Some asymptotic properties of point estimation with n-dimensional fuzzy data

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Abstract

Some asymptotic properties of point estimation with n-dimensional fuzzy data with respect to a special L_2-metric ρ are investigated in this paper. It is shown that the collection of all n-dimensional fuzzy data endowed with the ρ-metric is a complete and separable space. Some criterions for point estimation in such fuzzy environments are proposed and the sample mean and variance and covariance with n-dimensional fuzzy data under these criterions are further studied.

keywords: Fuzzy random variables; support function; expectation; variance; point estimation.

1 Introduction

The theory of statistical inference with vague data has been developed extensively in recent years (see Feng(2001), Kruse and Meyer(1987), Körner(1997, 2000), Näther(1997,2000), Lubiano et al (2000) etc. and their references). Kruse and Meyer(1987) investigated some asymptotical statistics with one-dimensional fuzzy random variables with respect to the Hausdorff metric. Extension of it to n-dimensional fuzzy random variables has been discussed by Näther(2000) preliminarily, and some results like that the sample mean and variance with n-dimensional fuzzy data are consistent unbiased estimators of the expectation and variance of population concerned in a fuzzy observation had been obtained via a SLLN (Strong Law of Large Number) of fuzzy random variables given by Klement et al.(1986).

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Also a ρ-metric as a unification of different metric applied in statistical inference with fuzzy data was introduced by Näther(2000). As far as we are aware there is a few literatures on the properties of statistical estimation with n-dimensional fuzzy data under this metric ρ. Some SLLN for fuzzy random variables have been achieved under some metric such as d_1, d_p and ρ_p, $1 \leq p < \infty$ (see Diamond and Kloeden (1994)), and recently under d_∞, (see Proske(1998), Molchanov(1999), Colubi et al.(1999)).

Based on the results mentioned above we re-examin some asymptotic properties of point estimation with n-dimensional fuzzy data under this ρ-metric in this paper. The rest of this article is orgnized as follows. In section 2 we recall some definitions and results which will be used in the sequel, some criterions such as unbiasedness, consistency, uniform mean square error, uniform minimum variance unbiasedness as well as efficiency for estimation with n-dimensional fuzzy data under ρ-metric are re-defined. In section 3 we investigate the structure of space $(E^n(K), \rho)$, and with these criterions some asymptotic properties and statistical relations of the sample mean and variance with n-dimensional fuzzy data are further studied.

2 Preliminaries

In this section,we give some notions for fuzzy random variables and fuzzy random samples.

Definition 1. A map $u : \mathbf{R}^n \to [0,1]$ is called a *normal compact convex fuzzy set* of $\mathbf{R}^n$, if u satisfies (i) u is normal, i.e. $\{x \in \mathbf{R}^n | u(x) = 1\} \neq \phi$; (ii) $\forall \alpha \in (0,1]$, $[u]^\alpha := \{x \in \mathbf{R}^n | u(x) \geq \alpha\}$ is compact and convex; (iii) $[u]^0 := cl\{x \in \mathbf{R}^n | u(x) > 0\}$, the support of u, is compact.

The set of all normal compact convex fuzzy sets of $\mathbf{R}^n$ is denoted by E^n. The addition and scalar multiplication in E^n are defined by

$$(u+v)(z) = \sup_{x+y=z} \min\{u(x), v(y)\}, \quad \forall z \in \mathbf{R}^n$$

$$au(x) = u(x/a), (a \neq 0), x \in \mathbf{R}^n$$

$$0u(x) = 0(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0 \end{cases}$$

where $u, v \in E^n, a \in \mathbf{R}$. It is easy to check that $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[au]^\alpha = a[u]^\alpha$.

Note that the space $(E^n, +, \cdot)$ is not a linear space under Minkovski addition $+$ and scalar multiplication $\cdot$.

Definition 2. For a normal compact convex fuzzy set u, the map u^* : $S^{n-1} \times [0,1] \to \mathbf{R}$, i.e. $(x, \alpha) \mapsto sup\{x \cdot a | a \in [u]^\alpha\}$, $x \in S^{n-1}$ is called a *support function* of u, where $\cdot$ is the inner product in $\mathbf{R}^n$, S^{n-1} the n-dimensional unit sphere $\{x \in \mathbf{R}^n | \parallel x \parallel = 1\}$ in $\mathbf{R}^n$ and $[u]^\alpha$ is α-cut of u for $\alpha \in [0,1]$.

For a non-fuzzy set B, the support function of which is uniquely determined as $B^*(x) := sup\{x \cdot b | b \in B\}$, $x \in S^{n-1}$ provided B is compact and convex. E^n can be embedded in a space of functions on $S^{n-1} \times [0,1]$ via the support function, namely, the mapping $u \to u^*$ is an isomophism of E^n on to the cone of continuous functions on $S^{n-1} \times [0,1]$ preserving the semi-linear structure

$$(\lambda u + \mu v)^* = \lambda u^* + \mu v^*, \lambda \geq 0, \mu \geq 0.$$

and u, $v \in E^n$. For further details on the properties of support function of a set the reader is referred to Näther(2001) and its references.

It is known that there are various definitions of distance between two normal compact convex fuzzy sets such as d_p, $\rho_p, d_\infty, 1 \leq p < \infty$ as follows: For $u, v \in E^n$

$$d_p(u,v) = \left(\int_0^1 (d_H([u]^\alpha, [v]^\alpha))^p d\alpha \right)^{1/p},$$
$$\rho_p(u,v) = \left(\int_0^1 (\delta_p([u]^\alpha, [v]^\alpha))^p d\alpha \right)^{1/p},$$
$$d_\infty(u,v) = \sup_{\alpha \in (0,1]} \{d_H([u]^\alpha, [v]^\alpha)\}$$

where d_H is the Hausdorff metric, i.e.

$$d_H(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x-y\|, \sup_{y \in B} \inf_{x \in A} \|x-y\|\}$$

and

$$\delta_p(A,B) = \left(\int_{S^{n-1}} |A^*(x) - B^*(x)|^p \mu(dx) \right)^{1/p} \text{ for compact sets } A, B \subset \mathbf{R}^n.$$

However, we are interested in the metric ρ proposed by Näther(2000), since some metrics applied in a statistical inference with vague data could be unified.

