Stopping Game Problem for Dynamic Fuzzy Systems

Yuji Yoshida
Faculty of Economics and Business Administration
The University of Kitakyushu
Kitakyushu 802-8577, Japan
yoshida@kitakyu-u.ac.jp

Masami Yasuda
Department of Mathematics and Informatics
Chiba University, Inage-ku
Chiba 263-8522, Japan;
yasuda@math.s.chiba-u.ac.jp

Masami Kurano
Department of Mathematics
Chiba University
Inage-ku, Chiba 263-8522, Japan
kurano@faculty.chiba-u.jp

Jun-ichi Nakagami
Department of Mathematics and Informatics
Chiba University
Inage-ku, Chiba 263-8522, Japan
nakagami@math.s.chiba-u.ac.jp

Abstract

A stopping game problem is formulated by cooperating with fuzzy stopping time in a decision environment. The dynamic fuzzy system is a fuzzification version of a deterministic dynamic system and the move of the game is a fuzzy relation connecting between two fuzzy states. We define a fuzzy stopping time using several degrees of levels and instances under a monotonicity property, then an "expectation" of the terminal fuzzy state via the stopping time. By inducing a scalarization function (a linear ranking function) as a payoff for the game problem we will evaluate the expectation of the terminal fuzzy state. In particular, a two-person zero-sum game is considered in case its state space is a fuzzy set and a payoff is ordered in a sense of the fuzzy max order. For both players, our aim is to find the equilibrium point of a payoff function. The approach depends on the interval analysis, that is, manipulating a class of sets arising from α-cut of fuzzy sets. We construct an equilibrium fuzzy stopping time under some conditions.

1 Introduction

Optimal stopping on stochastic processes contributes to an essential problem in the sequential decision problem. It is a simple and interesting one because its decision has only two forms, that is, stop or continue. Their applications by many authors are well-known to various fields, economics, engineering etc. The game version of the stopping problem was originated by Dynkin [2] and then Neveu [3] whose pioneering work is named *Dynkin Game*. See [13,11,5] for more references. The results are described clearly and are very attractive, however, we cannot avoid an uncertainty modeling the real problem.

On the other hand the fuzzy theory was founded by Zadeh [17] and then there have been many papers on applications and modeling for extending the results of the classical systems. For example, the fuzzy random variable was studied by Puri and Ralescu in [9].

Here we will discuss a stopping problem concerned with dynamical fuzzy systems. The main discussion is regarding the following two points: One is to define a game value in the zero-sum matrix game under fuzzification and the next is to formulate a fuzzy stopping game using this fuzzy game value for the sequence.

Dynamic fuzzy systems [6] are an extension of Markov decision processes induced by fuzzy configuration. That is, the transition law depending on the state and the action corresponds to a fuzzy relational equation.

In Section 1 the formulation of the fuzzy dynamic system (FDS) and the fuzzy stopping time (FST) are described in order to define a composition of FDS and FST. This base process corresponds to each player's payoff and will be evaluated using the following scalarization called a linear ranking function. In Section 2 we define a game value of a matrix whose elements are fuzzy numbers. A fuzzy stopping model is formulated in Section 3, provided by the previous notions. Also the equilibrium strategy of the model and its game value are obtained under a suitable assumption.

1.1 Preliminaries on Fuzzy Sets

A brief sketch of the notation using here is given as follows: A fuzzy set $\widetilde{a} = \widetilde{a}(x) : x \in \mathbb{R} \to [0, 1]$ on $\mathbb{R}$ is normal, upper semi-continuous, fuzzy convex such that $\widetilde{a}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{a}(x), \widetilde{a}(y)\}$ for $\lambda \in [0, 1]$. It may be called a *fuzzy number* instead of fuzzy set because we are considering the real number of the space $\mathbb{R}$. The operation of sum $+$ and a scalar product $\cdot$ for fuzzy sets are defined by $(\widetilde{a} + \widetilde{b})(x) := \sup_{x=x_1+x_2}\{\widetilde{a}(x_1) \wedge \widetilde{b}(x_2)\}$ and $(\lambda \cdot \widetilde{a})(x) := \widetilde{a}(x/\lambda)$ if $\lambda > 0$, $:= 1_{\{0\}}(x)$ if $\lambda = 0$, where $\wedge = \min$ and $1_{\{\cdot\}}$ means the characteristic function. The α-cut ($\alpha \in [0, 1]$) of the fuzzy set $\widetilde{a}$ is denoted by

$$\widetilde{a}_\alpha := \{x \in E \mid \widetilde{a}(x) \geq \alpha\} \ (\alpha > 0) \tag{1}$$

and $\widetilde{a}_0 := \mathrm{cl}\{x \in E \mid \widetilde{a}(x) > 0\}$, where 'cl' denotes the closure of a set. We frequently use the α-cut in order to define the model and analyze an existence of strategies.

