Monotone set-valued functions defined by set-valued Choquet integrals

Sun Hongxia

Department of Mathematics, Dezhou University, Dezhou, Shandong 253023, China

Li Jun*

School of Science, Communication University of China, Beijing, 100024, China

Masami Yasuda

Department of Mathematics & Informatics, Faculty of Science, Chiba University, Chiba 263-8522, Japan

Abstract

In this paper, some properties of the monotone set-valued function defined by the set-valued Choquet integral are discussed. It is shown that several important structural characteristics of the original set function, such as null-additivity, strong order continuity, property(S) and pseudometric generating property, etc., are preserved by the new set-valued function. It is also shown that integrable assumption is inevitable for the preservation of strong order continuous and pseudometric generating property. Several kind of absolute continuity of set-valued function with respect to set function are also discussed.

Keywords: Monotone set-valued function; set-valued Choquet integrals; Choquet integral

1 Introduction

Given a measurable space $(X, \mathcal{A})$, a nonnegative monotone set function μ on $\mathcal{A}$ with $\mu(\emptyset) = 0$, and a nonnegative measurable function f, then the set function v_f defined by the Chouqet integral

$$v_f(A) = (C)\int_A f d\mu \quad (\forall A \in \mathcal{A})$$

**E-mail address*: lijun@cuc.edu.cn

is also nonnegative and monotone on $\mathcal{A}$ with $v_f(\emptyset) = 0$[1]. Wang and Klir ([1]) have discussed that $v_f(A)$ preserved some important structure characteristics of the original function μ, such as null-additivit, subadditivity, autocontinuity, e.t.c. Ouyang and Li ([2]) further discussed that v_f preserved other important structure characteristics of μ, such as strong order continuity, order continuity, property(S), pseudometric generating property.

Similarly to the single-valued fuzzy measure, we can define the structural characteristics for a set-valued fuzzy measure, whether these characteristics can be preserved by the set-valued set functions defined by set-valued Choquet integral?

In this paper, we give the definition of these characteristics of set-valued set function, and we prove that a new monotone set-valued function defined by set-valued Choquet integral also preserved the characteristics of original set-valued set function. We also discuss several kind of absolute continuity of set-valued function with respect to set function.

2 Preliminaries

Throughout this paper, we suppose that $(X, \mathcal{A})$ is a measurable space, f is a nonnegative measurable function on $(X, \mathcal{A})$, and μ is a monotone set function with $\mu(\emptyset) = 0$. R^+ denotes the interval $[0,\infty]$, $\mathcal{P}_0(R^+)(\mathcal{P}_f(R^+)$, resp.) denotes the class of all nonempty subsets(closed, resp.) of R^+.

Definition 2.1 ([3]) *The Choquet integral of f on A with respect to μ, denoted by $(C)\int_A f d\mu$, is defined as*

$$(C)\int_A f d\mu = \int_0^\infty \mu(F_\alpha \cap A) d\alpha$$

where $F_\alpha = \{f \geq \alpha\} = \{x | f(x) \geq \alpha\}$ for any $\alpha \geq 0$ and the right-hand side is the Lebesgue integral.

Definition 2.2 ([5]) *A set-valued function $F : X \to \mathcal{P}_0(R^+)$ is said to be measurable if its graph is measurable, that is ,*

$$G_r(F) = \{(x, r) \in X \times R^+ : r \in F(x)\} \in \mathcal{A} \otimes Borel(R^+)$$

where $Borel(R^+)$ is the Borel field of R^+. F is said to be closed-valued if it values in $\mathcal{P}_f(R^+)$

Definition 2.3 ([4]) *A set-valued function F is said to be Choquet integrably bounded if there is a Choquet integrable function g such that*

$$\|F(x)\| = \sup_{r\in F(x)} |r| \leq g(x)$$

Definition 2.4 ([5]) *Let F be a set-valued function and $A \in \mathcal{A}$. The Choquet integral of F on A with respect to μ, denote by $(C)\int_A Fd\mu$, is defined as*

$$(C)\int_A Fd\mu = \{(C)\int_A fd\mu : f \in S(F)\}$$

where S(F) is the family of μ-a.e. measurable selection of F.

Instead of $(C)\int_X Fd\mu$, we will write $(C)\int Fd\mu$. Obviously, $(C)\int Fd\mu$ may be empty. A set-valued function F is said to be integrable existing if $(C)\int Fd\mu \neq \emptyset$, and F is said to be integrable if $(C)\int Fd\mu$ exists and does not include ∞.