Definition 3(Näther (2000)). For any two u, $v \in E^n$, the distance ρ between u and v is defined as

$$\rho(u,v) := \left(\iiiint_S \{(u^* - v^*)(x,\alpha)\} \{(u^* - v^*)(y,\beta)\} \, dK(x,\alpha,y,\beta) \right)^{1/2}$$

where $S = S^{n-1} \times [0,1] \times S^{n-1} \times [0,1]$ and K is a symmetric and positive definite kernel. The distance ρ satisfies the following properties:

(i) $\rho(tu, tv) = t\rho(u, v),\ t \geq 0$;

(ii) $\rho(u + w, v + w) = \rho(u, v)$;

(iii) $\rho(u + v, u_1 + v_1) \leq \rho(u, u_1) + \rho(v, v_1) + 2(u - u_1) \odot (v - v_1)$.

where

$$(u - u_1) \odot (v - v_1) \\ := \left(\iiiint_S \{(u^* - u_1^*)(x, \alpha)\} \{(v^* - v_1^*)(y, \beta)\} \, dK(x, \alpha, y, \beta) \right)^{1/2}.$$

Note that d_p is equivalent to ρ_p (Diamond and Kloeden(1994)) and for each nonempty compact subset C of R^n both $(E^n(C), d_p), (E^n(C), \rho_p)$ are complete and separable metric space, where $E^n(C) := \{u \in E^n : [u]^0 \subseteq C\}$. Later in section 3 we prove that $(E^n(C), \rho)$ also be a complete and separable metric space.

Definition 4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, $\mathbf{R}^n, \mathcal{B})$ be a measurable space. The mapping $X : \Omega \to E^n$ is called *a fuzzy random variable* if X satisfies that for any measurable subset $D \in \mathbf{R}^n$, $\{\omega | X_\alpha(\omega) \bigcap D \neq \phi\} \in \mathcal{A}$, where $X_\alpha(\omega) := \{x \in \mathbf{R}^n | X(\omega)(x) \geq \alpha\} = [X(\omega)]^\alpha$, $\alpha \in [0, 1]$, $\omega \in \Omega$. All fuzzy random variables from Ω to E^n is denoted by $L(\Omega, E^n)$.

Puri and Ralescu(1986) have proposed an expectation of fuzzy random variables, an advantage of which lies in its linear property.

Definition 5. The *expectation* $\mathbb{E}X$ of a fuzzy random variable X is a normal compact fuzzy set of $\mathbf{R}^n$ with the property of

$$(\mathbb{E}X)_\alpha = \mathbb{E}X_\alpha,\ \forall \alpha \in [0, 1], \quad \mathbb{E}X^* < \infty,$$

where $\mathbb{E}X_\alpha$ is the Aumann-expectation of the random set X_α defined by

$$\mathbb{E}X_\alpha = \{\mathbb{E}\eta \mid \eta(\omega) \in X_\alpha(\omega) \quad a.e., \eta \in L(\Omega, \mathbf{R})\}$$

here $L(\Omega, \mathbf{R})$ is the set of all real random variables with existing expectation defined on Ω.

Note that in the case of one dimensional fuzzy random variables the expectation described above is same as the expectation of fuzzy random variables proposed by Kruse and Meyer(1987), and also Feng(2001).

There are several different definitions on variance of fuzzy random variables (see Kruse and Meyer(1987), Feng(2001), Körner(1997) etc.). In our case, by the definition of fuzzy random variables in this paper, we favuor that the variance of fuzzy random variables (here the fuzzy random variables are different from that of Kwakernaak-Kruse-Meyer) is an accurate measure of the spread or dispersion of the fuzzy random variables about its mean from a viewpoint of the essentiality of the classical variance, also the covariance or the correlation coefficient of two fuzzy random variables must measure their linear interdependence, they should have no fuzziness.

Definition 6. Let $X \in L(\Omega, E^n)$ and $\mathbb{E}\|X\|_\rho^2 < \infty$. The *variance* $Var(X)$ of X is defined as $\mathbf{Var}(X) := \mathbb{E}\rho^2(X, \mathbb{E}X)$ where $\|X\|_\rho = \rho(X, \{0\})$.

Feng (2001) proposed a variance under the metric d_2 for the one dimensional fuzzy random variables, which can be viewed as a special case of the variance in definition 6 approximately.

Lemma 2.1. (Näther (2001)). *If* $X \in L(\Omega, E^n)$ *then* $(\mathbb{E}X)^* = \mathbb{E}X^*$ *and*

$$Var(X) = \iiiint_S Cov(X^*(x,\alpha), X^*(y,\beta)) dK(x,\alpha,y,\beta).$$

For the fuzzy random variables X and Y, let $\langle X, Y\rangle$ be

$$\langle X, Y\rangle := \iiiint_S X^*(x,\alpha) Y^*(y,\beta) dK(x,\alpha,y,\beta),$$

where $S = S^{n-1} \times [0,1] \times S^{n-1} \times [0,1]$, then $\langle X, Y\rangle$ is a "real" random variable and the variance, covariance of the fuzzy random variables can be defind in a similar form as

(i) $\mathbf{Cov}(X,Y) := \mathbb{E}\langle X, Y\rangle - \langle \mathbb{E}X, \mathbb{E}Y\rangle$;

(ii) $\mathbf{Var}(X) = \mathbb{E}\langle X, X\rangle - \langle \mathbb{E}X, \mathbb{E}X\rangle$;

(iii) $R(X,Y) := \mathbf{Cov}(X,Y)/\sqrt{\mathbf{Var}(X)\mathbf{Var}(Y)}$.

It is easy to prove that the operation $\langle \cdot, \cdot \rangle$ posseses the following properties:

(i) $\langle X, X\rangle = \|X\|_\rho^2$ is a non-negative real valued random variable.

(ii) $\langle X, Y\rangle = \langle Y, X\rangle$,

(iii) $\langle \lambda X + \mu Y, Z\rangle = \lambda\langle X, Z\rangle + \mu\langle X, Z\rangle$,

where X, Y, $Z \in L(\Omega, E^n)$ and λ, $\mu \in [0, \infty)$.