If the operations $+, \cdot$ for any non-empty closed intervals A, B in $\mathbb{R}$ are defined as $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda \cdot A := \{\lambda x \mid x \in A\}$ and especially $A + \emptyset = \emptyset + A := A$ and $\lambda \cdot \emptyset := \emptyset$, then the following two properties are known.

(1) (Interchanges) Interchanging the operation for fuzzy sets and α-cut is useful in the following discussion. $(\widetilde{a}+\widetilde{b})_\alpha = \widetilde{a}_\alpha + \widetilde{b}_\alpha$ and $(\lambda \cdot \widetilde{a})_\alpha = \lambda \cdot \widetilde{a}_\alpha$ ($\alpha \in [0, 1]$) holds.
(2) (Relation between α-cuts and a fuzzy set) The relation between a fuzzy set and its α-cut $\widetilde{a}(x) = \sup_\alpha\{\alpha \wedge \mathbf{1}_{\widetilde{a}_\alpha}(x)\}$, $x \in \mathbb{R}$, holds.

The following construction of a fuzzy set from a family of subsets was given by Zadeh [17]. So when the family of subsets is given, it can be constructed as a fuzzy set provided the condition are satisfied.

Proposition 1.1. *(Representation Theorem) For a given family of $\{M_\alpha\}$ in $\mathbb{R}$, if (i) $\alpha \le \beta \Rightarrow M_\alpha \supset M_\beta$, (ii) $\alpha_n \uparrow \alpha \Rightarrow M_\alpha = \bigcap_n M_{\alpha_n}$, then there exists a fuzzy set $\widetilde{M}(x)$ such that*

$$\widetilde{M}(x) = \sup_\alpha\{\alpha \wedge \mathbf{1}_{M_\alpha}(x)\} \tag{2}$$

for $x \in \mathbb{R}$.

There are special fuzzy sets, whose α-cuts become closed intervals as follows:

(i) Interval case: There exist two real numbers such that

$$\widetilde{a}_\alpha = \{x \in \mathbb{R} \mid \widetilde{a}_\alpha^- \le x \le \widetilde{a}_\alpha^+\},$$

where $\widetilde{a}_\alpha^+ = [\widetilde{a}]_\alpha^+ = \sup\{x \mid \widetilde{a}(x) \ge \alpha\}$ and $\widetilde{a}_\alpha^- = [\widetilde{a}]_\alpha^- = \inf\{x \mid \widetilde{a}(x) \ge \alpha\}$.
(ii) Triangular-type symmetric L-fuzzy number: (a) $L(x) = L(-x)$, (b) $L(x) = 1 \Longleftrightarrow x = 0$, (c) $L(x) \downarrow 0$ ($x \nearrow \infty$), (d) its support is finite.

For an example, if

$$\widetilde{a}(x) := \begin{cases} L((m-x)/k) & \text{if, } x \ge m \\ L((x-m)/k) & \text{otherwise,} \end{cases}$$

where $L(x) := \max\{1 - |x|, 0\}$, then its α-cut equals to

$$\widetilde{a}_\alpha = \{x \in \mathbb{R} \mid m - (1-\alpha)k \le x \le m + (1-\alpha)k\}.$$

The number m is called a center and k is a spread.

1.2 Fuzzy Dynamic System (FDS)

A *fuzzy dynamic system* is a sequence of fuzzy states generated by the pair of an initial state $\tilde{s}(x)$ and a convex fuzzy relation $\tilde{q}(x, y)$. The state space herein is $\mathbb{R} = (-\infty, \infty)$ of real numbers and an initial fuzzy set is a fuzzy number $\tilde{s} = \tilde{s}(x), x \in \mathbb{R}$. These are assumed to be given.

