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Construction of Fuzzy Control Charts Based on Weighted Possibilistic Mean

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The problem of constructing control charts for fuzzy data has been considered in literature. The proposed transformation approaches and direct fuzzy approaches have their advantages and disadvantages. The representative values charts based on transformation methods are often recommended in practical application. For representing a fuzzy set by a crisp value, the weight of importance of the members assigned with some membership levels in a fuzzy set should be considered, and the possibility theory can be employed to deal with such problem. In this article, we propose to employ the weighted possibilistic mean (WPM), weighted interval valued possibilistic mean (WIVPM) of fuzzy number as a sort of representative values for the fuzzy attribute data, and establish new fuzzy control charts with WPM and WIVPM. The performance of the charts is compared to the existing fuzzy charts with a fuzzy c-chart example via newly defined average number of inspection for variation of control state.

Keywords Control chart; Fuzzy number; Weighted possibilistic mean; Weighted possibilistic variance.

Mathematics Subject Classification 62A86; 62C86; 68K86.

1. Introduction

Statistical process control (SPC) is a methodology for monitoring processes to identify special causes of variation and to signal the need for corrective action. SPC is widely employed throughout industry, and is a proven technique for improving quality and productivity.

The most common SPC technique is the statistical control chart. Control charts have found widespread application in engineering and services for monitoring process stability and capability. Typically, a control chart consists of a center line(CL) and two control lines,

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referred to as the upper control limit (UCL) and the lower control limit (LCL), respectively. The center line represents an estimate of the process level, while the two control limits denote the boundaries of normal variability, and are specified such that the majority of the observations lie within their bounded range when the process is under control. Samples drawn from the process are plotted as points on the control chart. The control charts are constructed concurrently with the statistical hypotheses testing process. Essentially, the control chart tests the hypotheses that the process in the state of statistical control. Accordingly, a plotted data point falling within the control limits confirm the hypotheses of statistical control, while a point falling outside of the control limits indicates a rejection of this hypotheses.

It was well known that control charts are based on data representing one or several quality-related characteristics of the product or service. If these characteristics are measurable on real-valued number or vector, then variable control charts are used. If the quality-related characteristics cannot be easily represented in numerical form, then attribute control charts such as p-chart, c-chart are useful (Gülbay and Kahraman, 2006; Wang and Raz, 1990). However, the key process characteristic sometimes may be much more complicated in SPC such as the area of economic quality control, acceptance samplings for attributes, where due to the complication of continuous process quality, sample data based on the observations and measurements are characterized in impreciseness (Hryniewicz, 2008; Viertl, 2008). For instance, the color-intensity quality of produced pictures or screens and the reading-precise quality shown on analog measurement equipments, apparently inheres with imprecise character, whose samples data are collected by taking certain imprecise information into consideration, known as interval valued or vague data. Also, the vague data may come from the judgment with humans' partial knowledge or subjectivity on categories or attributes of the inspected item, such a judgment may result in some verbal, form which cannot be denoted by a numerical scale appropriately (Gülbay and Kahraman, 2007; Kanagawa and Ohta, 1993; Wang and Raz, 1990). Therefore, based on the imprecise sample data to monitor the process, the traditional control charts have to be expanded so as to possibly carry out process monitoring in this fuzzy environment. Possibly, the extended attribute control chart such as χ^2 -control chart for multi-distinct categories, the control charts with grouped data methods may be suitable for this purpose, if the vague data could be expressed approximately in distinct categorical form and the distributions of the underlying process variables are known (Woodall et al., 1997). However, the uncertainty of the vague data is essentially non-statistical in nature so that the conditions mentioned above are hardly satisfied.

The fuzzy set theory and possibility theory provide useful tools for dealing with fuzzy data (Zadeh, 1978). For monitoring process quality with fuzzy sample data, few researchers have proposed the construction of fuzzy control charts based on fuzzy set theory and control chart methods (Cheng, 2005; Faraz, 2010; Gülbay and Kahraman, 2007; Kanagawa and Ohta, 1993, Wang and Raz, 1990, etc.) An illustration of a general fuzzy control chart is shown in Fig. 1.

The concepts and methods of fuzzy control charts are well documented in the available literature. Basically, the control rule of fuzzy control charts can be sorted into probability rule and possibility rule. Hryniewicz (2008) gives a short overview on basic problems of statistical quality control in fuzzy environment, in which he pointed out that using the ordinary charts to transformed fuzzy data is an easy way for construction of fuzzy control charts. Raz and Wang (1990) and Wang and Raz (1990) pointed out that linguistic data can provide more information than the binary classification used in control charts by attributes, and they proposed some fuzzy control charts for linguistic data. Their proposed methods



Figure 1. Fuzzy control chart.

was to convert the fuzzy sets associated with linguistic values into scalars referred to as representative values and plot them on an ordinary control chart. Like the measures of central tendency used in descriptive statistics, the proposed calculation formulas for transforming the linguistic data into its corresponding representative values as follows: *fuzzy mode*

$$f_{mode} = \{x | A(x) = 1\},$$

fuzzy midrange

$$f_{mr}(\alpha) = (\inf A_{\alpha} + \sup A_{\alpha})/2,$$

fuzzy median f_{med} satisfies

$$\int_{-\infty}^{f_{med}} A(x) dx = \int_{f_{med}}^{\infty} A(x) dx$$

fuzzy average

$$f_{avg} = \int_{-\infty}^{\infty} x A(x) dx / \int_{-\infty}^{\infty} A(x) dx$$

where A(x) is the membership function of a normal, convex and compact fuzzy set A on real line \mathbb{R} (refer to Definition 2.1) and A_{α} ($0 \le \alpha \le 1$) the α -cut of A (Wang and Raz, 1990).

Kanagawa et al. (1993) proposed a more general way for construction of a fuzzy control chart where the representative values—*barycenter*—are defined based on Zadeh's probability measure of fuzzy events as

$$Rep(A) = \int_{-\infty}^{\infty} x A(x) p(x) dx / \int_{-\infty}^{\infty} A(x) p(x) dx$$

for fuzzy number A, where p(x) is the probability density of the ground variable. This method may be used not only for monitoring the fuzzy process mean, but also for monitoring the process variability. The main difficulty of this approach is that the unknown distribution p(x) cannot be determined easily. Based on the concept of fuzzy random variables (Puri

and Ralescu, 1986, Gil et al., 2006). Wang (2006) propose to view a fuzzy quality data as the triangular fuzzy number-valued random variable (i.e., a fuzzy random variable), whose central variable that represents the randomness of the data is normally distributed, and the degree of fuzziness of the fuzzy numbers represents the fuzziness of the data. A novel representative values called *fuzzy random sum* is defined as the sum of the fuzzy mode and a measure of fuzziness mentioned above, and a cumulative sum (CUSUM) chart is established with this *fuzzy random sum*. This method does not depend on the values of α -level, however, it only suits triangular fuzzy numbers, and also the formula for computing the degree of fuzziness of the data is not unique, thus, different decisions will be made when we use different formula. It should be pointed out that there is no theoretical basis supporting any representative values specifically. The selection among them should be mainly based on the ease of computation or the user's preference.

Taleb (2009) suggests to assign a representative value to a given fuzzy sample with one of the four methods proposed by Wang and Raz (1990) when doing applications of multivariate attribute control chart in decorated porcelain production. Senturk and Erginel (2009) use the α -level *fuzzy midrange* transformation technique to construct the fuzzy $\overline{X} - \widetilde{R}$ and $\overline{X} - \widetilde{S}$ control charts, which rely heavily on the properties of the normal distribution. Gülbay and Kahraman (2005, 2006, 2007) have not only explained why we require fuzzy control charts, but also discussed the charts relying on the normal distribution with respect to the *fuzzy mode*, *fuzzy midrange* and *fuzzy median* on α level carefully, the inspection becomes tighter as the α -level is set higher. Also without any defuzzification they propose a direct fuzzy approach (DFA) for constructing fuzzy control charts, in which, by comparing the percentage area that the sample remains inside the fuzzy control limits, a decision on whether the process is in control is made in preference of the operators. This direct fuzzy charting is a completely novel reasonable method in the area of SPC. However, it somewhat is complicated for practical application due to the computations of the sample's area which depends on the shape of the sample and the selected α -level. Faraz and Shapiro (2010) propose another DFA based on theory of fuzzy random variables, in which under the given significant level, the fuzzy in-control region (FIR) is first determined, and then a proper fuzzy inclusion operator to determine the degree that fuzzy sample group are excluded from the FIR is selected . This work has a significant worth for constructing control charts in fuzzy environments. Cheng (2005) propose a construction of fuzzy control charts for a process with fuzzy outcomes derived from the subjective quality ratings provided by a group of experts. The process capability analyses is an important aspect in SPC. Kaya and Kahraman (2010a, b, c) consider the fuzzy process capability analysis approach and some fuzzy $\overline{X} - \widetilde{R}$, $\overline{X} - \widetilde{S}$ charts are proposed based on extension principle (Zadeh, 1978), here the basic quality characteristics should follow the normal distribution. Fuzzy sets approaches are also applied in crisp control charts technique. Alaeddini et al. (2009) propose a hybrid fuzzy-statistical clustering approach to estimate the change time in fixed and variable sampling control charts. Fazel and Alaeddini (2010) also consider a general fuzzy-statistical clustering approach for estimating the time of change in variable sampling control charts. Demirli and Vijayakumar (2010) consider a fuzzy logic based assignable cause diagnosis using control chart patterns. Fazel et al. (2008) propose a hybrid fuzzy adaptive sampling run rules for Shewhart control charts.