Lemma 2.2. (Näther(2000), Feng(2001)). *Let* X, Y *be fuzzy random variables with* $\mathbb{E}\|X\|_\rho^2 < \infty$, *Then for any positive squared integrable random variable* ξ, *any* $a \in E^n$ *and any real numbers* λ, μ *it holds that*

(1) $\mathbb{E}(\lambda X + \mu Y) = \lambda\mathbb{E}X + \mu\mathbb{E}Y$,

(2) $\mathbb{E}(\lambda X \cdot \mu Y) = \lambda\mu\mathbb{E}X\mathbb{E}Y$ *if* X *and* Y *are independent (fuzzy random variable* X *and* Y *are independent if and only if* X_α *and* Y_α *are independent random sets,* $\alpha \in [0,1]$),

(3) $\mathbf{Var}(X) = E \parallel X \parallel_\rho^2 - \parallel EX \parallel_\rho^2$,

(4) $\mathbf{Var}(\lambda X) = \lambda^2 \mathbf{Var} X,\ \lambda \geqslant 0$,

(5) $\mathbf{Var}(\xi a) = \parallel a \parallel_\rho^2 Var\xi$,

(6) $\mathbf{Var}(a + X) = \mathbf{Var} X$,

(7) $\mathbf{Var}(\xi X) = \mathbb{E} \parallel X \parallel_\rho^2 \mathbf{Var}(\xi) + \mathbb{E}\xi^2 \mathbf{Var}(X)$ *if* ξ *and* X *are independent,*

(8) $\mathbf{Var}(X + Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$ *if* X *and* Y *are independent,*

(9) $\mathbf{Cov}(X, Y) = 0$ *if* X *and* Y *are independent,*

(10) $\mathbf{Cov}(\lambda X + u, \mu Y + v) = \lambda\mu \mathbf{Cov}(X, Y)$ *where* $\lambda\mu \geq 0$.

Let $X_1, \ldots, X_m$ be *a simple random samples (or n-dimensional fuzzy data)* of size m from a population represented by a fuzzy random variable X, namely, $X_1, \ldots, X_m$ are independent and identically distributed (i.i.d.) with X. Let θ be an unknown parameter with respect to the distribution of X and let T_m be a statistic with respect to the random sample $X_1, \ldots, X_m$ from the fuzzy random variable X and T_m can be used to estimate the unknown parameter θ.

Remark 2.1. (1). T_m can be obtained from a given ordinary statistic depending real random sample by using Zadeh's extension principle.

(2).The distribution of the fuzzy random variable X here means that theoretically there exist a distribution for the fuzzy random variable X, however, it is not easy to define a concrete distribution for X in detail. we favour that the distribution of a fuzzy random varible defined in this paper can be followed as a capacity functional of a random compact set (cf. Matheron (1975) , Molchanov (1998)).

(3). The concerned unknown parameter θ in the distribution of the population X is assumed as a fuzzy set on the parameter space.

Definition 7. $T_m(X_1, \ldots, X_m)$ is called

(i) *an unbiased estimator* for θ in ρ if it satisfies

$$\mathbb{E}(T_m(X_1, \ldots, X_m)) = \theta;$$

(ii) *a weak consistent estimator* for θ in ρ if it satisfies

$$\lim_{m \to \infty} P(\rho(T_m(X_1, \ldots, X_m), \theta) > \varepsilon) = 0 \quad \text{for any} \quad \varepsilon > 0.$$

(iii) *a strong consistent estimator* for θ in ρ if it satisfies

$$\rho(T_m(X_1, \ldots, X_m), \theta) \to 0 \quad a.s.\, (m \to \infty)$$

(iv) A sequence $\{T_m\}$ of estimators with respect to random samples from fuzzy random variable X is called *a uniform mean square error estimator* for unknown fuzzy parameter θ in ρ if

$$\lim_{m\to\infty} \mathbb{E}(\rho^2(T_m, \theta)) = 0$$

.

For the random variable $\rho(T(X_1, \ldots, X_m), \theta)$ provided whose expectation exist, the Tchebyshev inequality obviously holds, i.e.

$$P(\{\rho(T_m(X_1, \ldots, X_m), \theta) > \varepsilon\}) \leq \frac{\mathbb{E}\rho^2(T_m(X_1, \ldots, X_m), \theta)}{\varepsilon^2}$$

for any $\varepsilon > 0$, therefore, the uniform mean square error estimation implies the consistent estimation in metric ρ.

Definition 8.

(i) Let T_{m1} and T_{m2} be two unbiased estimates with respect to random samples from the fuzzy random variable X for unknown parameter θ, T_{m1} is said to be *more efficient* than T_{m2} if $\mathbf{Var}(T_{m1}) \leq \mathbf{Var}(T_{m2})$.

(ii) An unbiased estimator T_m for θ with respect to random samples from the fuzzy random variable X is said to be an *uniform minimum variance unbiased estimator(UMVUE)* if and only if $ET_m = \theta$ and $\mathbf{Var}(T_m) \leq \mathbf{Var}(U)$ for any other unbiased estimator U of θ with respect to random samples from the fuzzy random variable X.

Obviously an UMVUE is more efficient than any other unbiased estimator and also is asymptotically efficient.

In the following, for statistical studies of fuzzy random variables in metric ρ, we state some useful limit theroems for fuzzy random variable.

The law of large number(LLN) for fuzzy random variables under the metric d_1 has been obtained by Klement et al (1986), and under the metric d_p, ρ_p, $1 \leq p < \infty$ by Diamond and Kloeden (1994).

Lemma 2.3. (Klement et al.(1986), Diamond and Kloeden (1994)) *Let $X_1, X_2, \ldots$ be independent and identically distributed fuzzy random variables with $\mathbb{E}\|X_1\| < \infty$, Then $\overline{X_m} = \sum_{i=1}^m X_m/m$ is an unbiased and consistent estimator of the expectation, i.e.*

$$\mathbb{E}\overline{X_m} = EX_1$$

and

$$\overline{X_m} \xrightarrow{a.s.} EX_1 \quad (n \to \infty)$$

with respect to metrics d_1, d_p *and* ρ_p*,* $1 \leq p < \infty$*.*

A direct consquence of the LLN applied to statistical inference with fuzzy random variables has been obtained as follows.