Then a finite sequence $\{\tilde{s}_t; t = 1, 2, \ldots, N\}$ is generated by the fuzzy transition law Q recursively as

$$\begin{aligned} \tilde{s}_1 &:= \tilde{s}, \\ \tilde{s}_{t+1} &:= Q(\tilde{s}_t) := \sup_{x\in\mathbb{R}}\{\tilde{s}(x) \wedge \tilde{q}(x, y)\}, \end{aligned} \tag{3}$$

where $\tilde{q} = \tilde{q}(x, y) : \mathbb{R} \times \mathbb{R} \to [0, 1]$ is a convex fuzzy relation, that is, a fuzzy number defined by two variables, which satisfies

$$\tilde{q}(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geq \tilde{q}(x_1, y_1) \wedge \tilde{q}(x_2, y_2) \tag{4}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$.

We call the sequence $\{\tilde{s}_t, t = 1, 2, \ldots, N\}$ a fuzzy dynamic system.

1.3 Fuzzy Stopping Time (FST)

A *fuzzy stopping time* $\tilde{\sigma} = \tilde{\sigma}(t)$ on a time-index set $t \in \{1, 2, \ldots, N\}$ is

(a) a fuzzy number (set) on $\{1, 2, \ldots, N\}$,

(b) and non-increasing, that is,

$$\tilde{\sigma}(t) \geq \tilde{\sigma}(s) \quad \text{for} \quad 1 \leq t < s \leq N \tag{5}$$

with $\tilde{\sigma}(1) = 1$. The interpretation of a fuzzy stopping time $\tilde{\sigma}$ means a degree of continuity, explicitly as the next three kinds

- $\tilde{\sigma}(t) = 1$ is to continue at time t,
- $\tilde{\sigma}(t) = \alpha$ $(0 < \alpha < 1)$ is an intensity of degree for continuity with level α and degree for stopping with level $1 - \alpha$,
- $\tilde{\sigma}(t) = 0$ is to stop at time t.

Since the real value $\tilde{\sigma}(t)$ for $t \in \{1, 2, \ldots N\}$ should decrease with time, we impose the requirement of (b). By the definition of α-cut for $\tilde{\sigma}$, clearly

$$\tilde{\sigma}_\alpha = \{1, 2, \ldots, \sigma_\alpha\} \tag{6}$$

provided that $\sigma_\alpha := \max\{t \in N \mid \tilde{\sigma}(t) \geq \alpha\}$ for $0 < \alpha$ and $\sigma_0 := \mathrm{cl}\,\{t \in N \mid \tilde{\sigma}(t) > 0\}$ so that $\tilde{\sigma}_\alpha (0 \leq \alpha \leq 1)$ is a connected subset of $\{1, 2, \ldots, N\}$.

1.4 Composition of *FDS* and *FST*

Now we consider, by using α-cut and then a representation theorem, a composition of $\{\widetilde{s}_t\}$, $\{\tilde{\sigma}(t)\}$ for $t = 1, 2, \ldots, N$ which are the fuzzy dynamic system (FDS) and the fuzzy stopping time(FST) respectively.

Because α-cut of a fuzzy set $\widetilde{s}$ on $\mathbb{R}$ equals a closed interval, denoted by the superscript $()^-$, $()^+$,

$$\widetilde{s}_\alpha = [\widetilde{s}_\alpha^-, \widetilde{s}_\alpha^+].$$

Conversely, if a family $[\widetilde{s}_\alpha^-, \widetilde{s}_\alpha^+]$, $0 \leq \alpha \leq 1$, of bounded closed sub-intervals in $\mathbb{R}$ is given, we can construct a fuzzy number $\widetilde{s} = \widetilde{s}(x)$, $x \in \mathbb{R}$ by

$$\widetilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{[\widetilde{s}_\alpha^-, \widetilde{s}_\alpha^+]}(x)\}, \; x \in \mathbb{R}.$$

Definition 1.2. A composed fuzzy system $\widetilde{s}_{\tilde{\sigma}} = \widetilde{s}_{\tilde{\sigma}}(x)$, $x \in \mathbb{R}$ for a pair of a fuzzy dynamic system and a fuzzy stopping time $(\widetilde{s}_t, \tilde{\sigma}(t))$, $t = 1, 2, \ldots, N$, is defined in the following two steps:

Step 1. For each α, if α-cut of a fuzzy stopping time $\tilde{\sigma}$ is $\{1, 2, \ldots, t\}$, i.e. $\tilde{\sigma}_\alpha = \{1, 2, \ldots, t\}$, then we define α-cut of a composed fuzzy system $\widetilde{s}_{\tilde{\sigma}}$ by

$$(\widetilde{s}_{\tilde{\sigma}})_\alpha := \widetilde{s}_{\tilde{\sigma},\alpha} := \widetilde{s}_{t,\alpha} = [\widetilde{s}_{t,\alpha}^-, \widetilde{s}_{t,\alpha}^+]. \tag{7}$$

Step 2. By letting $S_\alpha := \widetilde{s}_{\tilde{\sigma},\alpha}$, $\alpha \in [0, 1]$, the presentation theorem is applied to a family of S_α:

$$\widetilde{s}_{\tilde{\sigma}}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{S_\alpha}(x)\}, \; x \in \mathbb{R}. \tag{8}$$

Thus a composed fuzzy system $\widetilde{s}_{\tilde{\sigma}}$ of FDS $\{\widetilde{s}_t; t = 1, 2, \ldots, N\}$ and FST $\{\tilde{\sigma}(t); t = 1, 2, \ldots, N\}$ is obtained.

2 Game Value for Fuzzy Matrix Game

The usual sequential decision problems consist of several decisions but the simplest one is two cases, that is, Stopping problem – 2-decision (*Stop, Conti*). Considering the straightforward version in a two-person zero sum game for players PL_I(max), PL_{II}(min), one may adapt the following problem: When either of the player declares "stop", then the system stops and each can get rewards. In case of this stop rule the next three cases occur,

$$\text{Decision table of } (PL_I, PL_{II}) = \begin{pmatrix} (Stop, Stop) & (Stop, Conti) \\ (Conti, Stop) & Next \end{pmatrix}$$

and the corresponding value of the matrix game (zero sum) is

$$v_t = val \begin{pmatrix} r_{(S,S)} & r_{(S,C)} \\ r_{(C,S)} & v_{t+1} \end{pmatrix}, \quad t = 1, 2, \ldots$$

where v_t is a payoff at time t and val means a value of the matrix provided their payoffs $r_{(\cdot,\cdot)}$ are given.

From now we will consider the fuzzy version of this sequential decision problem. First define a value of matrix whose elements are fuzzy numbers.

2.1 Fuzzy Game Value

Definition 2.1. For a matrix $\widetilde{A}$ with each (i, j) element $\widetilde{a}_{ij} = \widetilde{a}_{ij}(x)$, $x \in \mathbb{R}$, is a fuzzy number, define

$$\widetilde{val}(\widetilde{A}) = \widetilde{val}(\widetilde{A})(x) = \widetilde{val} \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} \end{pmatrix} (x) \tag{9}$$

as a map $x \in \mathbb{R} \longmapsto [0, 1]$ in the following steps.

Step 1. For $0 \leq \alpha \leq 1$, let the α-cut of each element $\left(\widetilde{a}_{ij}\right)_\alpha := [a_{ij}^-, a_{ij}^+]$ and define the α-cut of the matrix by

$$\widetilde{A}_\alpha := \left((\widetilde{a}_{ij})_\alpha\right) = \begin{pmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] \\ [a_{21}^-, a_{21}^+] & [a_{22}^-, a_{22}^+] \end{pmatrix}.$$

Step 2. For each i, j, two real numbers a_{ij}^-, $a_{ij}^+ \in (\widetilde{a}_{ij})_\alpha$, consider each value of the matrix game

$$val\left(a_{ij}^-\right) \quad \text{and} \quad val\left(a_{ij}^+\right)$$

as $\min_j \max_i \left(a_{ij}^\pm\right) = \max_i \min_j \left(a_{ij}^\pm\right)$ in the usual sense.

Step 3. Using these values, define a closed interval, denoted by $val(\widetilde{A}_\alpha)$, as

$$val(\widetilde{A}_\alpha) := \left[val\left(a_{ij}^-\right), val\left(a_{ij}^+\right)\right] = \left[val \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix}, val \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix}\right]$$

for each α.