From a viewpoint of weighting function of real valued data, the representative values methods proposed in literature such as fuzzy mode, fuzzy midrange, fuzzy median, fuzzy average, barycenter, and fuzzy random sum can be characterized as follows. Some of them emphasize only a certain α -cut of the fuzzy data, such as fuzzy mode and fuzzy midrange

method. Some focus on the partial levels that equal to or higher than α , such as fuzzy median method. And some view all of the levels as the same, such as fuzzy average, barycenter and fuzzy random sum methods, etc. As pointed out in Gülbay and Kahraman (2006), the assumptions in Raz and Wang (1990) and Wang and Raz (1990) are not very realistic, and such kinds of defuzzified values of sample statistics may result in losing some important information. It probably due to that the whole information contained in different α -cuts of fuzzy data are not fully and accurately captured. Different levels of α -cuts, in fact, probably represent the different sense of information represented by fuzzy set in daily productions processes and service processes, most of the situation may be that the higher the level of α -cut of the fuzzy set, the more valuable information it contains, and therefore, the more attention we must pay to.

Though the conversion of fuzzy sets into crisp values may results in loss of information of the fuzzy data in some sense (Gülbay and Kahraman, 2006, 2007), the representative values methods are still common in practical applications (Taleb, 2009). It is important to propose some more accurate and more powerful representative values for the considered fuzzy or imprecise information in SPC.

Recently, possibility theory has received much attention in the area of uncertainty modeling. Carlsson and Fullér (2001) proposed the possibilistic mean and possibilistic variance for fuzzy numbers, these concepts behave properly in measuring central tendency of fuzzy numbers based on a ranking of fuzzy numbers by the desire to give less importance to the lower levels of fuzzy numbers. Fullér and Majlender (2003) introduced the weighted possibilistic mean (WPM) and weighted possibilistic variance (WPV) for fuzzy numbers, which are the possibilistic mean and variance with weighting functions that give corresponding importance to different α -levels of fuzzy numbers. Saeidifar and Pasha (2009) further investigate the properties of the WPM and WPV in the metric space of fuzzy numbers, and some higher order possibilistic moments of fuzzy numbers are proposed. An important conclusion is obtained in their work, that is, for a given fuzzy number, the weighted possibilistic interval valued mean is the nearest weighted interval to the fuzzy number and the WPM is the nearest weighted point approximation to the fuzzy number. Thavaneswarana et al. (2009) present a further research on the properties of WPM and WPV, and apply it to the fuzzy coefficients of GARCH model of time series, which makes the model much more improved and softened. However, there is no literature concerning construction of control charts by using weighted possibilistic moments of fuzzy or imprecise data. In view of representative values of fuzzy numbers, the WPM and WPV have the advantages of more flexible and comprehensive over previously proposed representative values of fuzzy data. Specifically, it has the feature that the higher the levels of α -cuts of a fuzzy number, the larger the weight be assigned to. Therefore, it may emphasize the information contained in fuzzy data more accurately if we properly choose the weighting function. Inspired by the advantages of WPM and WPV of fuzzy number, in this article, we propose a novel approache. to construct a control chart for the fuzzy count of nonconformity under probabilistic rule.

This article is organized as follows, in Sec. 2, we focus on the new representative value—the WPM, the weighted interval valued possibilistic mean (WIVPM) and their relevant notions as well as some related important theorems. In Sec. 3, the representative values fuzzy *c*-charts are established, and a numerical example is given. In Sec. 4, comparisons with the existing fuzzy control charts are given by case performance simulation and general sensitivity analysis based on actual data set and average number of inspection for variation of control state (ANIVCS). In Sec. 5, a conclusion remark for our proposal is presented.

2. The WPM and WPV

In this section, we will consider some novel transformation methods that convert the considered fuzzy number into its WPM and WIVPM or WPV. By \mathbb{R} we denote the set of all real numbers associated with the usual distance between real numbers, \mathbb{R} is a usual metric space and has the usual topology consists of all open intervals under this distance. The concept of fuzzy numbers will be introduced by considering this topological structure of \mathbb{R} .

Definition 2.1. A mapping $A : \mathbb{R} \to [0, 1]$ is said to be a fuzzy number on \mathbb{R} if

- (1) $\{x|A(x)=1\} \neq \emptyset;$
- (2) A(x) is upper semicontinuous ;
- (3) Each α -cut $A_{\alpha} = \{x | A(x) \ge \alpha\}$ is a compact bounded interval for all $\alpha \in (0, 1]$;
- (4) A₀ := cl{x|A(x) > 0}, the support of A, is compact and bounded, where cl denotes the closure of a set. Fuzzy number sometimes is also called normal convex compact fuzzy set on ℝ, and A(·) is said to be the membership function of fuzzy number A. By F(ℝ) we denote the set of all fuzzy numbers.

A fuzzy number A is called an LR-fuzzy number, if its membership function can be written as the following form:

$$A(x) = \begin{cases} L\left(\frac{m-x}{l}\right) & m-l \le x \le m, \\ 1 & m \le x \le n, \\ R\left(\frac{x-n}{r}\right) & n \le x \le n+r, \\ 0 & else. \end{cases}$$
(2.1)

Here the closed interval [m, n] is the peak of fuzzy number A, i.e., the fuzzy mode f_{mode} of A, m, n is the lower, upper modal values, respectively; L, $R : [0, 1] \rightarrow [0, 1]$ are non-increasing and left-continuous functions satisfying that L(0) = R(0) = 1 and L(1) = R(1) = 0, and called left, right shape function, respectively. Such an *LR*-fuzzy number can be abbreviated by $A = (m, n, l, r)_{LR}$. The *LR*-fuzzy numbers presented in this article are assumed to be strictly decreasing so that its α -cuts can be simply computed by

$$A_{\alpha} = [m - lL^{-1}(\alpha), n + rR^{-1}(\alpha)], \ \forall \alpha \in [0, 1],$$
(2.2)

where $L^{-1}(\alpha) := \sup\{x | L(x) \ge \alpha\}, R^{-1}(\alpha) := \sup\{x | R(x) \ge \alpha\}$. *LR*-fuzzy number, where $L(x) = R(x) = \max\{0, 1 - x\}$, is often used to characterize the fuzzy quality data (Wang, 2006).

Definition 2.2. (Viertl, 2008) An imprecise number *B* is a fuzzy set of \mathbb{R} whose membership function $B(\cdot)$ satisfying that for any $\alpha \in (0, 1]$ the α -cut B_{α} is non-empty and a finite union of compact intervals $[B_{\alpha,j}^{-}, B_{\alpha,j}^{+}]$,

$$B_lpha = igcup_{j=1}^{k_j} [B^-_{lpha,j}, B^+_{lpha,j}]$$

Here fuzzy subset *B* of \mathbb{R} may not be convex, so that the α -cut of which may not be an interval, whereas a union of a number of mutually separated closed intervals $[B_{\alpha,j}^-, B_{\alpha,j}^+]$,

 $j = 1, \ldots, k_j$, where $B_{\alpha,j}^-, B_{\alpha,j}^+$ are the left, right end points of the *j*th closed interval, k_j is a natural number representing the number of intervals included in the α -cut B_{α} . A fuzzy number is a special imprecise number.