Lemma 2.4. (Näther (2000)). *Let* $X_1, X_2, \ldots$ *be a sequence of independent and identically distributed fuzzy random variables with* $\mathbb{E}\|X_1\| < \infty$*, Then*

$$S_m^{*2} = \frac{1}{m-1}\sum_{k=1}^{m} \rho^2(X_k, \overline{X_m})$$

is an unbiased and consistent estimator of $\mathbf{Var}(X_1)$*, i.e.*

$$\mathbb{E}(S_m^{*2}) = Var(X_1), \qquad S_m^{*2} \xrightarrow{a.s.} \mathbf{Var}(X_1) \quad (m \to \infty).$$

Some important SLLN for fuzzy random variables had been given by Körner(1997), Proske(1998), Molchanov (1999), Colubi et al.(1999). These SLLN establish a theoretical foundation for the research on the consistency of the estimation with n-dimensional fuzzy data.

3 Main results

In this section we shall investigate some asymptotical properties of statistical estimations with fuzzy data under the ρ-metric with an assumption: For $X \in L(\Omega, E^n)$ and positive integer r, $\mathbb{E}\|X\|_\rho^r < \infty$.

Theroem 3.1. $(E^n(C), \rho)$ *is a complete and separable metric space for each nonempty compact subset* C *of* $\mathbb{R}^n$*.*

Proof. By Cauchy-Schwarz inequality of integral and the property of kernel K we have

$$\begin{aligned}
& \rho^2(u,v) \\
= & \iiiint_S \{(u^* - v^*)(x,\alpha)\}\{(u^* - v^*)(y,\beta)\}\, dK(x,\alpha,y,\beta) \\
\leq & \left(\iiiint_S (u^* - v^*)^2(x,\alpha) dK(x,\alpha,y,\beta)\right)^{1/2} \\
& \quad \times \left(\iiiint_S (u^* - v^*)^2(y,\beta) dK(x,\alpha,y,\beta)\right)^{1/2} \\
= & \iint_{S^{n-1}\times[0,1]} (u^* - v^*)^2(x,\alpha) \iint_{S^{n-1}\times[0,1]} dK(x,\alpha,y,\beta) \\
= & \iint_{S^{n-1}\times[0,1]} (u^* - v^*)^2(x,\alpha) m(x,\alpha)\mu(dx)d\alpha \\
\leq & M\rho_2^2(u,v)
\end{aligned}$$

where $S = S^{n-1} \times [0,1] \times S^{n-1} \times [0,1]$, $m(x,\alpha)$ is an integrable function and $|m(x,\alpha)| \leq M$, M is a positive real number. By the Proposition 7.2.6 and Proposition 7.3.3 of Diamond and Kloeden(1994), we see that $(E^n(C), \rho_2)$ is a complete and separable metric space. Thus obviously $(E^n(C), \rho)$ is also a complete space. The separability of the space $(E^n(C), \rho_2)$ implies that for any $u \in E^n(C)$ there exists some countable subset $U = \{u_1, \ldots, u_n, \ldots\}$ of $E^n(C)$ such that $cl(U) = E^n(C)$, i.e. $\lim\limits_{n\to\infty} \rho_2(u_n, u) = 0$, then we have that $\lim\limits_{n\to\infty} \rho(u_n, u) = 0$, which means $(E^n(C), \rho)$ be separable. This completes the proof of Theorem 3.1.

We obviously see that LLN and SLLN also hold for the metric ρ since $\rho(u,v) \leq \rho_2(u,v) \leq d_\infty(u,v)$ for $u, v \in E^n$ from the proof of Theorem 3.1. (cf. Diamond and Kloeden(1994), Körner(1997), Lyashenko(1982)).

In the following, based on the LLN and SLLN with respect to the metric ρ, we shall present a limit theorem for the r order moment of fuzzy random variables.

For a fuzzy random variable X with $\mathbb{E}\|X\|^r < \infty$, let $X^r(\omega) := (X(\omega))^r$ $\omega \in \Omega$ and let

$$y = (y_1, \ldots, y_n), \quad x = (x_1, \ldots, x_n)$$

be elements of $\mathbb{R}^n$. By abuse of notation, we define

$$y^r := (y_1^r, \ldots, y_n^r).$$

using Zadeh's extension principle, we have

$$X^r(\omega)(x) = \sup_{x=y^r} \min\{X(\omega)(y), \ldots, X(\omega)(y)\}.$$

By $x = y^r$, we have that $x_i = y_i^r$, $i = 1, \ldots, n$. Assume that $y_i = \sqrt[r]{x_i} \geqslant 0$ if r is a positive even number, and $y_i = \sqrt[r]{x_i} \in \mathbb{R}$ if r is a positive odd number. We define that

$$\sqrt[r]{x} := (\sqrt[r]{x_1}, \ldots, \sqrt[r]{x_n})$$

Under these assumptions, we obtain that

$$X^r(\omega)(x) = \begin{cases} X(\omega)(\sqrt[r]{x}), & r = 2l, x \in \mathbb{R}_+^n, \\ X(\omega)(\sqrt[r]{x}), & r = 2l-1, x \in \mathbb{R}^n. \end{cases}$$

$l \in \mathbf{N}$.