Step 4. Construct a fuzzy set (number) $\widetilde{val}(\widetilde{A})$ on $\mathbb{R}$ from a family of $\{val(\widetilde{A}_\alpha); 0 \leq \alpha \leq 1\}$ as

$$\widetilde{val}(\widetilde{A})(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{val(\widetilde{A}_\alpha)}(x)\}$$

for $x \in \mathbb{R}$.

Proposition 2.1. *From the definition of $\widetilde{val}$, it holds that*

$$\left(\widetilde{val}(\widetilde{A})\right)_\alpha = val(\widetilde{A}_\alpha). \tag{10}$$

Proof. From the definition of the α-cut of $\widetilde{val}$, the result is immediately obtained. □

A *fuzzy max order* (Ramík and Řimánek [10]), between fuzzy numbers is a partial order which defined by the order for interval of α-cut in fuzzy numbers.

Definition 2.2. For two fuzzy numbers in $\mathbb{R}$, $\widetilde{b} \preceq \widetilde{a}$ if and only if the next inequality $\widetilde{b}_\alpha^- \leq \widetilde{a}_\alpha^-$ and $\widetilde{b}_\alpha^+ \leq \widetilde{a}_\alpha^+$ hold in both for all α.

Proposition 2.2. *(Special case with order of elements) If* $\widetilde{A} = \begin{pmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{c} & \widetilde{d} \end{pmatrix}$ *with*

$$\widetilde{b} \preceq \widetilde{a} \preceq \widetilde{c} \tag{11}$$

where $\preceq$ means a fuzzy max order, then

$$\widetilde{val}(\widetilde{A}) = \begin{cases} \widetilde{b} & \text{if } \widetilde{d} \preceq \widetilde{b} \\ \widetilde{d} & \text{if } \widetilde{b} \preceq \widetilde{d} \preceq \widetilde{c} \\ \widetilde{c} & \text{if } \widetilde{c} \preceq \widetilde{d}. \end{cases} \tag{12}$$

Proof. The proof depends on the usual case that a matrix game with each element is a real number. Because of assumption (11) the game has a pure strategy and the element "$\widetilde{a}$" / (*Stop,Stop*) does not become an equilibrium. So this result is extended easily to the interval version and this fuzzy value. □

2.2 A Linear Ranking Function (Scalarization)

In this section we will discuss the evaluation for a fuzzy set. This leads us to define an objective function and an equilibrium strategy for each player.

The first is to consider a function of the scalarization (evaluation) from an interval to a real number. Let a map g from an interval in $\mathbb{R}$ to $\mathbb{R}$ which satisfies

(i) $g(A + B) = g(A) + g(B)$,
(ii) $g(\lambda A) = \lambda g(A)$, $\lambda > 0$,
(iii) $A = [a_1, a_2] \Rightarrow a_1 \leq g([a_1, a_2]) \leq a_2$.

This map is called a *linear ranking function* (Fortemps and Roubens [4]) and it is adapted to the correspondence from the α-cut of a fuzzy number to its reduced-scalar.

Lemma 2.1. *The following three assertions are equivalent.*

(a) A map g is a linear ranking function.

(b) (Affine property) For $\lambda \geq 0$, μ,

$$g(\lambda[0, 1] + \mu) = \lambda g([0, 1]) + \mu.$$

(c) By letting $k := g([0, 1])$,

$$g([a_1, a_2]) = a_1(1 - k) + a_2 k.$$

Lemma 2.2. *If* $\widetilde{a} \preceq \widetilde{b}$, *then* $\|\widetilde{a}\|_g \leq \|\widetilde{b}\|_g$ *where*

$$\|\widetilde{a}\|_g := \int_0^1 g(\widetilde{a}_\alpha)\, d\alpha. \tag{13}$$

3 A Fuzzy Stopping Game and Equilibrium Strategies

Our stopping model for the zero-sum case is based on FDS in Section 1.2 and the linear ranking function in Section 2.2 for its evaluation in order to define an objective function and discuss an equilibrium strategy. The data are generated sequentially to define the model by fuzzy transition laws Q, FDS $\{\widetilde{r}^t_{(\cdot,\cdot)};\ t = 1, 2, \ldots, N\}$. A stopping strategy is a pair of FSTs $\tilde{\sigma}_I, \tilde{\sigma}_{II}$ defined by (5). Associated with a stopping strategy, we consider a payoff function and thus an equilibrium of the integral (16).