Definition 2.5 ([6]) *Let A, $B \in \mathcal{P}_0(R^+)$, then $A \leq B$ means that*

(1) *For each $x_0 \in A$,there exists $y_0 \in B$, such that $x_0 \leq y_0$;*

(2) *For each $y_0 \in A$,there exists $x_0 \in B$, such that $x_0 \leq y_0$.*

Definition 2.6 *A set-valued function $\pi : \mathcal{A} \to \mathcal{P}_0(R^+)$ is said to be a monotone set-valued function if it satisfies*

(1) $\pi(\emptyset) = 0$.

(2) $A \subset B$ implies $\pi(A) \leq \pi(B)$

Definition 2.7 *The set-valued function $(C)\pi_F$ is defined as*

$$(C)\pi_F(A) = (C)\int_A Fd\mu(A \in \mathcal{A})$$

where F is a set-valued function with $S(F) \neq \emptyset$, μ is a monotone set function.

Remark 2.1 *By proposition 3.5[5] , we know that $(C)\pi_F$ is a monotone set-valued function.*

Definition 2.8 ([6]) *Let $\{A_n\} \subset \mathcal{P}_0(R^+)$, we write*

$$\limsup_{n\to\infty} A_n = \{x \in R^+ : x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}(k \geq 1)\}$$

$$\liminf_{n\to\infty} A_n = \{x \in R^+ : x = \lim_{k\to\infty} x_n, x_n \in A_n \ (n \geq 1)\}$$

If $\lim_{n\to\infty} \sup A_n = \lim_{n\to\infty} \inf A_n = A$, then $\{A_n\}$ is said to be convergent to A and it is simply noted with $A_n \to A$.

Similarly to the single-valued function, we will defined several important structural characteristics for a set-valued function.

Definition 2.9 *Let π is a set-valued function.*

(1) *π is called null-additive if for any $A,\ B\in\mathcal{A}$, $\pi(A)=\{0\}$ implies $\pi(A\cup B)=\pi(B)$.*

(2) *π is called weakly null-additive if for any $A,\ B\in\mathcal{A}$,$\pi(A)=\pi(B)=\{0\}$ implies $\pi(A\cup B)=\{0\}$.*

(3)*π is said to be strongly order continuous if for any $A_n\in\mathcal{A}$, $A_n\downarrow A$ and $\pi(A)=\{0\}$ implies $\lim\limits_{n\to\infty}\pi(A_n)=\{0\}$*

(4) *π is said to be order continuous if for any $\{A_n\}\subset\mathcal{A}$, $A_n\downarrow\emptyset$ implies $\lim\limits_{n\to\infty}\pi(A_n)=\{0\}$*

(5) *π is said to have the property (S) if for any $A_n\in\mathcal{A}$ that satisfies $\lim\limits_{n\to\infty}\pi(A_n)=\{0\}$ there exists a subsequence $\{A_{n_k}\}\subset\{A_n\}$ such that $\pi(\lim\limits_{k\to\infty}\sup A_{n_k})=\{0\}$.*

(6) *π is said to have the pseudometric generating property, abbreviated as p.g.p. if for any $\epsilon>0$, there exists $\delta>0$ such that $\pi(A\cup B)\subset[0,\epsilon]$ whenever $A,B\in\mathcal{A}$, $\pi(A)\subset[0,\delta]$ and $\pi(B)\subset[0,\delta]$.*

It is easy to prove that the above definition is the generalization of single-valued case.

3 Preservation of structural characteristics

In this section, we suppose that F is a measurable closed-valued function.

Theorem 3.1 *If μ is null-additive, then so is $(C)\pi_F$.*

Proof. For any A, B$\in\mathcal{A}$ with $(C)\pi_F(B)=\{0\}$, i.e. $(C)\int_B Fd\mu=\{0\}$, then for any $f\in S(F)$, we have $(C)\int_B fd\mu=0$. Hence by theorem 7 [1] , we have

$$(C)\int_{A\cup B} fd\mu=(C)\int_A fd\mu.$$

Let $a_0\in(C)\pi_F(A\cup B)$, then exists $f\in S(F)$, such that

$$a_0=(C)\int_{A\cup B} fd\mu=(C)\int_A fd\mu\in(C)\pi_F(A).$$

Similarly, for $b_0\in(C)\pi_F(A)$, we can prove that $b_0\in(C)\pi_F(A\cup B)$.

The proof is completed.