Note that X^r is a fuzzy random variable but may no longer preserve the convexity, however, it is not difficult to see that X^r takes on values in the set of normal compact fuzzy subsets of $\mathbb{R}^n$. We also assume that expectation $\mathbb{E}X^r$ is one in the sense of Puri-Ralescu. Then

Theorem 3.2. *Let $\{X_i\}$ be a squence of independent and identically distributed fuzzy random variables with $\mathbb{E}\|X_i\|_\rho^r < \infty$, r be a positive integer and $\sqrt[r]{x}$ is defined as in the preceding. Then*

(1) $\overline{X_m^r} = \sum_{k=1}^m X_k^r/m$ *is an unbiased estimate of the expectation* $\mathbb{E}X_1^r$, *i.e.* $\mathbb{E}\overline{X_m^r} = \mathbb{E}X_1^r$

(2) $\overline{X_m^r} \xrightarrow{a.s.} \mathbb{E}(co(X_1^r))\,(m \to \infty)$ *in the metric* ρ*;*

(3) $S_m^{r*2} = \sum_{k=1}^m \rho^2(X_k^r, \overline{X_m^r})/(m-1)$ *is an unbiased estimate of* $\mathbf{Var}(X_1^r)$, *i.e.* $\mathbb{E}(S_m^{r*2}) = \mathbf{Var}(X_1^r)$,

(4) $S_m^{r*2} \xrightarrow{a.s.} \mathbb{E}\|X_1^r\|_\rho^2 - \|\mathbb{E}(co(X_1^r))\|_\rho^2\,(n \to \infty)$ *in the metric* ρ*.*

Here $co(X)$ denotes a convex hull of X (see Klement et al (1986)).
Proof. (1) and (2). It follows from Theorem 5.1 of Klement et al (1986) and the relationship between two members of the metrics $d_p, \rho_p, and \rho$ (Diamond and Kloeden (1994) and Theorem 3.1).
(3) and (4). The proof of theorem 6 of Näther (2000) and (1), (2) yield the conclusion (3) and (4). This completes the proof.

Note that conclusion (1) and (2) also holds for the metrics $d_p, \rho_p, 1 \leq p < \infty$.

Theorem 3.3. *Let* X, Y *be fuzzy random variables as in Lemma 2.2, then it holds that*

(i) $|R(X, Y)| \leqslant 1$.

(ii) *If* $R(X, Y) = 1$ *then* $P(\{\rho(Y + \lambda\mathbb{E}X, \mathbb{E}Y + \lambda X) = 0\}) = 1$*; if* $Y + \lambda\mathbb{E}X = \mathbb{E}Y + \lambda X$ *then* $R(X, Y) = 1$, *where* $\lambda = \sqrt{\mathbf{Var}Y/\mathbf{Var}X}$ *;*

(iii) *If* $R(X, Y) = -1$ *then* $P(\{\rho(Y + \lambda X, \mathbb{E}Y + \lambda\mathbb{E}X) = 0\}) = 1$*; if* $Y + \lambda X = \mathbb{E}Y + \lambda\mathbb{E}X$ *then* $R(X, Y) = -1$, *where* $\lambda = \sqrt{\mathbf{Var}Y/\mathbf{Var}X}$.

Proof. (i) By the classical Fubini theorem and the definition of $\langle X, Y\rangle$ and the assumption of the theorem, it holds that

$$\mathbb{E}\langle X, \mathbb{E}Y\rangle = \langle \mathbb{E}X, \mathbb{E}Y\rangle,$$

we prove that

$$\begin{aligned} f(t) &= t^2\mathbf{Var}X - 2t\mathbf{Cov}(X, Y) + \mathbf{Var}Y \\ &= \begin{cases} \mathbb{E}\rho^2(Y + t\mathbb{E}X, \mathbb{E}Y + tX) & , t \geqslant 0 \\ \mathbb{E}\rho^2(Y + |t|X, \mathbb{E}Y + |t|\mathbb{E}X) & , t < 0 \end{cases} \end{aligned}$$

holds for all $t \in \mathbb{R}$.

In fact, let $t \geqslant 0$, then

$$
\begin{aligned}
&\mathbb{E}\rho^2(Y+t\mathbb{E}X,\, \mathbb{E}Y+tX) = \\
&= \mathbb{E}\Big[\int_{(\mathbf{S}^{d-1})^2\times[0,1]^2} \big((Y+t\mathbb{E}X)^*(x,\alpha)-(\mathbb{E}Y+tX)^*(x,\alpha)\big) \\
&\quad \times\big((Y+t\mathbb{E}X)^*(y,\beta)-(\mathbb{E}Y+tX)^*(y,\beta)\big)dK(x,\alpha,y,\beta)\Big] \\
&= \mathbb{E}\|Y\|_\rho^2+2t\mathbb{E}\langle Y,\mathbb{E}X\rangle+t^2\|\mathbb{E}X\|_\rho^2-2\mathbb{E}\langle Y,\mathbb{E}Y\rangle-2t\mathbb{E}\langle X,Y\rangle \\
&\quad -2t^2\mathbb{E}\langle X,\mathbb{E}X\rangle-2t\mathbb{E}\langle \mathbb{E}X,\mathbb{E}Y\rangle+2t\mathbb{E}\langle X,\mathbb{E}Y\rangle+\mathbb{E}\|\mathbb{E}Y\|_\rho^2+t^2\mathbb{E}\|X\|_\rho^2 \\
&= t^2(\mathbb{E}\|X\|_\rho^2-\|\mathbb{E}X\|_\rho^2)-2t(\mathbb{E}\langle X,Y\rangle-\langle \mathbb{E}X,\mathbb{E}Y\rangle)+\mathbb{E}\|Y\|_\rho^2-\|\mathbb{E}Y\|_\rho^2 \\
&= t^2\mathbf{Var}X-2t\mathbf{Cov}(X,\,Y)+\mathbf{Var}Y.
\end{aligned}
$$