Definition 2.3. A function $f:[0,1] \to \mathbb{R}$ is said to be a weighting function, if f is nonnegative and monotonically increasing and fulfill $\int_0^1 f(x)dx = 1$. Especially, let us introduce a family of weighting function $\mathbf{1} : [0, 1] \to \mathbb{R}$ which is

defined as

$$\mathbf{l}(\alpha) = \begin{cases} 1/(1-\alpha_0) & \text{if } \alpha \in (\alpha_0, 1], \\ 0 & \text{if } \alpha \in [0, \alpha_0], \end{cases}$$
(2.3)

where $\alpha_0 \in [0, 1]$ is an arbitrary real number. It can easily be proved that **1** is a weighting function. Obviously, $f(\alpha) = 2\alpha$, $3\alpha^2$ ($\alpha \in [0, 1]$) are also weighting functions.

The concepts of WPM and WPV (Fullér and Majlender, 2003) are from the concepts of possibilistic mean and variance proposed by Carlsson and Fullér (2001).

Definition 2.4. (Fullér and Majlender, 2003, Thavaneswarana et al., 2009) Let A be a fuzzy number and let f be a weighting function. The f-WPM of A is defined as

$$\overline{M}_f(A) = \int_0^1 f(\alpha) \frac{A_\alpha^- + A_\alpha^+}{2} d\alpha,$$

where $A_{\alpha}^{-} = \inf\{x | A(x) \ge \alpha\}, A_{\alpha}^{+} = \sup\{x | A(x) \ge \alpha\}, A_{\alpha}^{-} \le A_{\alpha}^{+}$. According to the definition of *fuzzy midrange*, the WPM of fuzzy number A can be written by $\int_0^1 f(\alpha) f_{mr}(\alpha) d\alpha$, which indicates that the WPM is a general weighted midrange. The WPM of fuzzy number is also more accurate and flexible in representing the fuzzy numbers than fuzzy average, fuzzy median, barycenter and fuzzy random sum, because there are no weighting functions measuring the importance of α levels in the latter, i.e., they view all of the α -levels as the same one.

Definition 2.5. Let A be a fuzzy number and let f be a weighting function. The f-WPV of A's defined as

$$Var_f(A) = \frac{1}{2} \int_0^1 f(\alpha) \left[(A_\alpha^- - \overline{M}_f(A))^2 + (A_\alpha^+ - \overline{M}_f(A))^2 \right] d\alpha.$$

Following definitions of WPM and WPV of fuzzy number A, we can propose the definitions of WPM and WPV for the imprecise number B as follows:

$$\overline{M}_{f}(B) = \int_{0}^{1} \frac{1}{k_{j}} f(\alpha) \left(\sum_{j=1}^{k_{j}} \frac{B_{\alpha,j}^{-} + B_{\alpha,j}^{+}}{2} \right) d\alpha,$$
$$Var_{f}(B) = \int_{0}^{1} \left(\frac{1}{k_{j}} f(\alpha) \left(\sum_{j=1}^{k_{j}} [(B_{\alpha,j}^{-} - \overline{M}_{f}(B))^{2} + (B_{\alpha,j}^{+} - \overline{M}_{f}(B))^{2}] \right) \right) d\alpha.$$

where k_i is a positive integer.

Example 1. Let *B* be an imprecise number defined as

$$B(x) = \begin{cases} 0, & x < 1\\ (x-1)^2, & 1 \le x \le 2\\ 1.5 - 0.25x, & 2 \le x \le 3\\ 0.25x, & 3 \le x \le 4\\ -2x + 9, & 4 \le x \le 4.5\\ 0, & x \ge 4.5. \end{cases}$$

Then, $\overline{M}_{3x^2}(B) = \int_0^{0.75} 1.5\alpha^2 (2.75 + 0.5\sqrt{\alpha} - 0.25\alpha) d\alpha + \int_{0.75}^1 1.5\alpha^2 (5.75 + 0.5\sqrt{\alpha} - 0.25\alpha) d\alpha = 2.3626. \ Var_{3x^2}(B) = \int_0^{0.75} 1.5\alpha^2 [(\sqrt{\alpha} - 1.3626)^2 + (2.1374 - 0.5\alpha)^2] d\alpha + \int_{0.75}^1 [(\sqrt{\alpha} - 4\alpha + 4.6374)^2 + (2.1374 + 3.5\alpha)^2] d\alpha = 10.4701.$

Definition 2.6. (Fullér and Majlender, 2003) The f-WIVPM of the fuzzy number A can be defined as

$$M_f(A) = [M_f^-(A), M_f^+(A)],$$
(2.4)

and f-WPM can be written as

$$\overline{M}_{f}(A) = \frac{1}{2} [M_{f}^{-}(A) + M_{f}^{+}(A)], \qquad (2.5)$$

where $M_f^-(A) = \int_0^1 f(\alpha) A_\alpha^- d\alpha$ and $M_f^+(A) = \int_0^1 f(\alpha) A_\alpha^+ d\alpha$ are defined to be the lower and upper *f*-WPM of *A*, respectively. And for an *LR*-fuzzy number $A = (m, n, l, r)_{LR}$ we have

$$M_{f}^{-}(A) = m - l \int_{0}^{1} L^{-1}(\alpha) f(\alpha) d\alpha,$$
(2.6)

$$M_{f}^{+}(A) = n + r \int_{0}^{1} R^{-1}(\alpha) f(\alpha) d\alpha.$$
 (2.7)

Let

$$\phi_f := \int_0^1 L^{-1}(\alpha) f(\alpha) d\alpha, \qquad (2.8)$$

$$\psi_f := \int_0^1 R^{-1}(\alpha) f(\alpha) d\alpha, \qquad (2.9)$$

then the WIVPM and the WPM of A, respectively, can be calculated as

$$M_f(A) = [m - l\phi_f, n + r\psi_f],$$
(2.10)

$$\overline{M}_{f}(A) = \frac{1}{2}(m+n-l\phi_{f}+r\psi_{f}).$$
 (2.11)

Let $A = (m, n, l, r)_{LR}$ be a *LR*-fuzzy number of trapezoidal form in Carlesson and Fullér (2001), Fullér and Majlender (2003), where $L(x) = R(x) = \max\{0, 1 - x\}$.

$$A_{\alpha} = [m - l(1 - \alpha), n + r(1 - \alpha)], \ \alpha \in [0, 1].$$
(2.12)

Then

$$M_f(A) = \left[m - l \int_0^1 (1 - \alpha) f(\alpha) d\alpha, n + r \int_0^1 (1 - \alpha) f(\alpha) d\alpha\right], \qquad (2.13)$$

$$\overline{M}_{f}(A) = \frac{1}{2}(m+n-(l-r)\int_{0}^{1}(1-\alpha)f(\alpha)d\alpha).$$
(2.14)

The WIVPM and the WPM of *A* are also considered to be weighted expected values (Fullér and Majlender, 2003). From which we can infer a familiar fact that points with small membership degrees are considered to be less important. Saeidifar and Pasha (2009) have pointed out that if we select the metric on $\mathcal{F}(\mathbb{R})$ as

$$D_f(A_1, A_2) = \frac{1}{2} \int_0^1 f(\alpha) [(A_{1\alpha}^- - A_{2\alpha}^-)^2 + (A_{1\alpha}^+ - A_{2\alpha}^+)^2] d\alpha,$$

where f is the weighting function, $A_1, A_2 \in \mathcal{F}(\mathbb{R})$, then the following theorems hold:

Theorem 2.1. (Saeidifar and Pasha, 2009). Let A be a fuzzy number, let f be a weighting function. Then, the interval $M_f(A) = [M_f^-(A), M_f^+(A)]$ is the nearest f-weighted possibilistic interval to fuzzy number A with respect to the f-weighted possibilistic distance D_f .

Theorem 2.2. (Saeidifar and Pasha, 2009). Let A be a fuzzy number, let f be a weighting function. Then, $\overline{M}_f(A)$, the WPM of A is the nearest f-weighted possibilistic point to fuzzy number A which is unique with respect to the f-weighted possibilistic distance D_f .

Theorem 2.3. Let B be an imprecise number, let f be a weighting function. Then, $\overline{M}_f(B)$, the WPM of B is the nearest f-weighted possibilistic point to imprecise number B which is unique with respect to the f-weighted possibilistic distance D_f .

