Explicit Dynamic Quadratic Programming creating Fibonacci number and Golden ratio

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Contents

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We list several classes of Dynamic Programming, especially, Linear Quadratic Control problem. These are simple optimizations but interesting since they are related to famous subjects:

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- **Division of numbers with quadratic criterion.**
- **Golden optimal solution in quadratic programming.**
- **Matrix of threefold diagonal form with Fibonacci components.**
- **Multi-variate stopping with a monotone rule for three persons.**

These are basically connected with Fibonacci Number and Golden Ratio, which means recurrence, iterative structure, duality and so on.

Rules and criterion

(1) Given c, divide it into two.
(2) By selection a x, one is x and the other is $c - x$.
(3) Criterion is the each of quadratics; $x^2 + (c - x)^2$.
(4) Next is consider $c - x$ and so divide it until the final stop.
(5) Aim is to minimize the total sum of division

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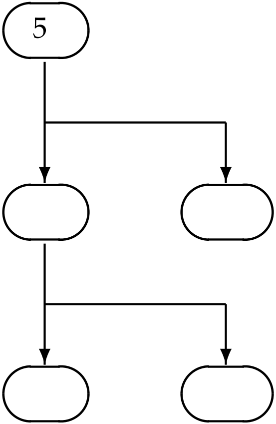

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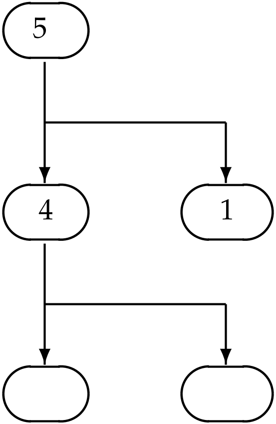

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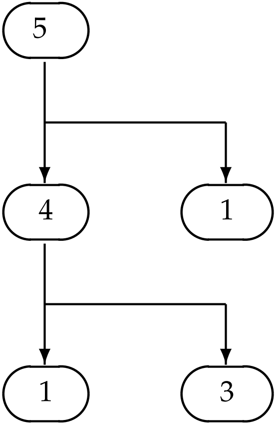

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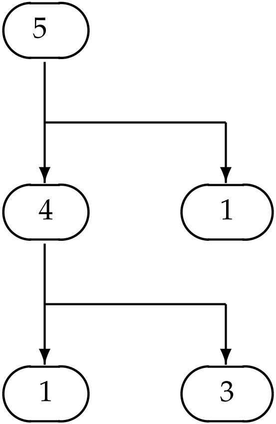
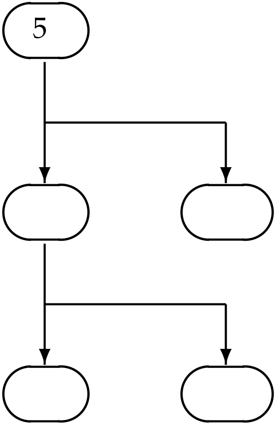


$\mathbf{1^2 + 4^2 + 3^2 + 1^2 \ = \ 27}$(Try another!)

Optimal partition is



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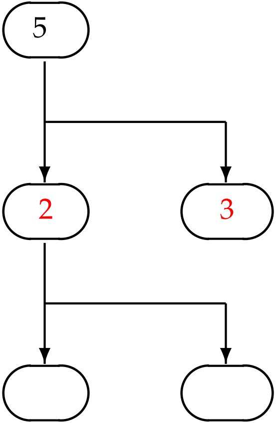

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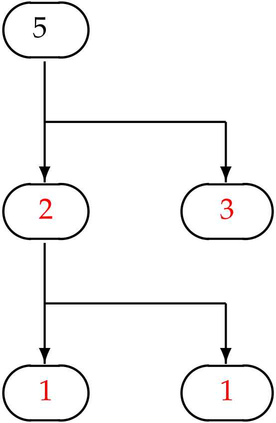

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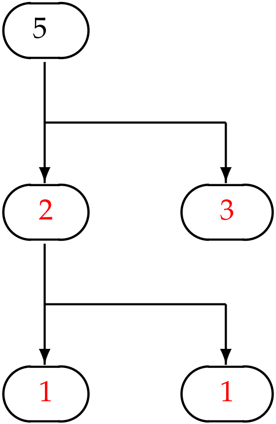

$$\mathbf{3^2 + 2^2 + 1^2 + 1^2 = 15}\text{(Optimal Partition!)}$$

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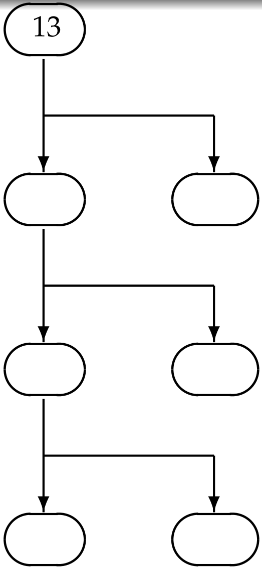