Theorem 3.2 *If μ is weakly null-additive , then so is $(C)\pi_F$.*

Proof. It is similar to the proof of Theorem 3.1.

Theorem 3.3 *Let F is integrable. If μ is strongly order continuous, then so is $(C)\pi_F$.*

Proof. For any $\{A_n\} \subset \mathcal{A}$ with $A_n \downarrow A$ and $(c)\pi_F(A) = \{0\}$, then for any $f \in S(F)$, $v_f(A) = (C)\int_A f d\mu = 0$, hence by Theorem 3.(2) ([2]), we have

$$\lim_{n\to\infty} v_f(A_n) = 0, \quad for \;\; any \;\; f \in S(F).$$

Hence

$$\limsup_{n\to\infty}(C)\pi(F)(A_n) = \liminf_{n\to\infty}(C)\pi(F)(A_n) = \{0\}.$$

So

$$\lim_{n\to\infty}(C)\pi_F(A_n) = \{0\}.$$

Theorem 3.4 *Let F is integrable. If μ is order continuous, then so is $(C)\pi_F$.*

Proof. Similar to Theorem 3.3.

Theorem 3.5 *Let F is integrable. If μ has p.g.p, then so is $(C)\pi_F$.*

Proof. Since F is integrable, hence $(C)\int f d\mu < \infty$. Since μ have p.g.p., there exists $\delta > 0$, such that $\mu(E \cup F) < \epsilon$ whenever $E, F \in \mathcal{A}$ and $\mu(E) \vee \mu(F) < \delta$.

We will prove that $(C)\pi_F(A \cup B) \subset [0, \epsilon]$ whenever $A, B \in \mathcal{A}$, $(c)\pi_F(A) \subset [0, \delta]$ and $(c)\pi_F(B) \subset [0, \delta]$. In fact, for any $f \in S(F)$, we have $(C)\int_A f d\mu \vee (C)\int_B f d\mu < \delta$, by Theorem 5 ([2]), we have $(C)\int_{A\cup B} f d\mu < \epsilon$. Hence$(C)\pi_F(A \cup B) \subset [0, \epsilon]$.

Remark 3.1 Observe that Theorem 3.3 to Theorem 3.5 are based on the assumption that F is integrable. The following examples show that the conclusions in Theorem 3.3 to Theorem 3.5 may not be true when the assumption is abandoned.

Example 3.1 Let $X = \{1, 2, ..., \}$, $\mathcal{A} = \mathcal{P}(X)$. μ is defined as

$$\mu(E) = \begin{cases} 0, & E = \emptyset, \\ \\ max\{1/i | i \in E\}, & E \neq \emptyset. \end{cases}$$

It is easy to see that μ is strongly order continuous (so μ is order continuous) monotone set function. Let $F(X) = \{x\}(x \in X)$, then we have $(C)\pi_F(X) = \{\infty\}$, hence F is not Choquet integrable. we will show that $(C)\pi_F$ is not order continuous (so $(C)\pi_F$ is not strongly order continuous).

In fact, let $E_n = \{n, n+1, ...,\}$, then $E_n \downarrow \emptyset$. For any $f \in S(F)$, $f(x) = x$.

$$(C)\int_{E_n} f d\mu = \int_0^\infty \mu(E_n \cap F_\alpha) d\alpha \geq \int_0^n \mu(E_n \cap F_\alpha) d\alpha = 1.$$

Hence $(C)\pi_F$ is not order continuous.

Example 3.2 Let $X_1 = \{1, 3, ...,\}, X_2 = \{2, 4, ...\}, X = X_1 \bigcup X_2, \mathcal{A} = \mathcal{P}(X)$. μ is defined as

$$\mu(E) = \begin{cases} 0, & \text{if } \ E = \emptyset, \\ max\{1/i^2 | i \in E\}, & \text{if } \ E \in X_1 or X_2 \\ max\{1/i | i \in E\}, & otherwise \end{cases}$$

It is easy to see that μ have p.g.p.. Let $F(X) = \{x\}(x \in X)$, then we have $(C)\pi_F(X) = \{\infty\}$, hence F is not Choquet integrable. we will show that $(C)\pi_F$ do not have p.g.p..