Proof. By the definition of the imprecise number we have $B_{\alpha} = \bigcup_{j=1}^{k_j} [B_{\alpha,j}^-, B_{\alpha,j}^+]$, and the distance function

$$D(b) = D_f(B, b) = \frac{1}{2} \int_0^1 f(\alpha) \Big[(B_\alpha^- - b)^2 + (B_\alpha^+ - b)^2 \Big] d\alpha$$
$$:= \frac{1}{2} \int_0^1 \bigg(\sum_{j=1}^{k_j} f(\alpha) \Big[(B_{\alpha,j}^- - b)^2 + (B_{\alpha,j}^+ - b)^2 \Big] \bigg) d\alpha.$$

We minimize the function D(b) with respect to b, so to solve $\frac{\partial D(b)}{\partial b} = 0$, i.e,

$$\frac{\partial D(b)}{\partial b} = \int_0^1 \left(\sum_{j=1}^{k_j} f(\alpha) \left[(b - B_{\alpha,j}^-) + (b - B_{\alpha,j}^+) \right] \right) d\alpha$$
$$= \int_0^1 \left(\sum_{j=1}^{k_j} f(\alpha) \left[2b - (B_{\alpha,j}^- + B_{\alpha,j}^+) \right] \right) d\alpha = 0,$$

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the solution is

$$b = \overline{M}_f(B) = \int_0^1 \frac{1}{k_j} f(\alpha) \left(\sum_{j=1}^{k_j} \frac{B_{\alpha,j}^- + B_{\alpha,j}^+}{2} \right) d\alpha.$$

since $\frac{\partial^2 D(b)}{\partial b^2} = 2k_j > 0$, the solution $\overline{M}_f(B)$ indeed minimizes D(b). Now we prove the uniqueness of the $\overline{M}_f(B)$. For any real number $c \neq \overline{M}_f(B)$,

$$\begin{split} D(c) &= D_f(B,c) := \frac{1}{2} \int_0^1 \bigg(\sum_{j=1}^{k_j} f(\alpha) \big[(B_{\alpha,j}^- - c)^2 + (B_{\alpha,j}^+ - c)^2 \big] \bigg) d\alpha \\ &= \frac{1}{2} \int_0^1 \bigg(\sum_{j=1}^{k_j} f(\alpha) [((B_{\alpha,j}^- - \overline{M}_f(B)) + (\overline{M}_f(B) - c))^2 \\ &+ ((B_{\alpha,j}^+ - \overline{M}_f(B)) + (\overline{M}_f(B) - c))^2 \big] \bigg) d\alpha \\ &= D(\overline{M}_f(B)) + 2k_j (\overline{M}_f(B) - c)^2 + 2(\overline{M}_f(B) - c) \\ &\times \sum_{j=1}^{k_j} \int_0^1 f(\alpha) (B_{\alpha,j}^- + B_{\alpha,j}^+ - 2\overline{M}_f(B)) d\alpha \\ &= D(\overline{M}_f(B)) + 2k_j (\overline{M}_f(B) - c)^2, \end{split}$$

thus, D(c) $D(M_f(B)) = 2k_j(M_f(B) - c)^2$ > 0., 1.e., D(c)

The above three theorems indicate that the WPM, WIVPM of fuzzy number A is the nearest point (number), nearest interval to A under the topological structure induced by the distance D_f , respectively, and so is the WPM of imprecise number B. From a view of quality engineering, using real number data or interval data is more convenient than using fuzzy data, and here the WPM and WIVPM just provide an appropriate way of transforming fuzzy data into real data or intervals. Therefore, the WIVPM may be a reasonable and accurate representative interval and the WPM a crisp representative value of A if we correctly choose the weighting function. In this article, we consider them as representative intervals and crisp representative values of A, respectively.

3. Control Charts for Fuzzy Attribute Data Based on WPM and WIVPM

It is well known that the classical count of nonconformity control chart (c-chart) is based on crisp attribute data (Barrie and Brown, 1991), and whose control limits can be determined by the 3σ rule, i.e.,

$$CL_c = \bar{c},\tag{3.15}$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}},\tag{3.16}$$

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}},\tag{3.17}$$

where \bar{c} is the mean of the count of nonconformity and the count of nonconformity is usually modeled by Possion distribution.

Now we consider the fuzzy case, for which we introduce a sort of fuzzy random variables and imprecise random variables. Let (Ω, \mathcal{A}, P) be a complete probability space, and let $F(\mathbb{R})$ be the set of all imprecise numbers on \mathbb{R} . Note that here $\mathcal{F}(\mathbb{R}) \subset F(\mathbb{R})$. A mapping $X : \Omega \to \mathcal{F}(\mathbb{R})$, which is measurable with respect to the measurable structure induced by the distance D_f , is said to be a fuzzy random variable. A mapping $X : \Omega \to F(\mathbb{R})$, which is measurable structure induced by the distance D_f , is said to be a fuzzy random variable. A mapping $X : \Omega \to F(\mathbb{R})$, which is measurable with respect to the measurable structure induced by the distance D_f , is said to be a nimprecise random variable. By $I_{[a,b]}(\mathbb{R})$ we denote the set of all closed intervals on \mathbb{R} , and the mappings $\xi_{WIVPM} : \mathcal{F}(\mathbb{R}) \to I_{[a,b]}(\mathbb{R})$, $A \mapsto [M_f^-(A), M_f^+(A)]$, and $\xi_{WPM} : F(\mathbb{R}) \to \mathbb{R}$, $B \mapsto \overline{M}_f(B)$. Then, the composed mapping $\xi_{WIVPM} \circ X : \Omega \to I_{[a,b]}(\mathbb{R})$ is a random interval, and $\xi_{WPM} \circ X : \Omega \to \mathbb{R}$ is a random variable.

Based on the concept of fuzzy quality (Wang, 2006), the conformity and nonconformity are actually fuzzy notions. Here we assume the count of nonconformity to be a fuzzy attribute data and expressed by *LR*-fuzzy number (for the detail and examples the readers are referred to Gülbay and Kahraman (2007), which can be viewed as a realization of the fuzzy random variable X_{LR} valued in $\mathcal{F}_{LR}(\mathbb{R})$, the set of all *LR*-fuzzy numbers (Wang, 2006). In this case, for the fuzzy samples of size *e*, which is represented by *LR*-fuzzy numbers $A^j = (m^j, n^j, l^j, r^j)_{LR}$, $j = 1, 2, \ldots, e$, the *f*-weighted possibilistic representative values of A^j can be calculated by

$$\overline{M}_f(A^j) = \frac{1}{2}(m^j + n^j - l^j\phi_f + r^j\psi_f), \ j = 1, 2, \cdots, e.$$
(3.18)

Which are the realizations of the random variable $\xi_{WPM} \circ X_{LR} : \Omega \to \mathbb{R}$ having Possion distribution approximately. Similar to the calculations of identities (2.15) - (2.17), the control limits (CL \overline{CL}_f , LCL \overline{LCL}_f and UCL \overline{UCL}_f) of the control chart based on WPM of fuzzy attribute data can be calculated under the 3σ rule by

$$\overline{CL}_f = \frac{1}{e} \sum_{j=1}^{e} \overline{M}_f(A^j) = \frac{1}{2} (\overline{m} + \overline{n} - \overline{l}\phi_f + \overline{r}\psi_f), \qquad (3.19)$$

$$\overline{LCL}_f = \overline{CL}_f - 3\sqrt{\overline{CL}_f}, \qquad (3.20)$$

$$\overline{UCL}_f = \overline{CL}_f + 3\sqrt{\overline{CL}_f}.$$
(3.21)

After plotting representative value $\overline{M}_f(A)$ of the sample A on the control chart, we can make a decision on the process.