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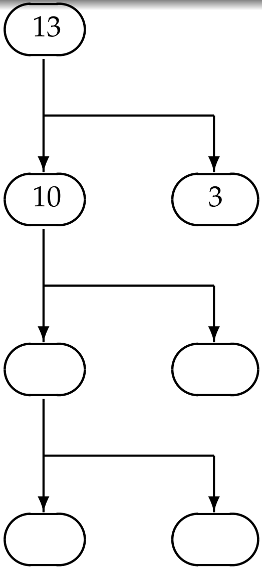

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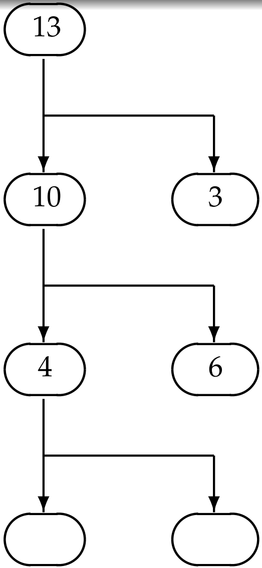

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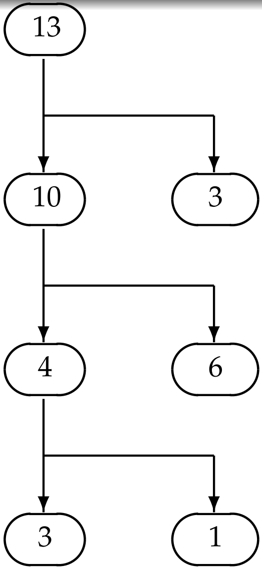

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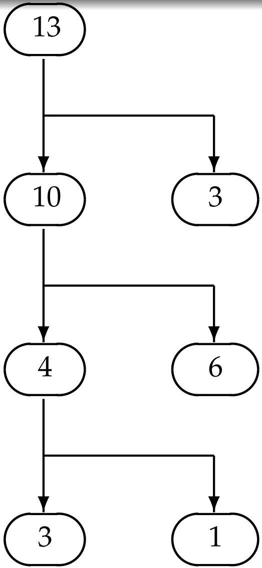
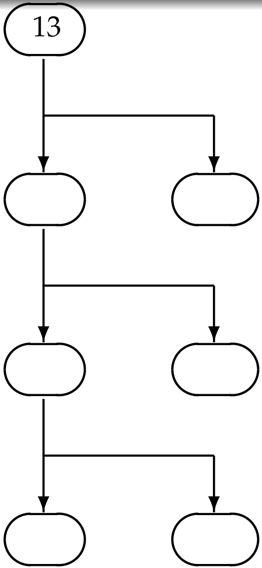


$3^2 + 10^2 + 6^2 + 4^2 + 1^2 + 3^2 = 171$(Try another!)

Start number is "13"



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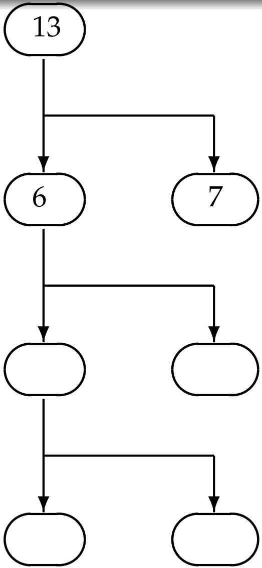

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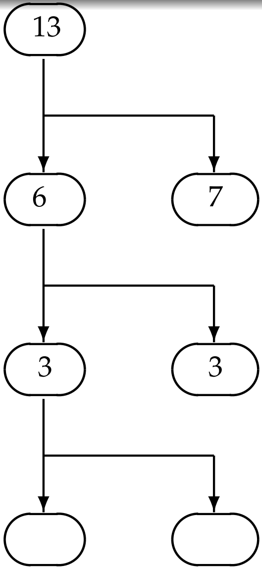