(1) Initial fuzzy state defined on $\mathbb{R}$:

$$\widetilde{r}_{(S,S)},\ \widetilde{r}_{(C,S)},\ \widetilde{r}_{(S,C)}$$

where $\widetilde{r}_{(\cdot,\cdot)} = \widetilde{r}_{(\cdot,\cdot)}(x)$, $x \in \mathbb{R}$.

(2) Fuzzy translation law:

$$Q_{(S,S)},\ Q_{(C,S)},\ Q_{(S,C)}$$

where fuzzy translation laws $Q_{(\cdot,\cdot)}$ are generated by some convex fuzzy relations $\widetilde{q}_{(\cdot,\cdot)}(x, y)$, $x, y \in \mathbb{R}$.

(3) FDS $\{\widetilde{r}^t_{(\cdot,\cdot)};\ t = 1, 2, \ldots, N\}$ generated by Q similar to (3):

$$\widetilde{r}^t_{(S,S)},\ \widetilde{r}^t_{(C,S)},\ \widetilde{r}^t_{(S,C)}$$

where

$$\begin{cases} \widetilde{r}^t_{(\cdot,\cdot)} & := Q_{(\cdot,\cdot)}\widetilde{r}^{t-1}_{(\cdot,\cdot)}, \quad t = 2, 3, \ldots, N, \\ \widetilde{r}^1_{(\cdot,\cdot)} & := \widetilde{r}_{(\cdot,\cdot)}. \end{cases} \tag{14}$$

(4) Stopping strategy: Two FSTs $\tilde{\sigma}_I, \tilde{\sigma}_{II}$ for each player defined by (5).

(5) Payoff function: $\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})$ whose α-cuts are

$$\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})_\alpha := \begin{cases} \widetilde{r}^t_{(C,S),\alpha} & \text{if} \quad t = \tilde{\sigma}_{I,\alpha} \le \tilde{\sigma}_{II,\alpha}, \\ \widetilde{r}^t_{(S,C),\alpha} & \text{if} \quad t = \tilde{\sigma}_{II,\alpha} \le \tilde{\sigma}_{I,\alpha}, \\ \widetilde{r}^t_{(S,S),\alpha} & \text{if} \quad t = \tilde{\sigma}_{I,\alpha} = \tilde{\sigma}_{II,\alpha}. \end{cases} \tag{15}$$

(6) Objective functions:

$$\min_{\tilde{\sigma}_{II}} \max_{\tilde{\sigma}_I} \|\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})\|_g \text{ and } \max_{\tilde{\sigma}_I} \min_{\tilde{\sigma}_{II}} \|\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})\|_g$$

where

$$\|\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})\|_g := \int_0^1 g(\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})_\alpha)\mathrm{d}\alpha \tag{16}$$

with a given linear ranking function g similar to (13).

Thus we have defined a stopping game problem for dynamic fuzzy systems and we shall look for the equilibrium of (16). In order to avoid analytical difficulty, the game values for the zero-sum matrix are restricted within only pure strategies, that is, this is enough to consider the class of FST. Refer to [13]. Explicitly we need the following assumption, which is assumed in several papers [3,5] as

Assumption 3.1. (Dynkin Game) For each t,

$$\widetilde{r}^t_{(S,C)} \preceq \widetilde{r}^t_{(S,S)} \preceq \widetilde{r}^t_{(C,S)}. \tag{17}$$

Consider the next in backward induction

$$\widetilde{v}_t = \widetilde{val} \begin{pmatrix} \widetilde{r}^t_{(S,S)} & \widetilde{r}^t_{(S,C)} \\ \widetilde{r}^t_{(C,S)} & \widetilde{v}_{t+1} \end{pmatrix}, \quad t = N-1, \ldots, 2, 1, \tag{18}$$

by $\widetilde{val}$ in a fuzzy sense of (9) and

$$\widetilde{v}_N := \widetilde{r}^N_{(S,S)}.$$

Lemma 3.1. *(i) For $t = N-1, \ldots, 2, 1$, and each α,*

$$g(\widetilde{v}_{t,\alpha}) = val \begin{pmatrix} g(\widetilde{r}^t_{(S,S),\alpha}) & g(\widetilde{r}^t_{(S,C),\alpha}) \\ g(\widetilde{r}^t_{(C,S),\alpha}) & g(\widetilde{v}_{t+1,\alpha}) \end{pmatrix} \tag{19}$$

by the scalarization. Here val means in the normal usage.