Let $A_n = \{2n\}, B_n = \{2n+1\}$,then

$$(C)\int_{A_n} f d\mu = \int_0^\infty \mu(A_n \cap F_\alpha) d\alpha = \int_0^{2n} 1/(2n)^2 d\alpha = 1/2n \to 0 (n \to \infty)$$

Similarly,

$$(C)\int_{B_n} f d\mu \to 0 \quad (n \to \infty)$$

But

$$\begin{aligned} (C)\int_{A_n \cup B_n} f d\mu &= \int_0^\infty \mu((A_n \cup B_n) \cap F_\alpha) d\alpha \\ &= \int_0^{2n} \mu((A_n \cup B_n) \cap F_\alpha) d\alpha + \int_{2n+1}^{2n+2} \mu((A_n \cup B_n) \cap F_\alpha) d\alpha \\ &= \int_0^{2n} 1/2n d\alpha + \int_{2n+1}^{2n+2} 1/(2n+1)^2 d\alpha \\ &= 1 + 1/(2n+1)^2 \to 1 (n \to \infty). \end{aligned}$$

Hence $(C)\pi_F$ do not have p.g.p..

Theorem 3.6 *Let F is Choquet bounded and μ is continuous . If μ have property (S), then so is $(C)\pi_F$.*

Proof. For any For any $\{A_n\} \subset \mathcal{A}$ with $\lim_{n\to\infty}(C)\pi_F(A_n) = \{0\}$, we have

$$\lim_{n\to\infty} d(0,(C)\pi_F(A_n)) = 0$$

i.e.

$$\liminf_{n\to\infty}{}_{f\in S(F)} \mid (C)\int_{A_n} f d\mu \mid = 0$$

Then from the definition of limit inferior, we have that

$$\lim_{m\to\infty}(C)\int_{A_n} f_{n_{(m)}} d\mu = \inf_{f\in S(F)} \mid (C)\int_{A_n} f d\mu \mid \quad f_{n_{(m)}} \in S(F)$$

Since F is integrably bounded , so there exists a Choquet integrable function g , such that $\mid f(x) \mid \leq g(x)$ for any $f \in S(F)$. Hence we have

$$\lim_{m\to\infty}(C)\int_{A_n} f_{n_{(m)}} d\mu = (C)\int_{A_n} \lim_{m\to\infty} f_{n_{(m)}} d\mu.$$

Denote $f_0 = \lim_{m\to\infty} f_{n_{(m)}} d\mu$, then $\lim_{n\to\infty}(C)\int_{A_n} f_0 d\mu = 0$. By Theorem 2 in [2], there exists $\{A_{n_k}\} \subset A_n$, such that $v_{f_0}(\lim_k supA_{n_k}) = 0$. i.e.

$$(C)\int_{\lim_k supA_{n_k}} f_0 d\mu = \int_0^\infty \mu(\limsup_k A_{n_k} \cap \{f_0 \geq \alpha\})) = 0.$$

Hence for any $a_0 > 0$, we have

$$\mu(\lim_k supA_{n_k} \cap \{f_0 \geq \alpha\}) = 0 \quad a.e.$$

Denote $B_{n_k} = A_{n_k} \cap \{f_0 \geq \alpha_0\}$, then

$$\mu(\limsup_k B_{n_k}) = 0 \quad a.e.$$

Therefore, for any $f \in S(F)$

$$(C)\int_{\limsup_k B_{n_k}} f d\mu = \int_0^\infty \mu(\limsup_k A_{n_k} \cap \{f \geq \alpha\})) = 0.$$

Hence

$$(C)\pi_F(\limsup_k B_{n_k}) = \{0\}.$$

Definition 3.1 ([3]) *A set $N \in \mathcal{A}$ is called a null set (with respect to μ) if $\mu(A \cup N) = \mu(A)$, for all $A \in \mathcal{A}$.*

Definition 3.2 ([3]) *A fuzzy measure μ is said to be m-continuous if there exists a complete and finite measure m, such that $\mu \ll m$, i.e. m(A)=0 implies A is a null set (with respect to μ).*

Lemma 1 *$(C)\int_A F d\mu = (C)\int X_A \circ F d\mu$ where X_A is the characteristic function of A, and*

$$X_A \circ F(x) = \begin{cases} F(x), & x \in A, \\ \\ 0, & otherwise. \end{cases}$$

Proof. It is easy to be proved.

Theorem 3.7 *Let μ is m-continuous and F is Choquet integrable bounded. If μ is continuous, then so is $(C)\pi_F$.*

Proof. It is similar to the proof of Proposition 3 in [7].

4 Absolutely continuity

Absolutely plays an important role in measure theorem, we expect that most of the classical results, such as Radon-Nikodym theorem, etc., remain valid for monotone set-valued function. In this section, we will introduce several absolutely continuous.