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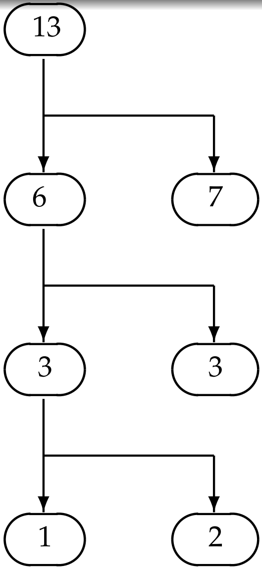

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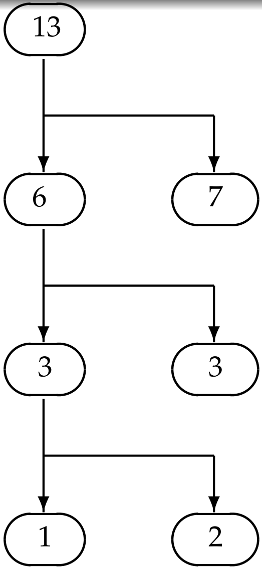

$\mathbf{7^2 + 6^2 + 3^2 + 3^2 + 2^2 + 1^2 = 108}$(Try another!)

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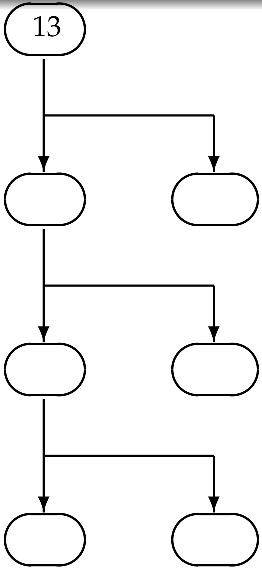

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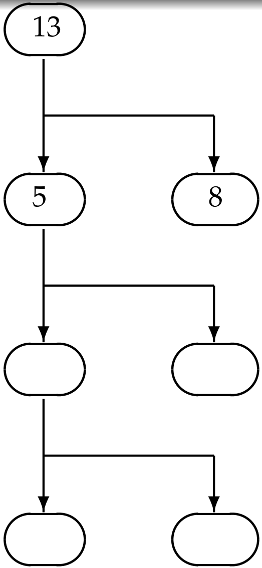

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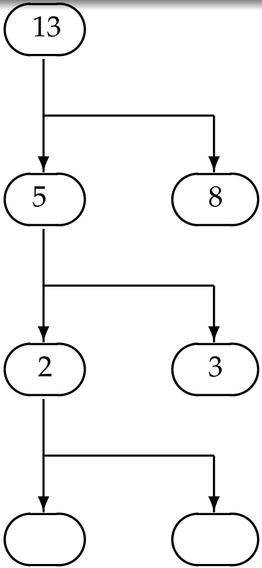

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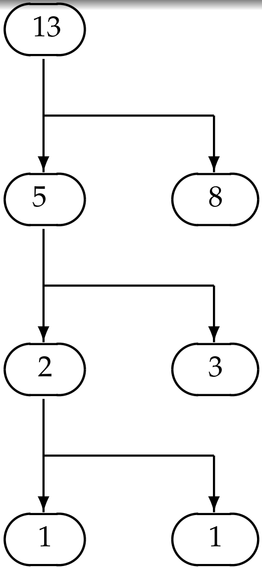

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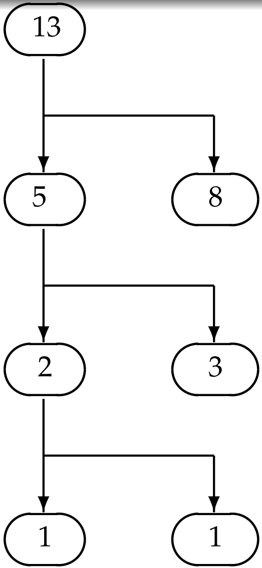

$$8^2 + 5^2 + 3^2 + 2^2 + 1^2 + 1^2 = 104 = 13 \times 8$$ (Optimal !!)

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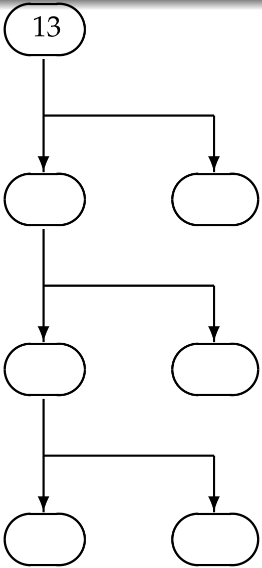