let $t<0$, then

$$
\begin{aligned}
&\mathbb{E}\rho^2(Y+|t|X,\, \mathbb{E}Y+|t|\mathbb{E}X) = \\
&= \mathbb{E}\Big[\int_{(\mathbf{S}^{d-1})^2\times[0,1]^2} \big((Y^*+|t|X^*)(x,\alpha)((\mathbb{E}Y)^*+|t|(\mathbb{E}X)^*)(x,\alpha)\big) \\
&\quad \times\big((Y^*+|t|X^*)(y,\beta)((\mathbb{E}Y)^*+|t|(\mathbb{E}X)^*)(y,\beta)\big)dK(x,\alpha,y,\beta)\Big] \\
&= \mathbb{E}[\langle Y,Y\rangle+2|t|\langle X,Y\rangle|t|^2\langle X,X\rangle-\langle \mathbb{E}Y,Y\rangle-|t|\langle \mathbb{E}Y,X\rangle-|t|\langle \mathbb{E}X,Y\rangle-|t|^2\langle \mathbb{E}X,X\rangle \\
&\quad -\langle Y,\mathbb{E}Y\rangle-|t|\langle X,\mathbb{E}Y\rangle-|t|\langle Y,\mathbb{E}X\rangle-|t|^2\langle X,\mathbb{E}X\rangle+\langle \mathbb{E}Y,\mathbb{E}Y\rangle \\
&\quad +|t|\langle \mathbb{E}X,\mathbb{E}Y\rangle+2|t|\langle Y,Y\rangle+|t^2|\langle \mathbb{E}X,\mathbb{E}X\rangle] \\
&= \mathbb{E}\|Y\|_\rho^2-\|\mathbb{E}Y\|_\rho^2+2|t|(\mathbb{E}\langle X,Y\rangle-\langle \mathbb{E}X,\mathbb{E}Y\rangle)+|t|^2(\mathbb{E}\|X\|_\rho^2-\|\mathbb{E}X\|_\rho^2) \\
&= \mathbf{Var}Y+2|t|\mathbf{Cov}(X,Y)+|t|^2\mathbf{Var}X \\
&= \mathbf{Var}Y-2t\mathbf{Cov}(X,Y)+|t|^2\mathbf{Var}X.
\end{aligned}
$$

thus, obviously $f(t) \geqslant 0$, whence

$$(2\mathbf{Cov}(X,Y))^2-4\mathbf{Var}X\mathbf{Var}Y \leqslant 0.$$

i.e.,

$$|R(X,\,Y)| \leqslant 1.$$

This completes the proof of (i).
(ii). Assume that $R(X,Y)=1$, then there exist unique real number

$$t_0=\frac{-2\mathbf{Cov}(X,\,Y)}{2\mathbf{Var}X}=\sqrt{\mathbf{Var}Y/\mathbf{Var}X}>0$$

such that

$$f(t_0)=\mathbb{E}\rho^2(Y+\lambda\mathbb{E}X,\,\mathbb{E}Y+\lambda X)=0,$$

where $\lambda=\sqrt{\mathbf{Var}Y/\mathbf{Var}X}$. By the Tchebyshev inequality

$$P(\{\rho(Y+\lambda\mathbb{E}X,\,\mathbb{E}Y+\lambda X)>\epsilon\}) \leqslant \frac{\mathbb{E}\rho^2(Y+\lambda\mathbb{E}X,\,\mathbb{E}Y+\lambda X)}{\epsilon^2},$$

for any $\epsilon > 0$, we obtain that

$$P(\{\rho(Y + \lambda \mathbb{E}X,\, \mathbb{E}Y + \lambda X) = 0\}) = 1$$

where $\lambda = \sqrt{\mathbf{Var}Y/\mathbf{Var}X}$.

Assume that $Y+\lambda\mathbb{E}X = \mathbb{E}Y+\lambda X$, then by the property (viii) of Lemma 2.2, it follows that

$$\begin{aligned}\mathbf{Cov}(X,Y) &= \frac{1}{\lambda}\mathbf{Cov}(\lambda X + \mathbb{E}Y, Y + \lambda\mathbb{E}X)\\ &= \frac{1}{\lambda}\mathbf{Cov}(Y + \lambda\mathbb{E}X, Y + \lambda\mathbb{E}X)\\ &= \frac{1}{\lambda}\mathbf{Cov}(Y,\, Y) = \sqrt{\mathbf{Var}X\mathbf{Var}Y},\end{aligned}$$

thus, $R(X,Y) = 1$. This completes the proof of (ii).
(iii). Assume that $R(X,Y) = -1$, then there exist unique real number

$$t_0 = \frac{\mathbf{Cov}(X,Y)}{\mathbf{Var}X} = \frac{-\sqrt{\mathbf{Var}X\mathbf{Var}Y}}{\mathbf{Var}X} = -\sqrt{\mathbf{Var}Y/\mathbf{Var}X} < 0,$$

such that

$$f(t_0) = \mathbb{E}\rho^2(Y + |t_0|X, \mathbb{E}Y + |t_0|\mathbb{E}X) = 0.$$

then by Tchebyshev ineuality, we also obtain that

$$P(\{\rho(Y + \lambda X,\, \mathbb{E}Y + \lambda\mathbb{E}X) = 0\}) = 1$$

where $\lambda = |t_0| = \sqrt{\mathbf{Var}Y/\mathbf{Var}X}$.

Assume that $Y + \lambda X = \mathbb{E}Y + \lambda\mathbb{E}X$, then by the properties of the operation $\langle\cdot,\cdot\rangle$ and the classical Fubini theorem, it follows that

$$\begin{aligned}\mathbf{Cov}(X,Y) &= \mathbb{E}\langle X,Y\rangle - \langle\mathbb{E}X,\mathbb{E}Y\rangle\\ &= \mathbb{E}\langle Y+\lambda X, X\rangle - \lambda\mathbb{E}\langle X,X\rangle - \langle\mathbb{E}X,\mathbb{E}Y\rangle\\ &= \mathbb{E}\langle\mathbb{E}Y+\lambda\mathbb{E}X, X\rangle - \lambda\mathbb{E}\langle X,X\rangle - \langle\mathbb{E}X,\mathbb{E}Y\rangle\\ &= \lambda\langle\mathbb{E}X,\mathbb{E}X\rangle - \lambda\mathbb{E}\langle X,X\rangle\\ &= -\lambda\mathbf{Var}X = -|t_0|\mathbf{Var}X\\ &= -\sqrt{\mathbf{Var}X\mathbf{Var}Y},\end{aligned}$$

thus $R(X,Y) = -1$. This completes the proof of (iii).

