim_{n \to \infty} \tilde{E}_{\tilde{a}^n}(\tilde{a}^n), \tag{13}$$

where $\tilde{E}_{\tilde{a}^n}(\tilde{a}^n)$ is defined in (12) and $\{\tilde{a}^n\}_{n=1}^\infty (\subset \mathcal{R}_c)$ is a sequence of fuzzy numbers whose membership functions are continuous and satisfy $\tilde{a}^n \downarrow \tilde{a}$ pointwise as $n \to \infty$. We call (13) well-defined if their limiting values are independent of selection of the sequences $\{\tilde{a}^n\}_{n=1}^\infty \subset \mathcal{R}_c$. Then, the following lemma is trivial from (12), (13) and Assumption M.

Lemma 1. *Suppose Assumption M holds. Let a fuzzy number $\tilde{a} \in \mathcal{R}$. Then, the mean value (12) is well-defined and it is represented as follows.*

$$\tilde{E}(\tilde{a}) = \int_0^1 \rho(\alpha) \, \frac{\tilde{a}_\alpha^- + \tilde{a}_\alpha^+}{2} \, \mathrm{d}\alpha \Big/ \int_0^1 \rho(\alpha) \, \mathrm{d}\alpha. \tag{14}$$

The mean value $\tilde{E}(\cdot)$ has the following natural properties for fuzzy numbers (Yoshida et al. (preprint)).

Lemma 2. *Suppose Assumption M holds. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, $\theta \in \mathbb{R}$ and $\zeta \geq 0$, the following (i) – (iv) hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$.
(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$.
(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.
(iv) *If $\tilde{a} \succeq \tilde{b}$, then $\tilde{E}(\tilde{a}) \geq \tilde{E}(\tilde{b})$ holds, where $\succeq$ is the fuzzy max order.*

3. A stopping game game of fuzzy random variables

Let (τ, σ) be a pair of finite stopping times. Now we consider the evaluation of the fuzzy random variable $\tilde{Z}[\tau, \sigma]$. The expectation is given by the closed interval $E(\tilde{Z}[\tau, \sigma])_\alpha$ and its evaluation is $m(E(\tilde{Z}[\tau, \sigma])_\alpha)$. For a pair of finite stopping times (τ, σ), we define a random variable

$$\begin{aligned}&\tilde{Z}_m[\tau, \sigma](\omega)\\ &:= \int_0^1 M_{\tilde{Z}[\tau,\sigma](\omega)}(\tilde{Z}[\tau, \sigma]_\alpha(\omega))\, m(\tilde{Z}[\tau, \sigma]_\alpha(\omega))\, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{Z}[\tau,\sigma](\omega)}(\tilde{Z}[\tau, \sigma]_\alpha(\omega))\, \mathrm{d}\alpha,\end{aligned} \tag{15}$$

$\omega \in \Omega$. Then we have the following relations.

Lemma 3. *Suppose Assumption M holds. For a pair of finite stopping times (τ, σ), it holds that*

$$\begin{aligned}&E\left(\int_0^1 M_{\tilde{Z}[\tau,\sigma](\cdot)}(\tilde{Z}[\tau, \sigma]_\alpha(\cdot))\, m(\tilde{Z}[\tau, \sigma]_\alpha(\cdot))\, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{Z}[\tau,\sigma](\cdot)}(\tilde{Z}[\tau, \sigma]_\alpha(\cdot))\, \mathrm{d}\alpha \right)\\ &= \int_0^1 M_{E(\tilde{Z}[\tau,\sigma])}(E(\tilde{Z}[\tau, \sigma])_\alpha)\, E(m(\tilde{Z}[\tau, \sigma]_\alpha))\, \mathrm{d}\alpha \Big/ \int_0^1 M_{E(\tilde{Z}[\tau,\sigma])}(E(\tilde{Z}[\tau, \sigma])_\alpha)\, \mathrm{d}\alpha\\ &= \int_0^1 M_{E(\tilde{Z}[\tau,\sigma])}(E(\tilde{Z}[\tau, \sigma])_\alpha)\, m(E(\tilde{Z}[\tau, \sigma])_\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 M_{E(\tilde{Z}[\tau,\sigma])}(E(\tilde{Z}[\tau, \sigma])_\alpha)\, \mathrm{d}\alpha\\ &= \tilde{E}_{E(\tilde{Z}[\tau,\sigma])}(E(\tilde{Z}[\tau, \sigma])).\end{aligned}$$

The expectation

$$J[\tau, \sigma] := E(\tilde{Z}_m[\tau, \sigma]) \tag{16}$$

means player 1's expected reward for a pair of the player's stopping times (τ, σ). Now we discuss a stopping game such that player 1 maximizes $J[\tau, \sigma]$ with respect to stopping time τ and player 2 minimizes it with respect to stopping time σ respectively.