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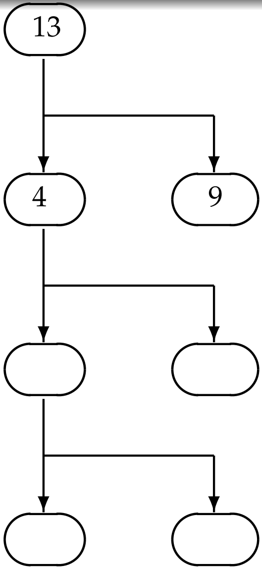

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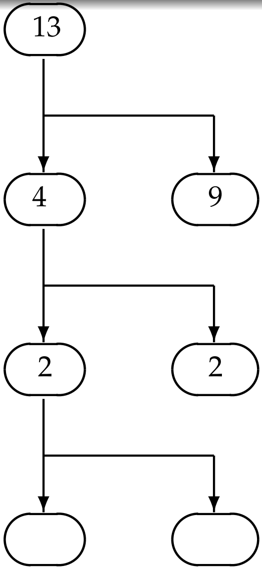

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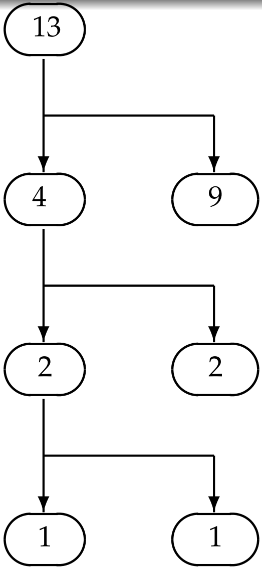

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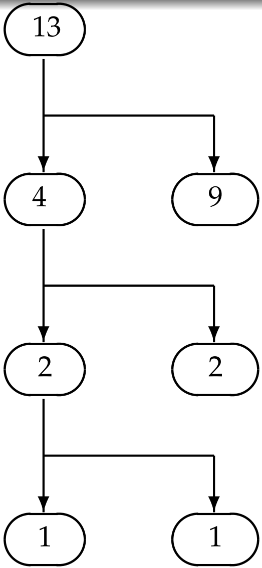
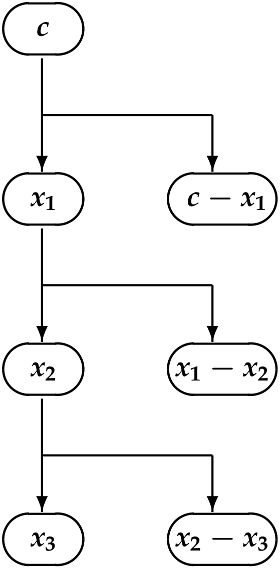


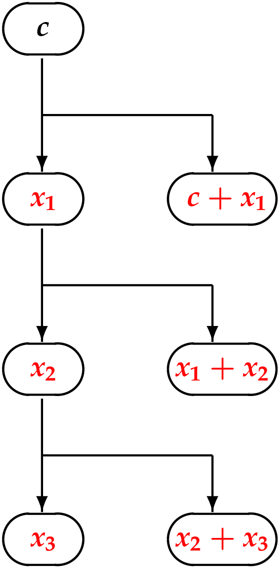
$$9^2+4^2+2^2+2^2+1^2+1^2 \ = \ 107 > 104 = 13 \times 8$$ (Not Optimal)

Change Sign



$$\text{minimize} \quad (c - x_1)^2 + x_1^2 + (x_1 - x_2)^2 + x_2^2 + (x_2 - x_3)^2 + x_3^2$$

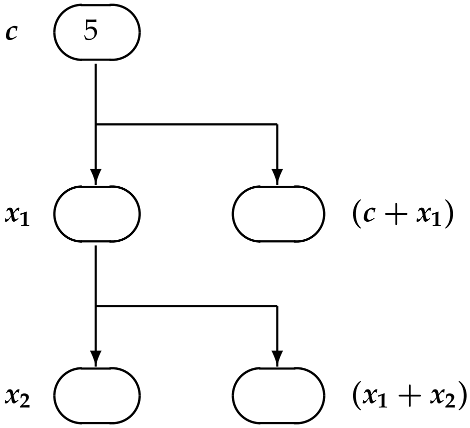
Change Sign



$$\text{minimize} \quad (c + x_1)^2 + x_1^2 + (x_1 + x_2)^2 + x_2^2 + (x_2 + x_3)^2 + x_3^2$$

Alternative sign Positive/Negative

Optimal partition:



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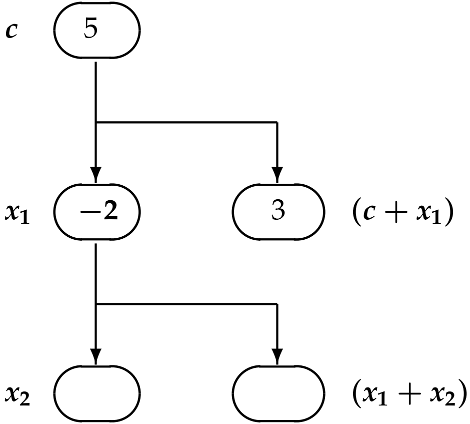

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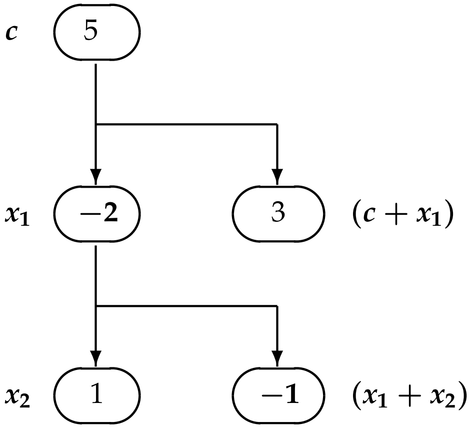

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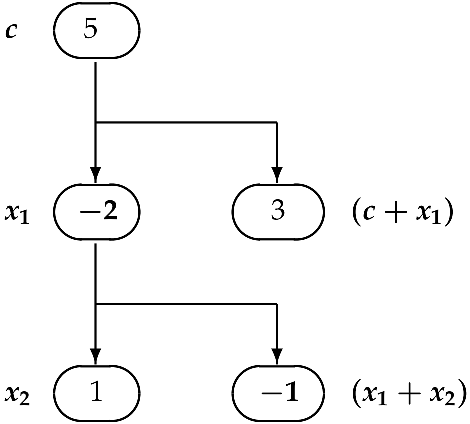
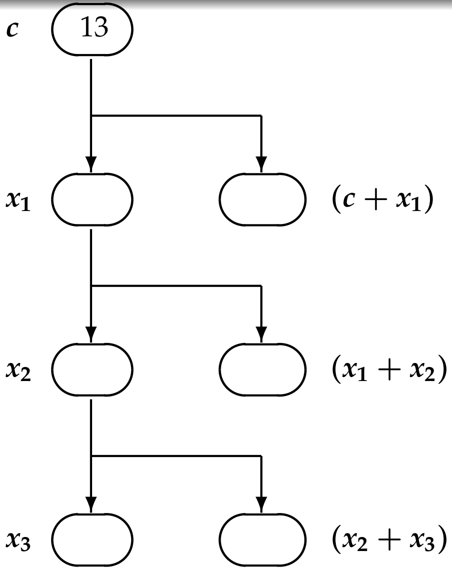
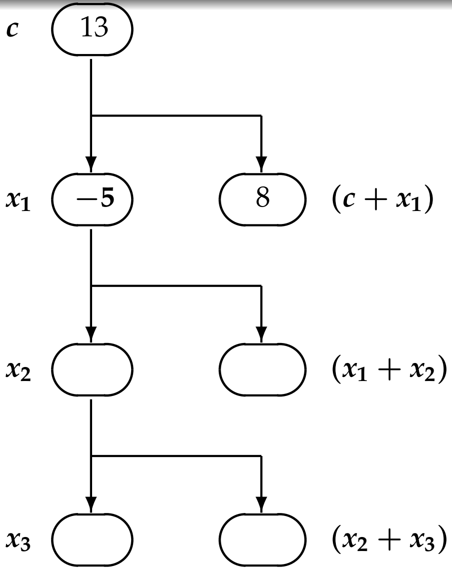


$$3^2 + (-2)^2 + (-1)^2 + 1^2 = 15 \text{Optimal}$$

Three steps



Three steps



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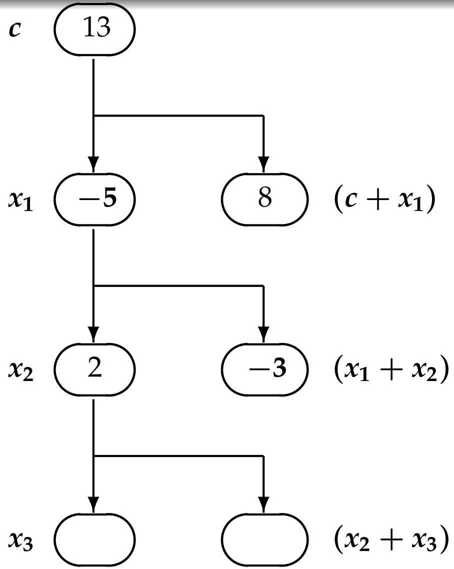