Alternatively, we can also use representative intervals WIVPM of the samples to construct some control chart. With identity (2.10), we can calculate the *f*-weighted possibilistic representative intervals of samples A^{j} , j = 1, 2, ..., e, by

$$M_f(A^j) = [m^j - l^j \phi_f, n^j + r^j \psi_f] := [a^j, b^j], a^j < b^j,$$
(3.22)

and we can obtain the intervals of the control limits by

$$CL_{f} = \left[\frac{1}{e}\sum_{j=1}^{e}M_{f}^{-}(A^{j}), \frac{1}{e}\sum_{j=1}^{e}M_{f}^{+}(A^{j})\right] := [CL_{f}^{-}, CL_{f}^{+}],$$
$$LCL_{f} = \left[CL_{f}^{-} - 3\sqrt{CL_{f}^{-}}, CL_{f}^{+} - 3\sqrt{CL_{f}^{+}}\right] := [LCL_{f}^{-}, LCL_{f}^{+}],$$

$$UCL_f = [CL_f^- + 3\sqrt{CL_f^-}, CL_f^+ + 3\sqrt{CL_f^+}] := [UCL_f^-, UCL_f^+].$$

Then, the percentage (β^j) that the representative interval of each sample falls inside the control limits can be calculated. The sample *j* is considered to be strictly in control when the interval of the sample is completely inside the control limits $(\beta^j = 1)$, and out of control when it is completely outside the control limits $(\beta^j = 0)$. Furthermore, for a sample whose representative interval partially fall inside the control limits, we may give some intermediate decisions as "rather in control" if $\beta^j \ge \beta$ or "rather out of control" if $\beta^j < \beta$, where β is predefined and

$$\beta^{j} = \begin{cases} 0 & a^{j} \ge UCL_{f}^{+}, \\ \frac{UCL_{f}^{+} - a^{j}}{b^{j} - a^{j}} & (LCL_{f}^{-} \le a^{j} \le UCL_{f}^{+}) \land (b^{j} > UCL_{f}^{+}), \\ 1 & (a^{j} \ge LCL_{f}^{-}) \land (b^{j} \le UCL_{f}^{+}), \\ \frac{b^{j} - LCL_{f}^{-}}{b^{j} - a^{j}} & (LCL_{f}^{-} \le b^{j} \le UCL_{f}^{+}) \land (a^{j} < LCL_{f}^{-}), \\ 0 & b^{j} \le LCL_{f}^{-}. \end{cases}$$
(3.23)

3.1. A Numerical Example

We employ the data of the example given in Gülbay and Kahraman (2007). The samples from a toy company producing a large-sized toys are taken every 4 h to control number of nonconformities. Because of the large dimensions of the toys, the number of nonconformities may also be large. Thirty subgroups for number of nonconformities are collected as linguistic data which here expressed by trapezoidal *LR*-fuzzy numbers shown in Table 1, also their corresponding WPM and WIVPM under weighting function $f(x) = 3x^2$, and the control limits are shown in Table 1.

Three kinds of different weighting functions $f(x) = \begin{cases} 2.5, & \alpha \in (0.6, 1] \\ 0, & \alpha \in [0, 0.6] \end{cases}$, $f(x) = 3x^2$,

 $f(x) = 5x^4$ are taken to make decisions based on WPM and WIVPM, the decision results are summarized in Table 2. As it is clearly shown in this table, the decisions made by the two-control state WPM fuzzy chart approach heavily depend on the weighting functions. For example, we have made different decisions by choosing different weighting functions on sample 8, 16, 27 based on WPM.

As an alternative method, we calculated the percentage $(\beta^j, j = 1, 2, ..., k)$ that the interval of each WIVPM remains inside the control limits with identity (2.26) and the decisions results are shown in Table 2. After setting a minimum acceptable percentage as $\beta = 0.7$ by the quality control expert, some intermediate decisions like "rather in control" or "rather out of control" can be made. There also exists some small differences on decision-making when choosing different weighting functions, for example, the samples no. 27 and no. 30, but comparatively speaking, the differences are not so sharp as the WPM method. The decisions made by the four-control state WIVPM fuzzy chart approach are subjected to the choice of both percentage β and the weighting functions, thus WIVPM chart is more flexible than WPM chart.

Table	1
Table	

Fuzzy number $(m, n, l, n)_{LR}$ representation of 30 subgroups
$(L(x) = R(x) = \max\{0, 1 - x\}, f(\alpha) = 3\alpha^2)$

No.	$(m, n, l, r)_{LR}$	$\overline{M}_f (= WPM)$	$M_f (= WIVPM)$
1	$(30, 30, 5, 5)_{LR}$	30	[28.75, 31.25]
2	$(20, 30, 5, 5)_{LR}$	25	[18.75, 31.25]
3	$(5, 12, 1, 3)_{LR}$	8.75	[4.75, 12.25]
4	$(6, 6, 3, 2)_{LR}$	5.875	[5.25, 6.5]
5	$(38, 38, 6, 7)_{LR}$	38.13	[36.5, 39.75]
6	$(20, 24, 4, 4)_{LR}$	22	[19, 25]
7	$(4, 8, 1, 4)_{LR}$	6.38	[3.75, 9]
8	$(36, 44, 9, 6)_{LR}$	39.63	[33.75, 45.5]
9	$(11, 15, 2, 5)_{LR}$	13.38	[10.5, 16.25]
10	$(10, 13, 3, 2)_{LR}$	11.38	[9.25, 13.5]
11	$(6, 6, 3, 4)_{LR}$	6.13	[5.25, 7]
12	$(32, 32, 5, 5)_{LR}$	32	[30.75, 33.25]
13	$(13, 13, 2, 2)_{LR}$	13	[12.25, 13.75]
14	$(50, 52, 11, 3)_{LR}$	50	[47.25, 52.75]
15	$(38, 41, 10, 4)_{LR}$	38.75	[35.5, 42]
16	$(40, 40, 7, 4)_{LR}$	39.63	[38.25, 41]
17	$(32, 50, 4, 10)_{LR}$	41.75	[31, 52.5]
18	$(39, 39, 6, 4)_{LR}$	38.75	[37.5, 40]
19	$(15, 21, 3, 17)_{LR}$	19.75	[14.25, 25.25]
20	$(28, 28, 5, 8)_{LR}$	28.38	[26.75, 30]
21	$(32, 35, 4, 7)_{LR}$	33.88	[31, 36.75]
22	$(18, 28, 4, 5)_{LR}$	23.13	[17, 29.25]
23	$(30, 30, 6, 4)_{LR}$	29.75	[28.5, 29]
24	$(25, 25, 5, 6)_{LR}$	25.13	[23.75, 26.5]
25	$(31, 41, 6, 5)_{LR}$	35.88	[29.5, 42.25]
26	$(10, 25, 3, 3)_{LR}$	17.5	[9.25, 25.75]
27	$(5, 14, 2, 6)_{LR}$	10	[4.5, 15.5]
28	$(28, 35, 5, 3)_{LR}$	31.25	[26.75, 35.75]
29	$(20, 25, 3, 4)_{LR}$	22.63	[19.25, 26]
30	$(8, 8, 3, 7)_{LR}$	8.5	[7.25, 9.75]
CL		$\overline{CL}_f = 24.877$	$CL_f = [21.56, 28.14]$
LCL		$\overline{LCL}_f = 9.913$	$LCL_f = [7.63, 12.23]$
UCL		$\overline{UCL}_f = 39.841$	$UCL_f = [35.49, 44.05]$

4. A Comparison with Other Results

4.1. A Comparison using an Actual Data Set

For the convenience of comparison, we use the data set given in the numerical example in Sec. 3. Control limits and representative values based on fuzzy transformation methods like WPM, WIVPM, fuzzy mode, fuzzy midrange and fuzzy median are shown in Table 3, where the concerned weighting function is taken as $f(x) = 3x^2$ and some data are taken

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Table 2 Decisions based on WPM and WIVPM $(L(x) = R(x) = \max\{0, 1 - x\}, \beta = 0.70)$