Problem G. Find a pair of the player's stopping times (τ^*, σ^*) such that

$$J[\tau, \sigma^*] \leq J[\tau^*, \sigma^*] \leq J[\tau^*, \sigma]. \tag{17}$$

for all stopping times τ and σ. Then, (τ^*, σ^*) is called a *g-saddle point* for the zero-sum stopping game with respect to a λ-weighting function g.

Then (τ^*, σ^*) is called a *g-saddle point* for the zero-sum stopping game with respect to a linear ranking function g. To consider this stopping problem, we define random variables

$$\begin{cases} \overline{V}_n := \text{ess inf}_{\sigma \geq n} \text{ ess sup}_{\tau \geq n} E(\tilde{Z}_m[\tau, \sigma] | \mathcal{M}_n), \\ \underline{V}_n := \text{ess sup}_{\tau \geq n} \text{ ess inf}_{\sigma \geq n} E(\tilde{Z}_m[\tau, \sigma] | \mathcal{M}_n), \end{cases}$$

$$X_n(\omega) := \int_0^1 M_{\tilde{X}_n(\omega)}(\tilde{X}_{n,\alpha}(\omega))\, m(\tilde{X}_{n,\alpha}(\omega))\, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{X}_n(\omega)}(\tilde{X}_{n,\alpha}(\omega))\, \mathrm{d}\alpha$$

and

$$Y_n(\omega) := \int_0^1 M_{\tilde{Y}_n(\omega)}(\tilde{Y}_{n,\alpha}(\omega))\, m(\tilde{Y}_{n,\alpha}(\omega))\, \mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{Y}_n(\omega)}(\tilde{Y}_{n,\alpha}(\omega))\, \mathrm{d}\alpha$$

for $n = 0, 1, 2, \cdots, \omega \in \Omega$. Then we have the following results in a similar approach in Ohtsubo (1986).

Theorem 1.

(i) $\overline{V}_n = \underline{V}_n$ *for all* $n = 0, 1, 2, \cdots$. *So, we put* $V_n := \overline{V}_n = \underline{V}_n$ *for* $n = 0, 1, 2, \cdots$. *The sequence of random variables* $\{V_n\}_{n=0}^{\infty}$ *satisfies*

$$V_n = \min\{\max\{X_n, E(V_{n+1}|\mathcal{M}_n)\}, Y_n\} \quad \text{a.s.} \quad \text{for } n = 0, 1, 2, \cdots. \tag{18}$$

(ii) *Define*

$$\tau^*(\omega) := \inf \{k \geq n \mid V_k(\omega) \leq X_k(\omega)\}, \quad \omega \in \Omega, \tag{19}$$

$$\sigma^*(\omega) := \inf \{k \geq n \mid V_k(\omega) \geq Y_k(\omega)\}, \quad \omega \in \Omega, \tag{20}$$

where the infimum of the empty set is understood to be $+\infty$. *If* $\min\{\tau^*, \sigma^*\} < \infty$ *a.s., then* (τ^*, σ^*) *is a g-saddle point of Problem G and*

$$E(V_0) = J[\tau^*, \sigma^*] = E(\tilde{Z}_m[\tau^*, \sigma^*]). \tag{21}$$

4. Concluding remarks

When we use fuzzy random variables in decision making problems, we finally need to evaluate fuzzy random variables. The most popular methods are the defuzzification and ordering of fuzzy numbers/fuzzy quantities. In the decision making, the meaning of estimation is important, and we have discussed it from the viewpoint of measure theory. The mean value by evaluation measures for fuzzy numbers, fuzzy random variables and fuzzy stochastic processes is valid for various decision making modeling. We hope that the presented method will be examined in the other types of decision making problems and it will be improved from application aspects. For example, we are untested in the following theme.

(a) Stopping Game with risk sensitivity derived from the mean value.
(b) Matrix Game with risk sensitivity derived from the mean value.

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