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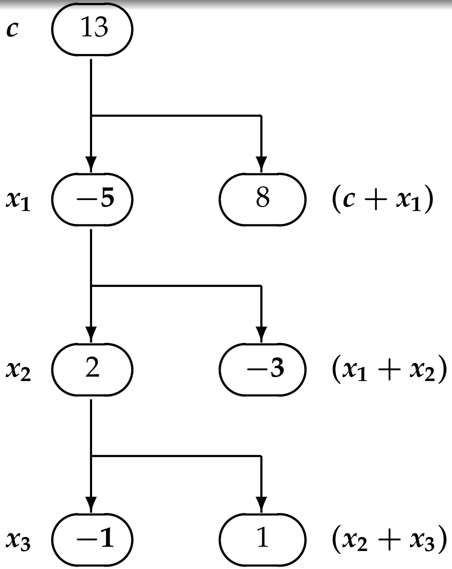

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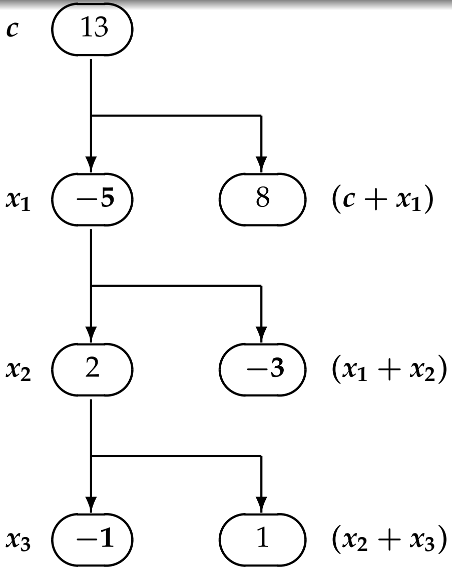

$$8^2 + (-5)^2 + (-3)^2 + 2^2 + 1^2 + (-1)^2 = 104$$ Optimal

Lucas Formula

(Lucas formula)

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144

Table 1: Fibonacci sequence $\{F_n\}$

Lucas Formula

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Table 1: Fibonacci sequence $\{F_n\}$

For exapmle, $n = 5$

$$\begin{aligned} & F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 \\ = \; & 1^2 + 1^2 + 2^2 + 3^2 + 5^2 \\ = \; & 40 \\ = \; & F_5 F_6 \end{aligned}$$

Optimal value $\boldsymbol{V9}$ of Number division problem is given by

$$\begin{aligned} \boldsymbol{V9} \;=\; & \left[(F_9 - F_7)^2 + (-F_7)^2\right] + \left[(-F_7 + F_5)^2 + F_5^2\right] \\ & + \left[(F_5 - F_3)^2 + (-F_3)^2\right] + \left[(-F_3 + F_1)^2 + F_1^2\right] \end{aligned}$$

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Clearly this leads to

$$\begin{aligned} &= F_8^2 + F_7^2 + F_6^2 + F_5^2 + F_4^2 + F_3^2 + F_2^2 + F_1^2 \\ &= F_8 \cdot F_9 = 104 \end{aligned}$$

by Lucas formula.

Quasi-cubic sum

Lemma (quasi-cubic sum)

For $\{F_k\}$, we have

$$2\sum_{k=1}^{n} F_k^2 F_{k+1} = F_n F_{n+1} F_{n+2}.$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
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Table 1: Fibonacci sequence $\{F_n\}$

This relates with a tiling problem in a plan or a space by using the unit of fibonacci square or cubes.

References I

S. Iwamoto, The Golden optimum solution in quadratic programming, Nonlinear Analysis and Convex Analysis (NACA05), Yokohama, 2007, pp. 199–205.

S. Iwamoto and Y. Kimura, The Alternately Fibonacci Complementary Duality in Quadratic Optimization Problem, Journal of Nonlinear Analysis and Optimization, Vol.2, No.1, 2011, pp.93–103.

S. Iwamoto and M. Yasuda, Golden optimal path in discrete-time dynamic optimization processes, Advanced Studies in Pure Mathematics 53, June 2009, pp.77–86.

Matrix form for recurrsiveness

Object function

$$I_3(x) = \sum_{i=0}^{3}\{x_i^2 + (x_i - x_{i+1})^2\}$$

is a quadratic of $x = (x_0, x_1,\ x_2,\ x_3), x_0 = c$:

$$I_3(x) = (x, Bx) + 2(b, x) + c^2$$

$$b \ = \ (c, 0, 0, 0)^T$$

provided starting number equals c. Expanding it, 4×4 matrix

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

becomes a posive definite and threefold diagonal form.