(ii) For $t = N-1, \ldots, 2, 1$,

$$\|\widetilde{v}_t\|_g = val \begin{pmatrix} \|\widetilde{r}^t_{(S,S)}\|_g & \|\widetilde{r}^t_{(S,C)}\|_g \\ \|\widetilde{r}^t_{(C,S)}\|_g & \|\widetilde{v}_{t+1}\|_g \end{pmatrix}$$

$$= \begin{cases} \|\widetilde{r}^t_{(S,C)}\|_g & \text{if} \quad \|\widetilde{v}_{t+1}\|_g \le \|\widetilde{r}^t_{(S,C)}\|_g \\ \|\widetilde{v}_{t+1}\|_g & \text{if} \quad \|\widetilde{r}^t_{(S,C)}\|_g \le \|\widetilde{v}_{t+1}\|_g \le \|\widetilde{r}^t_{(C,S)}\|_g \\ \|\widetilde{r}^t_{(C,S)}\|_g & \text{if} \quad \|\widetilde{r}^t_{(C,S)}\|_g \le \|\widetilde{v}_{t+1}\|_g. \end{cases} \tag{20}$$

Definition 3.1. For each α,

$$\begin{aligned} \sigma^*_{I,\alpha} &:= \inf\{1 \le t \le N \mid g(\widetilde{v}_{t,\alpha}) \le g(\widetilde{r}_{(S,C),\alpha})\}, \\ \sigma^*_{II,\alpha} &:= \inf\{1 \le t \le N \mid g(\widetilde{v}_{t,\alpha}) \ge g(\widetilde{r}_{(C,S),\alpha})\}, \end{aligned} \tag{21}$$

and define

$$\begin{aligned} \tilde{\sigma}^*_I = \tilde{\sigma}^*_I(t) &:= \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{[0,\sigma^*_{I,\alpha}]}(t)\}, \\ \tilde{\sigma}^*_{II} = \tilde{\sigma}^*_{II}(t) &:= \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathbf{1}_{[0,\sigma^*_{II,\alpha}]}(t)\} \end{aligned} \tag{22}$$

for $t = 1, 2, \dots, N$.

The next assumption is too technical. However we have to induce the class of strategy under the fuzzy configuration as a class of FST that needs to be well defined as (5).

Assumption 3.2. (Regularity of strategy) Each epoch of $\sigma^*_{I,\alpha}, \sigma^*_{II,\alpha}$ in (21) decrease monotonically in $\alpha \in [0, 1]$.

Theorem 3.1. *Under Assumptions 3.1 and 3.2,*

$$\sup_{\tilde{\sigma}_I} \inf_{\tilde{\sigma}_{II}} \|\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})\|_g = \inf_{\tilde{\sigma}_{II}} \sup_{\tilde{\sigma}_I} \|\widetilde{R}(\tilde{\sigma}_I, \tilde{\sigma}_{II})\|_g \tag{23}$$

holds and its equilibrium strategy $\tilde{\sigma}^*_{I,\alpha}, \tilde{\sigma}^*_{II,\alpha}$ for each player satisfies

$$\|\widetilde{R}(\tilde{\sigma}^*_I, \tilde{\sigma}^*_{II})\|_g = \|\widetilde{v}_1\|_g. \tag{24}$$

Proof. The proof is immediately obtained from the previous assumptions and lemmas. □

Remark 3.1. Here we do not show a concrete example, however, there are many examples in the crisp case, that is, $\alpha = 1$. Thus the assumptions are satisfied in the ordinary case.

Remark 3.2. An infinitely planned horizon case, should be considered a fixed point concerned with the fuzzy relational equation (18) of $\widetilde{val}$. Details are not discussed in this paper.

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