Theorem 3.4. *Let $X_1, X_2, \ldots, X_m$ be independent and identically distributed fuzzy random variables, put $\overline{X_m} := \sum_{k=1}^m X_k/m$, $S_m^2 := \sum_{k=1}^m \rho^2(X_k, \overline{X_n})/n$, $S_m^{*2} := \frac{m}{m-1}S_m^2$. Then*

(1) $\mathbf{Var}(S_m^{*2}) = 3\mathbb{E}\|X_1\|_\rho^4 - 4\mathbf{Var}(X_1)^2$,

(2) $\mathbf{Cov}(\overline{X_m}, S_m^2) = \frac{1}{2}\mathbb{E}\langle X_1, \|X_1\|_\rho^2\rangle - \mathbb{E}\langle X_1, \|\overline{X_m}\|_\rho^2\rangle + (1-\frac{1}{m})c\|\mathbb{E}X_1\|_\rho^2$, *where* $c = \mathbb{E}\int_{S^{m-1}\times[0,1]} X_1^*(x,\alpha)dK(x,\alpha)$.

Proof. It holds that $\mathbb{E}S_m^{*2} = \mathbf{Var}(X_1)$ from Lemma 4, whence

$$
\begin{aligned}
&\mathbf{Var}(S_m^{*2})\\
&= \mathbb{E}\langle S_m^{*2}, S_m^{*2}\rangle - \langle \mathbb{E}S_m^{*2}, \mathbb{E}S_m^{*2}\rangle\\
&= \mathbb{E}\left\langle \frac{1}{m-1}\sum_{k=1}^m \rho^2(X_k, \overline{X_m}), \frac{1}{m-1}\sum_{k=1}^m \rho^2(X_k, \overline{X_m})\right\rangle - \langle \mathbf{Var}(X_1), \mathbf{Var}(X_1)\rangle\\
&= \frac{1}{(m-1)^2}\left(\sum_{k=1}^m \mathbb{E}\langle \|X_k\|_\rho^2, \|X_k\|_\rho^2\rangle\right.\\
&\quad \left. +2\sum_{1\le i<j\le m}\mathbb{E}\langle \|X_i\|_\rho^2, \|X_j\|_\rho^2\rangle\right) - \frac{2m}{(m-1)^2}\sum_{k=1}^m \mathbb{E}\langle \|X_k\|_\rho^2, \|\overline{X_m}\|_\rho^2\rangle\\
&\quad + \frac{m^2}{(m-1)^2}\mathbb{E}\langle \|\overline{X_m}\|_\rho^2, \|\overline{X_m}\|_\rho^2\rangle - \langle \mathbf{Var}(X_1), \mathbf{Var}(X_1)\rangle\\
&= \frac{m^2}{(m-1)^2}\left(\mathbb{E}\langle \|\overline{X_m}\|_\rho^2, \|\overline{X_m}\|_\rho^2\rangle - \mathbb{E}\langle \|X_1\|_\rho^2, \|X_1\|_\rho^2\rangle\right) - 4\mathbf{Var}(X_1)^2\\
&= \frac{3-6m+3m^2}{(m-1)^2}\mathbb{E}\|X_1\|_\rho^4 - 4\mathbf{Var}(X_1)^2\\
&= 3\mathbb{E}\|X_1\|_\rho^4 - 4\mathbf{Var}(X_1)^2
\end{aligned}
$$

To prove (2), note that $\mathbf{Cov}(\overline{X_m}, S_m^2) = \mathbb{E}\langle \overline{X_m}, S_m^2\rangle - \langle \mathbb{E}\overline{X_m}, S_m^2\rangle$ and $S_m^2 = \frac{1}{m}\sum_{k=1}^m \|X_k\|_\rho^2 - \|\overline{X_m}\|_\rho^2$. Therefore

$$
\begin{aligned}
\mathbb{E}\langle \overline{X_m}, S_m^2\rangle &= \frac{1}{m^2}\mathbb{E}\left(\sum_{i=1}^m\sum_{k=1}^m \langle X_i, \|\overline{X}_k\|_\rho^2\rangle\right) - \frac{1}{m}\mathbb{E}\left(\sum_{k=1}^m \langle X_k, \|\overline{X_m}\|_\rho^2\rangle\right)\\
&= \frac{1}{m^2}\left[\sum_{j=1}^m \int_{S^{m-1}\times[0,1]} \mathbb{E}(\|X_j\|_\rho^2 X_j^*(x,\alpha))dK(x,\alpha)\right.\\
&\quad \left. +2\sum_{1\leqslant i<j\leqslant m}\mathbb{E}\|X_i\|_\rho^2 \int_{S^{m-1}\times[0,1]} \mathbb{E}X_j^*(x,\alpha)dK(x,\alpha)\right]\\
&\quad -\frac{1}{m}\sum_{k=1}^m \mathbb{E}\langle X_k, \|\overline{X_m}\|_\rho^2\rangle\\
&= \frac{1}{m}\mathbb{E}\langle X_1, \|X_1\|_\rho^2\rangle + \frac{n-1}{m}c\mathbb{E}\|X_1\|_\rho^2 - \mathbb{E}\langle X_1, \|\overline{X}_m\|_\rho^2\rangle
\end{aligned}
$$

and

$$
\langle \mathbb{E}\overline{X_m}, \mathbb{E}S_m^2\rangle = \left\langle \mathbb{E}X_1, \frac{m-1}{m}\mathbf{Var}(X_1)\right\rangle = \frac{m-1}{m}c\mathbf{Var}(X_1)
$$

where $c = \mathbb{E}\int_{S^{m-1}\times[0,1]} X_1^*(x,\alpha)dK(x,\alpha)$ is a constant. Thus we obtain the result of (2). □

Theorem 3.5. *Let $X_1, \ldots, X_m$ be fuzzy random variables, then*

(1) $\mathbf{Var}\big(\sum_{k=1}^m X_k\big) = \sum_{k=1}^m \mathbf{Var}(X_k) + 2\sum_{1\leqslant i<j\leqslant m} \mathbf{Cov}(X_i, X_j)$;

(2) $\mathbf{Var}\big(\overline{X_m}\big) = \frac{1}{m}\sum_{k=1}^m \mathbf{Var}(X_k) + \frac{2}{m^2}\sum_{1\leqslant i<j\leqslant m} \mathbf{Cov}(X_i, X_j)$;