Its inverse is

$$B^{-1} = \frac{1}{34} \begin{pmatrix} 13 & -5 & 2 & -1 \\ -5 & 15 & -6 & 3 \\ 2 & -6 & 16 & -8 \\ -1 & 3 & -8 & 21 \end{pmatrix},$$

$$\hat{x} = -B^{-1}b = -B^{-1} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{c}{34} \begin{pmatrix} -13 \\ 5 \\ -2 \\ 1 \end{pmatrix}.$$

That is,

$$\hat{x} = \frac{c}{34}\begin{pmatrix} -13 \\ 5 \\ -2 \\ 1 \end{pmatrix} = \frac{c}{F_9}\begin{pmatrix} -F_7 \\ F_5 \\ -F_3 \\ F_1 \end{pmatrix}.$$

The minimum value m_4 is given by

$$\begin{aligned} m_4 &= c^2 - (b, B^{-1}b) \\ &= c^2 - \frac{13}{34}\,c^2 = \frac{21}{34}\,c^2 \\ &= \frac{F_8}{F_9}\,c^2. \end{aligned}$$

When we starting with $c = F_9$, these becomes simple ones.

Its inverse equals

$$C^{-1} = \frac{1}{34}\begin{pmatrix} 21 & -8 & 3 & -1 \\ -8 & 16 & -6 & 2 \\ 3 & -6 & 15 & -5 \\ -1 & 2 & -5 & 13 \end{pmatrix}$$

Thus

$$\mu^* = C^{-1}b = \frac{c}{34}\begin{pmatrix} 21 \\ -8 \\ 3 \\ -1 \end{pmatrix} = \frac{c}{F_9}\begin{pmatrix} F_8 \\ -F_6 \\ F_4 \\ -F_2 \end{pmatrix}$$

the maximum value is

$$M_4 = (b, C^{-1}b) = \frac{21}{34}c^2 = \frac{F_8}{F_9}\,c^2$$

Notes

These are easyly extended in $n \times n$ form.
The symmetric matrix A_n of the objective:

$$I_n(x) = \sum_{i=0}^{n} \{x_i^2 + (x_i - x_{i+1})^2\}$$

is a quadratic of $x = (x_0, x_1, x_2, \cdots, x_{n+1}), x_0 = c$:

Theorem

The determinant satisfies that

1. $$|A_n| - 3|A_{n-1}| + |A_{n-2}| = 0, |A_1| = 1, |A_2| = 2$$
2. $$|A_n| = F_{2n-1} (2n - 1\text{-th fibonacci number})$$

Details are omitted.

Notes

Refer to
S.Iwamoto, A.Kira and M.Yasuda; " Golden Duality in Dynamic Optimization ", Proceeding of Kosen Workshop, MTE2008, 2008, 1-13.

MDP model

MDP (S, A, P, R):
$S; \{x_n\} \in R^1, A; \{u_n\} \in R^1, P : x_{n+1} = x_n + u_n, x_0 = c, R :$

$$\sum_{n=0}^{\infty} (x_n^2 + u_n^2) \quad \text{and} \quad \sum_{n=0}^{\infty} \left(u_n^2 + x_{n+1}^2\right)$$

with a given initial state $x_0 = c$.
This section minimizes two quadratic cost functions.

MDP

Both problems are solved as a control process with criterion:

$$\sum_{n=0}^{\infty} \left[x_n^2 + (x_n - x_{n+1})^2\right] \quad \text{and} \quad \sum_{n=0}^{\infty} \left[(x_n - x_{n+1})^2 + x_{n+1}^2\right].$$

MDP model

Let $v(c)$ be the minimum value with initial value c. Then the value function $v = v(x)$ satisfies Bellman equation:

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + v(x+u)\right].$$

This has a quadratic form $v(x)$.
A quadratic minimum value function $v(x) = \phi x^2$, where

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad u = -\frac{\phi}{1+\phi}x.$$

Since

$$\min_{0\le a\le x} \{Aa^2 + B(x-a)^2\} = \frac{x^2}{1/A + 1/B}$$

hold, so the continued fraction leads to Golden number.