Definition 4.1 *Let μ is a set function and π is a set-valued function , we say that*

(1) *π is absolutely continuous of type I with respect to μ, denote by $\pi \ll_I \mu$, if and only if $\pi(A) = \{0\}$ whenever $\mu(A) = 0$.*

(2) *π is absolutely continuous of type II with respect to μ, denote by $\pi \ll_{II_\alpha} \mu$, if and only if $\pi(A) = \pi(B)$ whenever $B \subset A$ and $\mu(A \setminus B) = 0$.*

(3) *π is absolutely continuous of type III with respect to μ, denote by $\pi \ll_{III_\alpha} \mu$, if and only if $\lim_{n\to\infty} \pi(A_n) = \{0\}$ whenever $A_n \downarrow \emptyset$ and $\lim_{n\to\infty} \mu(A_n) = 0$*

(4) *π is absolutely continuous of type IV with respect to μ, denote by $\pi \ll_{IV} \mu$, if and only if for any $\epsilon > 0$, there exists $\delta > 0$, such that $\pi(A) \subset [0, \epsilon]$ whenever $\mu(A) < \delta$.*

Theorem 4.1 *$(c)\pi_F \ll_I \mu$.*

Proof. For any $A \in \mathcal{A}$ with $\mu(A) = 0$. Then for any $f \in S(F)$

$$(C)\int_A f d\mu = \int_0^\infty \mu(A \cap F_\alpha) d\alpha = 0$$

Hence $(C)\pi_F(A) = 0$.

Theorem 4.2 *If μ is null additive,then $(C)\pi_F \ll_{II_a} \mu$;*

Proof. For any $A, B \in \mathcal{A}$ with $B \subset A$ and $\mu(A-B) = 0$. Let $a_0 \in (C)\pi_F(A)$, there exists $f \in S(F)$ such that

$$a_0 = (C)\int_A f d\mu = e\int_0^\infty \mu(F_\alpha \cap A) d\alpha$$

From the monotonicity of μ, we have $\mu((A-B) \cap F_\alpha) = 0$. Since μ is null additive , hence

$$\mu(A \cap F_\alpha) = \mu[((A-B) \cap F_\alpha) \cup (B \cap F_\alpha)] = \mu(B \cap F_\alpha)$$

So

$$a_0 = \int_0^\infty \mu(F_\alpha \cap B) d\alpha = (C)\int_B f d\mu \in \pi_F(B)$$

Theorem 4.3 *Let F is Choquet integrable .$(c)\pi_F \ll_{III_a} \mu$;*

Proof. For any $\{A_n\} \subset \mathcal{A}$, $A_n \downarrow \emptyset$. since $\lim_{n\to\infty} \mu(A_n) = 0$ for any $f \in S(F)$, so

$$\lim_{n\to\infty} \int f d\mu = \lim_{n\to\infty} \int \mu(F_\alpha \cap A_n) d\alpha = \int \lim_{n\to\infty} \mu(F_\alpha \cap A_n) d\alpha$$

hence

$$\limsup_{n\to\infty} (C)\pi(F)(A_n) = \liminf_{n\to\infty} (C)\pi(F)(A_n) = \{0\}$$

i.e.

$$\lim_{n\to\infty} (C)\pi_F(A_n) = \{0\}.$$

Theorem 4.4 *Let F is Choquet integrable. $(C)\pi_F \ll_{IV} \mu$.*

Proof. Since F is Choquet integrable, so $(C)\int f d\mu < \infty$ for any $f \in S(F)$, i.e.

$$\int_0^\infty \mu(F_\alpha) d\alpha < \infty$$

Hence for any $\epsilon > 0$, there exits $0 < a < b$, such that

$$(C)\int_0^a \mu(F_\alpha)d\alpha + \int_b^\infty \mu(F_\alpha)d\alpha < \frac{\epsilon}{2}$$

Let$\delta = \frac{\epsilon}{2(b-a)}$ and $\mu(A) < \delta$, then for any $f \in S(F)$

$$\begin{aligned}(C)\int f d\mu &= (C)\int_0^\infty \mu(F_\alpha \cap A)d\alpha \\ &= (C)\int_0^a \mu(F_\alpha \cap A)d\alpha + \int_b^\infty \mu(F_\alpha \cap A)d\alpha + \int_a^b \mu(F_\alpha \cap A)d\alpha \\ &< \frac{\epsilon}{2} + \mu(F_\alpha \cap A)(b-a) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Hence $\pi(A) \subset [0, \epsilon]$.

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