Dec.	$= 3\alpha^2$	IC	IC	ROC	oc	IC	IC	ROC	RIC	IC	IC	oc	IC	IC	oC	IC	IC	ROC	IC	RIC	IC	IC									
β^{j}	$f(\alpha)$	1.00	1.00	0.64	0.00	1.00	1.00	0.22	0.92	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.63	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.71	1.00	1.00	0.03
Dec.	$= 5\alpha^4$	IC	IC	ROC	S	IC	IC	ROC	RIC	IC	IC	S	IC	IC	00	IC	IC	ROC	IC	ROC	IC	IC	BOC								
β^{j}	$f(\alpha)$ =	1.00	1.00	0.60	0.00	1.00	1.00	0.16	0.83	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.59	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.31	1.00	1.00	0 22
	Dec.	IC	IC	ROC	б	IC	IC	ROC	RIC	IC	IC	б	IC	IC	С	IC	IC	ROC	IC	RIC	IC	IC	RIC								
	$\beta^{j[2]}$	1.00	1.00	0.62	0.00	1.00	1.00	0.21	0.87	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.61	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.70	1.00	1.00	0.82
Dec.	$5\alpha^4$	IC	IC	З	З	IC	IC	З	IC	IC	IC	З	IC	IC	oc	IC	IC	З	IC	б	IC	IC	00								
\overline{M}_{f}	$f(\alpha) =$	30.00	25.00	8.67	5.92	38.08	22.00	6.25	39.75	13.25	11.42	6.08	32.00	13.00	50.33	39.00	39.75	41.50	38.83	19.17	28.25	33.75	23.08	29.83	25.08	35.92	17.50	9.83	31.33	22.58	8.33
Dec.	$3\alpha^2$	IC	IC	OC	oc	IC	IC	oc	IC	IC	IC	oc	IC	IC	oc	IC	IC	oc	IC	00											
\overline{M}_{f}	$f(\alpha) =$	30.00	25.00	8.75	5.88	38.13	22.00	6.38	39.63	13.38	11.38	6.13	32.00	13.00	50.00	38.75	39.63	41.75	38.75	19.75	28.38	33.88	23.13	29.75	25.13	35.88	17.50	10.00	31.25	22.63	8.50
	Dec.	IC	IC	OC	oc	IC	IC	oc	oc	IC	IC	oc	IC	IC	oc	IC	oc	oc	IC	OC	IC	IC	00								
	$\overline{M}_{f}^{[1]}$	30.00	25.00	8.60	5.95	38.05	22.00	6.15	39.85	13.15	11.45	6.05	32.00	13.00	50.60	39.20	39.85	41.30	38.90	18.70	28.15	33.65	23.05	29.90	25.05	35.95	17.50	9.70	31.40	22.55	8.20
	r	5	5	ю	0	7	4	4	9	5	7	4	5	0	б	4	4	10	4	17	8	7	5	4	9	5	ю	9	ю	4	7
	1	5	5	1	б	9	4	1	6	0	Э	б	5	0	11	10	7	4	9	б	5	4	4	9	5	9	Э	0	5	ю	б
	и	30	30	12	9	38	24	8	4	15	13	9	32	13	52	41	40	50	39	21	28	35	28	30	25	41	25	14	35	25	8
	ш	30	20	5	9	38	20	4	36	11	10	9	32	13	50	38	40	32	39	15	28	32	18	30	25	31	10	S	28	20	8
Approximately	or between	30	20 - 30	5-12	9	38	20–24	4-8	36-44	11–15	10-13	9	32	13	50-52	38-41	40	32–50	39	15-21	28	32–35	18–28	30	25	31–41	10–25	5-14	28–35	20–25	8
	No.	1	7	ю	4	5	9	7	8	6	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

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Dec. is short for decisions. IC, OC, RIC, ROC is short for in control, out of control, rather in control, rather out of control, respectively. ^{[1],[2]}The weighting function here is

 $f(\alpha) = \begin{cases} 2.5, & \alpha \in (0.6, 1], \\ 0, & \alpha \in (0, 0.6]. \end{cases}$

(3.24)

	Control limits and represen	tative values b $(L(x) =$	ased on WPM, WIVI = $R(x) = \max\{0, 1 - 1$	PM, fuzzy mode, fuz - x }, $f(\alpha) = 3\alpha^2$)	zy midrange and fuzzy me	dian
	Fuzzy number $(m, n, l, r)_{LR}$	Fuzzy W P M	transformation WIVPM	method Mode	Midrange ($\alpha = 0.6$)	Median ($\alpha = 0.6$)
$\frac{\overline{\text{CL}}_f}{\underbrace{UCL}_f}$		24.877 9.913 39.841	[21.56, 28.14] [7.63, 12.23] [35.49, 44.05]			00
SC	$(22.01, 20.93, 4.54, 5.14)_{LR}$			[22.07, 20.93]	24.95	24.88
LCL	$(7.10, 12.65, 5.95, 6.64)_{LR}$			[7.10, 12.65]	10.05	9.96
ŨŨL	$(36.95, 42.5, 6.04, 6.55)_{LR}$			[36.95, 42.50]	38.95	39.79

Ę, . 1 2 đ 5 đ Table 3 F -2 Ę ÷.

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j	$\overline{M}_{f}^{[1]}$ D.	$\overline{M}_{3\alpha^2}$ D.	$\overline{M}_{5\alpha^4}$ D.	$M_{f}^{[2]}$ D.	$M_{3\alpha^2}$ D.	$M_{5\alpha^4}$ D.	$f_{mod,j}$ D.	$f_{mr,j}^{\alpha=0.6}$ D.	$f_{med,j}^{\alpha=0.6}$ D.	$DFA_{\alpha=0.6}^{\beta=0.7}$ D.
1	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
2	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
3	OC	OC	OC	ROC	ROC	ROC	RIC	OC	OC	RIC
4	OC	OC	OC	OC	OC	OC	OC	OC	OC	ROC
5	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
6	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
7	OC	OC	OC	ROC	ROC	ROC	ROC	OC	OC	ROC
8	OC	IC	IC	RIC	RIC	RIC	RIC	IC	IC	RIC
9	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
10	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
11	OC	OC	OC	OC	OC	OC	OC	OC	OC	RIC
12	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
13	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
14	OC	OC	OC	OC	OC	OC	OC	OC	OC	OC
15	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
16	OC	IC	IC	IC	IC	IC	IC	IC	IC	IC
17	OC	OC	OC	ROC	ROC	ROC	ROC	OC	OC	ROC
18	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
19	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
20	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
21	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
22	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
23	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
24	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
25	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
26	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
27	OC	IC	OC	RIC	RIC	ROC	RIC	IC	OC	RIC
28	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
29	IC	IC	IC	IC	IC	IC	IC	IC	IC	IC
30	OC	OC	OC	RIC	RIC	ROC	IC	OC	OC	IC
OC	10	7	8	3	3	3	3	7	8	1
ROC	0	0	0	3	3	5	2	0	0	3
IC	20	23	22	21	21	21	22	23	22	22
RIC	0	0	0	3	3	1	3	0	0	4

Table 4Comparison of alternative approaches: WPM, WIVPM, fuzzy mode, fuzzy midrange,
fuzzy median and DFA ($L(x) = R(x) = \max\{0, 1 - x\}$)

DFA is short for direct fuzzy approach. D. is short for decisions. IC, OC, RIC, ROC is short for in control, out of control, rather in control, rather out of control, respectively. Here [1][2] the weighting function.

$$f(\alpha) = \begin{cases} 2.5, & \alpha \in (0.6, 1], \\ 0, & \alpha \in (0, 0.6]. \end{cases}$$
(4.25)

from table 1, and the data with respect to fuzzy mode, fuzzy midrange and fuzzy median are taken from table 4 in Gülbay and Kahraman (2007). Also the charting methods of fuzzy mode, fuzzy midrange, fuzzy median and DFA are given in Gülbay and Kahraman (2007).

The overall comparison of WPM and WIVPM with fuzzy mode, fuzzy midrange, fuzzy median and DFA are summarized in Table 4, where some data are taken from Tables 1 and 2, and the data with respect to fuzzy mode, fuzzy midrange and fuzzy median are taken from Table 5 in Gülbay and Kahraman (2007)($\alpha = 0.6$), data with respect to DFA are taken

k	ANIVCS of WPM	ANIVCS of $f_{mr}^{\alpha=0.6}$	ANIVCS of $f_{med}^{\alpha=0.6}$	ANIVCS of WIVPM	ANIVCS of $f_{mod}^{\alpha=0.6}$	ANIVCS of $DFA_{\alpha=0.6}$
0.2	344.8	353	344.8	236.06	235.51	155.7
0.4	254	259.6	254	141.79	141.79	93.83
0.6	167	168.1	167	83.09	83.78	43.4
0.8	106.2	106.2	106.2	50.18	48.02	24.78
1	66.6	67.3	66.6	31.34	31.29	15.5
1.2	42.9	43.4	42.9	20.25	20.21	10.09
1.6	18.6	18.8	18.6	9.57	9.52	4.87
2	8.7	8.7	8.7	5.27	5.26	2.95
2.2	6.1	6.2	6.1	4.09	4.09	2.28
2.5	3.8	3.8	3.8	2.99	2.99	1.76
3	2	2	2	1.99	1.99	1.32