A monotone rule is introduced to sum up individual declarirations in a multi-variate stopping problem. The rule is defined by a monotone logical function and is equivalent to the winning class of Kadane('78). There given p-dimensional random process $\{X_n; n = 1, 2, \cdots\}$ and a stopping rule π by which the group decision determined from the declararion of p players at each stage. The stopping rule is p-variate $\{0, 1\}$-valued monotone logical function. We consider two cases of rules with $p = 3$ as follows:

$$\pi(x_1, x_2, x_3) = x_1 + x_2 \tag{1}$$

and

$$\pi(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3. \tag{2}$$

monotone rule

Monotone stopping rule:

x_1	x_2	$\pi(x_1, x_2, x_3)$
0	0	0
0	1	1
1	0	1
1	1	1

$\pi(x_1, x_2, x_3) = x_1 + x_2$ for any x_3

monotone rule

x_1	x_2	x_3	$\pi(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$\pi(x_1, x_2, x_3) = x_1x_2 + x_1x_3$$

Without loss of generality, we can assume that each X_n takes the uniformly distribution on $[0, 1]$. Then equilibrium expected values for each player is given as

	Player 1	Player 2	Player 3
$\pi(x_1, x_2, x_3) = x_1 + x_2$	$\frac{\sqrt{5}-1}{2}$	$\frac{\sqrt{5}-1}{2}$	0.5
$\pi(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3$	1	$\frac{\sqrt{5}-1}{2}$	$\frac{\sqrt{5}-1}{2}$

Table 1. The equilibrium expected value for each players.

In order to derive the value $\phi^{-1} = \frac{\sqrt{5}-1}{2}$, we consider an equilibrium stopping strategy of threshold type in the form $\{X_n > a\}$ for some a. Bellman type equation for this game version will be given as YNK('82). That is, each player declares "stop" or "continue" if the observed value exceeds some a or not. The event of the occurence is denoted by $D_n^i = \{\text{Player } i \text{ declares stop}\}$. Two trivial cases are the whole event Ω and the empty event $\emptyset$.

monotone rule

In generally a logical function is assumed "monotone" so its function can be written as

$$\begin{aligned} & \pi(x^1, \cdots, x^p) \\ = \; & x^i \cdot \pi(x^1, \cdots, \overset{i}{1}, \cdots x^p) \\ & + \overline{x^i} \cdot \pi(x^1, \cdots, \overset{i}{0}, \\ & \cdots x^p) \quad x^i \in \{0, 1\} \;\; \forall i \end{aligned}$$

where $\overline{x^i} = 1 - x^i$. Corresponding to this expression,

$$\begin{aligned} & \Pi(D^1, \cdots, D^p) \\ = \; & D^i \cdot \Pi(D^1, \cdots, \overset{i}{\Omega}, \cdots D^p) \\ & + \overline{D^i} \cdot \Pi(D^1, \cdots, \overset{i}{\emptyset}, \cdots D^p) \end{aligned}$$

where $\overline{D^i}$ is the complement of the event D^i.

The general equation for the expected for player i equals

$$E\left[(X_n^i - v^i)^+ \mathbf{1}_{\Pi(D_n^1,\cdots,\overset{i}{\Omega},\cdots D_n^p)}\right] + E\left[(X_n^i - v^i)^- \mathbf{1}_{\Pi(D_n^1,\cdots,\overset{i}{\emptyset},\cdots D_n^p)}\right] \tag{3}$$

where $D_n^i = \{X_n^i \geq v^i\}$ and
$(x)^+ = \max\{x, 0\}$, $(x)^- = \min\{x, 0\}$.

monotone rule

If we assume an independence case between player's random variable X_n^i for each i. The equation (3) becomes as

$$\beta_n^{\Pi(i)} E\left[(X_n^i - v^i)^+\right] - \alpha_n^{\Pi(i)} E\left[(X_n^i - v^i)^-\right] \tag{4}$$

where rule(a)Our objection is to find an equilibrium strategy and values of playes for a given monotone rule as the rule (1) and (2).

A sequence of expected value (a net gain) under the situation formulated in the section is obtained as

$$v_{n+1}^{i} = v_{n}^{i} + \beta_{n}^{\Pi(i)} E\left[(X_{n}^{i} - v_{n}^{i})^{+}\right] - \alpha_{n}^{\Pi(i)} E\left[(X_{n}^{i} - v_{n}^{i})^{-}\right] \quad (5)$$

for player $i = 1, \cdots, p$ and n denotes a time-to-go. The details refer to Theorem 2.1 in YKN[?]. Under these derivation, now we are able to calculate the optimal (equilibrium) value $v^{i} = \lim_{n} v_{n}^{i}$ for player $i = 1, 2, 3$ for the rule.

monotone rule

Refer to

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