Table 5ANIVCSs of WPM, f_{mr}^{α} , f_{med}^{α} , WIVPM, f_{mod} and DFA approaches $(L(x) = R(x) = \max\{0, 1 - x\})$

from the table 7 in Gülbay and Kahraman (2007) ($\alpha = 0.6$). The results shown in Table 4 indicate that different decisions can be made by using different fuzzy approaches for the same samples. For example, samples no. 30, no. 27, no. 17, no. 11, no. 8, no. 7, no. 4 and no. 3 are such cases, in which each sample is made with at least two different kinds of decision results. The total numbers of OC, ROC, IC and RIC for each fuzzy approaches shown in Table 4 indicate that our WPM (or WIVPM) approach performs more OC states (or OC, ROC cases) than that of fuzzy midrange and fuzzy median approaches (or fuzzy mode and DFA approaches). For instance, the $\overline{M}_{f}^{[1]}$ (WPM) approach performs 10 OC states, but the fuzzy midrange $f_{mr}^{\alpha=0.6}$ approach performs only 7 OC states. The $M_{5\alpha^4}$ (WIVPM) approach performs 3 OC and 5 ROC states, but the $DFA_{\alpha=0.6}^{\beta=0.7}$ approach performs only 1 OC and 3 ROC states. The $\overline{M}_{3\alpha^2}$ do the same performance with $f_{mr}^{\alpha=0.6}$ approach, so are the $\overline{M}_{5\alpha^4}$ and the $f_{med}^{\alpha=0.6}$ approaches. Thus, for the 30 subgroups toys data, WPM and WIVPM performed more sensitive than fuzzy midrange, fuzzy median and fuzzy mode, DFA, respectively. It is obvious that all the fuzzy approaches mentioned here are weak in robustness, since they are more or less affected by subjective parameters such as weighting functions, α -levels and the acceptable percentage β . Also the DFA, WIVPM and fuzzy mode approaches are more flexible than WPM, fuzzy midrange and fuzzy median approaches, since every approach in the former has four control states IC, RIC, ROC and OC as well as two subjective parameters β and weighting function, whereas every approach in the later has only two control states IC and OC as well as one subjective parameter the weighting function. Unlike the fuzzy mode, fuzzy midrange, fuzzy median and DFA approaches being suitable to only convex fuzzy data, the proposed WPM and WIVPM approaches are suitable for both convex and non-convex fuzzy data (imprecise data).

4.2. General Comparison Between Fuzzy Control Charts

The sensitivity of a control chart can be illustrated by the average run length (ARL), which is defined as the average number of inspected samples till the moment of an alarm. However, in the case of fuzzy approaches the conditions become much complicated, as there are four

control states OC, ROC, RIC and IC for WIVPM, fuzzy mode and DFA. We introduce the ANIVCS to measure the sensitivity of these fuzzy approaches. ANIVCS based on observation can be calculated by

And in general,

$$ANIVCS = \frac{1}{P(\text{variation of the control state})}$$
$$= \begin{cases} \frac{1}{P(OC)}, & \text{for WPM, } f_{mr}^{\alpha}, f_{med}^{\alpha}, \\ \frac{1}{P(OC \text{ or ROC or RIC})}, & \text{for WIVPM, } f_{mod}, \text{DFA.} \end{cases}$$

For calculating ANIVCS we need to calculate the probabilities P(OC or ROC or RIC) and P(OC) for the concerned approaches. Based on the central limit theorem, $\frac{X-\overline{X}}{\sqrt{\overline{X}}}$ is approximately governed by standard normal distribution if random variable X is modeled by Possion distribution, where \overline{X} denotes a large sample mean based on an observation on X. Since in our concerned issue the count of nonconformity is basically governed by Possion distribution, we may assume whose fuzzy observation, the fuzzy random variable $X_{LR} = (m, n, l, r)_{LR}$, $(L(x) = R(x) = \max\{0, 1 - x\})$, satisfies that the random variables m, n and $m - l\phi_f, n + r\psi_f$ can be modeled by Possion distribution approximately. Suppose that samples $A^j = (m^j, n^j, l^j, r^j)_{LR}$, e = 5000, $(L(x) = R(x) = \max\{0, 1 - x\})$ are taken from the in control fuzzy attribute process. Then, when the process is in control under each fuzzy control chart approach, respectively, we have

$$ANIVCS_{WPM}(0) = 370, ANIVCS_{f_{mr}^{\alpha}}(0) \approx 370, ANIVCS_{f_{med}^{\alpha}}(0) \approx 370, \qquad (4.24)$$
$$ANIVCS_{WIVPM}(0) \approx 370.6, ANIVCS_{f_{mod}}(0) \approx 370.6, ANIVCS_{DFA}(0) \approx 437.7. \qquad (4.25)$$

When the process is not in control state under some fuzzy approach, we need to calculate the probability of type two error for the fuzzy approach and to determine the corresponding ANIVCS. Here, we assume that the target value \overline{CL} of WPM approach shifts to $\overline{CL} - k\sqrt{\overline{CL}}$, the target value CL_{mr}^{α} of f_{mr}^{α} approach shifts to $CL_{mr}^{\alpha} - k\sqrt{CL_{mr}^{\alpha}}$, the target value CL_{med}^{α} of f_{med}^{α} approach shifts to $CL_{med}^{\alpha} - k\sqrt{CL_{med}^{\alpha}}$. For WIVPM approach we only consider the case that b^{j} deviated from the target $\overline{n} + \overline{r}\psi_{f}$ to $\overline{n} + \overline{r}\psi_{f} + k\sqrt{\overline{n} + \overline{r}\psi_{f}}$, for f_{mod} approach only the case that n^{j} deviated from the target \overline{n} to $\overline{n} + k\sqrt{\overline{n}}$ and for DFA only the case that n^{j} deviated from the target \overline{n} to $\overline{n} + k\sqrt{\overline{n}}$ and for DFA only the case that n^{j} deviated from the target \overline{n} to $\overline{n} + k\sqrt{\overline{n}} = 36$, $\overline{n} = 49$, $\overline{l} = 9$, $\overline{r} = 4$, and $l^{j} = 12$, $r^{j} = 6$. Then we can change k and calculate ANIVCSs of the fuzzy control charts approaches. The computational procedure of ANIVCSs of which are given in Appendix A, and the obtained ANIVCSs are shown in Table 5.

From Table 5 it can be clearly seen that ANIVCS of WPM takes the same values as ANIVCS of $f_{med}^{\alpha=0.6}$ (column 2, 4) for each value of k, but ANIVCS of WPM decreases more rapidly than ANIVCS of $f_{mr}^{\alpha=0.6}$ as the k increases (column 2, 3). This means that WPM is more sensitive than $f_{mr}^{\alpha=0.6}$, and WPM, $f_{med}^{\alpha=0.6}$ have the same sensitivity. ANIVCS

of DFA decreases more rapidly than ANIVCS of WIVPM and f_{mod} (column 7,5 and 6), ANIVCS of WIVPM decreases a little bit slowly than ANIVCS of f_{mod} (column 5,6) as the *k* increases. This means that DFA is more sensitive than both WIVPM and f_{mod} , WIVPM has almost the same sensitivity with f_{mod} . ANIVCSs of the later three approaches decrease more rapidly than ANIVCSs of the former three approaches as *k* increases, since their number of control states are more than that of the former.

5. Conclusion

In this article, the WPV and WIVPM of a fuzzy number (Carlsson and Fullér, 2001; Fullér and Majlender, 2003) are introduced to be representative values of a fuzzy attribute data, and fuzzy *c*-charts are established with WPV and WIVPM. The performance of the charts have been compared to existing fuzzy charts with a newly defined ANIVCS. The main conclusions are:

- (1) WPM and WIVPM approaches performed with a relatively good sensitivity as the previous proposed fuzzy approaches. WPM is more sensitive than fuzzy midrange. WIVPM has almost same sensitivity with fuzzy mode. It may be better to instead of fuzzy midrange and fuzzy median with WPM, and instead of fuzzy mode with WIVPM.
- (2) WPM and WIVPM approaches are suitable to both convex and non-convex fuzzy data. As a representative values of the concerned fuzzy data, they have better representativeness than the fuzzy midrange, fuzzy median and fuzzy mode, and their computation is much easier than DFA.
- (3) WPM and WIVPM approaches have a better robustness than that of fuzzy midrange, fuzzy median and DFA. It can be found from the computational procedure of the items in Table 5 that the choice of the weighting functions only have a small impact (which may be allowed to be ignored) on sensitivity of WPM and WIVPM. However, for the small size fuzzy data sample, the weighting functions will have a moderate impact on sensitivity (See Table 4). Therefore, determining a correct and reasonable weighting function is desired.

The construction of fuzzy control charts has faced to challenge in both theory and applications. The main difficulties come from the lack of appropriate distributional models for fuzzy data, therefore, the non-parametric statistical methods based on order, distance and operations for fuzzy data space maybe reasonable for the construction of fuzzy control charts. In the future research work, we will further consider the construction of control charts for the general imprecise data (non-convex fuzzy data) by using the WPM and WIVPM approaches, and a fuzzy control chart based on possibilistic variance. For obtaining a more powerful evaluation criteria for fuzzy control charts, we may further propose other methods which could be based on some fuzzy measure (non-additive measure).

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Appendix A

The computational procedure of ANIVCS when the process is in control states:

$$ANIVCC_{WPM}(0) = \frac{1}{1 - P(\overline{CL}_f - 3\sqrt{\overline{CL}_f} \leqslant \overline{M}_f \leqslant \overline{CL}_f - 3\sqrt{\overline{CL}_f})}$$

$$\approx \frac{1}{2(1 - \Phi(3))} = 370,$$

$$ANIVCC_{f_{mr}^{\alpha}}(0) = \frac{1}{1 - P(CL_{mr}^{\alpha} - 3\sqrt{CL_{mr}^{\alpha}} \leqslant f_{mr}^{\alpha} \leqslant CL_{mr}^{\alpha} + 3\sqrt{CL_{mr}^{\alpha}})}$$

$$\approx \frac{1}{2(1 - \Phi(3))} = 370,$$

$$ANIVCC_{f_{med}^{\alpha}}(0) = \frac{1}{1 - P(CL_{med}^{\alpha} - 3\sqrt{CL_{med}^{\alpha}} \leqslant f_{med}^{\alpha} \leqslant CL_{med}^{\alpha} + 3\sqrt{CL_{med}^{\alpha}})}$$

$$\approx \frac{1}{2(1 - \Phi(3))} = 370.$$

and

$$ANIVCC_{WIVPM}(0) = \frac{1}{1 - P(a^{j} \ge LCL_{f}^{-}, b^{j} \le UCL_{f}^{+})} \\ \approx \frac{1}{1 - \left[\Phi(3) - \Phi\left(-\sqrt{n} + \overline{r}\psi_{f}\right)\right] \left[\Phi\left(\frac{\overline{n} + \overline{r}\psi_{f} - (\overline{m} - \overline{l}\phi_{f}) + 3\sqrt{\overline{n} + \overline{r}\psi_{f}}}{\sqrt{\overline{m} - \overline{l}\phi_{f}}}\right) - \Phi(-3)\right]},$$

$$ANIVCC_{f_{mod}}(0) = \frac{1}{1 - P(m^{j} \ge \overline{m} - 3\sqrt{\overline{m}}, n^{j} \le \overline{n} + 3\sqrt{\overline{n}})} \\ \approx \frac{1}{1 - [\Phi(3) - \Phi(-\sqrt{\overline{n}})] \left[\Phi\left(\frac{\overline{n} - \overline{m} + 3\sqrt{\overline{n}}}{\sqrt{\overline{m}}}\right) + \Phi(-3)\right]},$$

$$ANIVCC_{DFA}(0) = \frac{1}{1 - [\Phi(3) - \Phi(-\sqrt{\overline{n}})] \left[\Phi\left(\frac{\overline{n} - \overline{m} + 3\sqrt{\overline{n}}}{\sqrt{\overline{m}}}\right) + \Phi(-3)\right]},$$

$$= \frac{1}{1 - P(A_{\alpha}^{j+} \ge \widetilde{UCL}_{\alpha}^{+}, n^{j} \le \overline{n} + 3\sqrt{\overline{n}}, \overline{m} - 3\sqrt{\overline{m}} \le m^{j}, \widetilde{LCL}_{\alpha}^{-} \le A_{\alpha}^{j-})},$$

where

$$\begin{split} & P\left(A_{\alpha}^{j+} \geqslant \widetilde{UCL}_{\alpha}^{+}, n^{j} \leqslant \overline{n} + 3\sqrt{\overline{n}}, \overline{m} - 3\sqrt{\overline{m}} \leqslant m^{j}, \widetilde{LCL}_{\alpha}^{-} \leqslant A_{\alpha}^{j-}\right) \\ &\approx [\Phi(3) - \Phi(-\sqrt{\overline{n}})] \Phi\left(\frac{3\sqrt{\overline{n}} + \frac{3}{2}\sqrt{\overline{r}}(1-\alpha)}{\sqrt{\overline{n} + \frac{\overline{r}}{2}}(1-\alpha)}\right) \\ &\times \left[\Phi\left(\frac{\overline{n} + 3\sqrt{\overline{n}} - l^{j}(1-\alpha) - (\overline{m} - \overline{l}(1-\alpha))}{\sqrt{\overline{m} - \overline{l}}(1-\alpha)}\right) \\ &- \Phi\left(\frac{-3(\sqrt{\overline{m}} + \sqrt{\overline{l}}(1-\alpha))}{\sqrt{\overline{m} - \overline{l}}(1-\alpha)}\right) \right]. \end{split}$$

The computational procedure of ANIVCS when the process is already not in control states:

$$\begin{split} ANIVCS_{WPM} &\approx \frac{1}{1 - \left[\Phi\left(\frac{(3+k)\sqrt{CL}}{\sqrt{CL} - k\sqrt{CL}}\right) - \Phi\left(\frac{(k-3)\sqrt{CL}}{\sqrt{CL} - k\sqrt{CL}}\right)\right]}, \\ ANIVCS_{f_{mr}^{\alpha}} &\approx \frac{1}{1 - \left[\Phi\left(\frac{(3+k)\sqrt{CL_{mr}^{\alpha}}}{\sqrt{CL_{mr}^{\alpha}} - k\sqrt{CL_{mr}^{\alpha}}}\right) - \Phi\left(\frac{(k-3)\sqrt{CL_{mr}^{\alpha}}}{\sqrt{CL_{mr}^{\alpha} - k\sqrt{CL_{mr}^{\alpha}}}}\right)\right]}, \\ ANIVCS_{f_{med}^{\alpha}} &\approx \frac{1}{1 - \left[\Phi\left(\frac{(3+k)\sqrt{CL_{med}^{\alpha}}}{\sqrt{CL_{med}^{\alpha}} - k\sqrt{CL_{med}^{\alpha}}}\right) - \Phi\left(\frac{(k-3)\sqrt{CL_{med}^{\alpha}}}{\sqrt{CL_{med}^{\alpha} - k\sqrt{CL_{med}^{\alpha}}}}\right)\right]}, \\ ANIVCS_{WIVPM} &\approx \frac{1}{1 - s_{1}}, ANIVCS_{f_{med}^{\alpha}} \approx \frac{1}{1 - s_{2}}, ANIVCS_{DFA} \approx \frac{1}{1 - s_{3}}, \end{split}$$

where

$$s_{1} = \left[\Phi\left(\frac{(3-k)\sqrt{n}+\overline{r}\psi_{f}}{\sqrt{n}+\overline{r}\psi_{f}+k\sqrt{n}+\overline{r}\psi_{f}}\right) - \Phi\left(-\sqrt{n}+\overline{r}\psi_{f}+k\sqrt{n}+\overline{r}\psi_{f}}\right)\right]$$

$$\times \left[\Phi\left(\frac{\overline{n}+\overline{r}\psi_{f}-(\overline{m}-\overline{l}\phi_{f})+3\sqrt{n}+\overline{r}\psi_{f}}{\sqrt{\overline{m}}-\overline{l}\phi_{f}}\right) - \Phi(-3)\right],$$

$$s_{2} = \left[\Phi\left(\frac{(3-k)\sqrt{n}}{\sqrt{\overline{n}+k\sqrt{n}}}\right) - \Phi\left(-\sqrt{\overline{n}+k\sqrt{n}}\right)\right] \left[\Phi\left(\frac{\overline{n}-\overline{m}+3\sqrt{\overline{n}}}{\sqrt{\overline{m}}}\right) - \Phi(-3)\right],$$

$$s_{3} = \left[\Phi\left(\frac{(3-k)\sqrt{\overline{n}}}{\sqrt{\overline{n}+k\sqrt{\overline{n}}}}\right) - \Phi\left(-\sqrt{\overline{n}+k\sqrt{\overline{n}}}\right)\right]$$

$$\times \Phi\left(\frac{(3-k)\sqrt{\overline{n}}+0.5(\overline{r}+3\sqrt{\overline{r}})(1-\alpha)-r^{j}(1-\alpha)}{\sqrt{\overline{n}+k\sqrt{\overline{n}}}}\right)$$

$$\times \left[\Phi\left(\frac{\overline{n}+3\sqrt{\overline{n}}-l^{j}(1-\alpha)-(\overline{m}-\overline{l}(1-\alpha))}{\sqrt{\overline{m}-\overline{l}(1-\alpha)}}\right) - \Phi\left(\frac{-3(\sqrt{\overline{m}}+\sqrt{\overline{l}}(1-\alpha))}{\sqrt{\overline{m}-\overline{l}(1-\alpha)}}\right)\right].$$