(3) $\mathbf{Var}\Big(\overline{X_m^r}\Big) = \frac{1}{m}\sum_{k=1}^m \mathbf{Var}(X_k^r) + \frac{2}{m^2}\sum_{1\leqslant i<j\leqslant m} \mathbf{Cov}(X_i^r, X_j^r)$.

where r is a positive integer with $\mathbb{E}\|X_i\|_\rho^r < \infty$, $i = 1, 2, \ldots, m$.
Proof. (1).

$$
\begin{aligned}
&Var\big(\sum_{k=1}^m X_k\big)\\
&= \mathbb{E}\langle\sum_{k=1}^m X_k,\, \sum_{k=1}^m X_k\rangle - \langle\mathbb{E}(\sum_{k=1}^m X_k),\, \mathbb{E}(\sum_{k=1}^m X_k)\rangle\\
&= \sum_{k=1}^m \mathbb{E}\langle X_k,\, X_k\rangle + 2\sum_{1\leqslant i<j\leqslant m} \mathbb{E}\langle X_i,\, X_j\rangle\\
&\quad -\sum_{k=1}^m\langle\mathbb{E}X_k,\, \mathbb{E}X_k\rangle - 2\sum_{1\leqslant i<j\leqslant m}\langle\mathbb{E}X_i,\, \mathbb{E}X_j\rangle\\
&= \sum_{k=1}^m \mathbf{Var}(X_k) + 2\sum_{1\leqslant i<j\leqslant m} \mathbf{Cov}(X_i, X_j).
\end{aligned}
$$

(2) and (3) can be followed similarly.

Theorem 3.6. *If $\{T_m\}$ is a consistent estimator of θ with respect to ρ and*

$$\lim_{m\to\infty} \mathbf{Var}(T_m) = 0,$$

then $\{T_m\}$ is an asymptotical unbiased estimator of θ, i.e.

$$\lim_{m\to\infty} P(\{\rho(\mathbb{E}(T_m), \theta) > \epsilon\}) = 0$$

for every $\epsilon > 0$.

Proof. Since $\{T_m\}$ be a consistent estimator of θ , thus for every $\epsilon > 0$, we have

$$\lim_{m\to\infty} P(\{\rho(T_m, \theta) > \epsilon\}) = 0.$$

Since here $\rho(\mathbb{E}(T_m), T_m)$ is a real integrably bounded random variable, by Tchebyshev inequality, we have that

$$\lim_{m\to\infty} \mathbf{Var}(T_m) = 0$$

implies that

$$\lim_{m\to\infty} P(\{\rho(\mathbb{E}(T_m), T_m) > \epsilon\}) = 0.$$

whence

$$\begin{aligned}\lim_{m\to\infty} P(\{\rho(\mathbb{E}(T_m), \theta) > \epsilon\}) &\leqslant \lim_{m\to\infty} P(\{\rho(\mathbb{E}(T_m), T_m) + \rho(T_m, \theta) > \epsilon\}) \\ &\leqslant \lim_{n\to\infty} P(\{\rho(\mathbb{E}(T_m), T_m\}) > \frac{\epsilon}{2}) \\ &\quad + \lim_{m\to\infty} P(\{\rho(T_m, \theta) > \frac{\epsilon}{2}\}) \\ &= 0.\end{aligned}$$

which completes the proof.

Theorem 3.7.

(1) *Let T_{mi} be an unbiased estimator of θ_i with respect to random sample from fuzzy random variable X, then their linear combination $\sum_{i=1}^k a_i T_{mi}$ defined by means of Zadeh's extension principle is an unbiased estimator of $\sum_{i=1}^k a_i\theta_i$, i.e.*

$$\mathbb{E}\left(\sum_{i=1}^k a_i T_{mi}\right) = \sum_{i=1}^k a_i\theta_i.$$

(2) *Let T_{mi} be a UMVUE estimator of θ_i with respect to random sample from fuzzy random variable X, $i = 1, \ldots, k$, and $R(T_{mi}, T_{mj}) = 0$, $i \neq j$, then their linear combination $\sum_{i=1}^k a_i T_{mi}$ defined by means of Zadeh's extension principle is a UMVUE of $\sum_{i=1}^k a_i\theta_i$.*

(3) *If T_m is a UMVUE of θ and U is an unbised estimator for fuzzy set 0, then $\mathbb{E}\langle T_m, U\rangle = 0$. Conversely, if T_m is an unbiased estimator for θ and for all unbiased estimators U of fuzzy set* 0 *it holds that* $\mathbf{Cov}(T_m, U) = \mathbb{E}\langle T_m, U\rangle = 0$, *then T_m is a UMVUM of θ.*

(4) *If T_{m1} and T_{m2} are two UMVUE of θ with respect to random sample from fuzzy random variable X, then $\rho^2(T_{m1}, \theta) = \rho^2(T_{m2}, \theta)$ almost surely.*

Proof. (1) and (2) can be obtained directly from the definitions of unbiased estimators and UMVUE and the linearity of the expectation of fuzzy random varibales.
(3).By the assumption of (3),if we assume $T := T_m + cU$, where $c > 0$, then we have $\mathbb{E}T = \mathbb{E}T_m + c\mathbb{E}U = \mathbb{E}T_m = \theta$, thus $\mathbf{Var}(T) \geqslant \mathbf{Var}(T_m)$, and

$$\mathbf{Var}T = \mathbf{Var}(T_m) + c^2\mathbf{Var}(U) + 2c(\mathbf{Cov}(T_m, U)),$$

whence

$$c^2\mathbf{Var}(U) + 2c(\mathbf{Cov}(T_m,\, U)) \geqslant 0,$$

which is impossible unless

$$\begin{aligned}\mathbf{Cov}(T_m,\, U) =& E\langle T_m,\, U\rangle - \langle \mathbb{E}T_m,\, \mathbb{E}U\rangle \\ &= \mathbb{E}\langle T_m,\, U\rangle = 0.\end{aligned}$$

thus the first assertion of (3) holds. Conversely, let T be an arbitrary unbiased estimator for θ and let $U = T - T_m$, then

$$\begin{aligned}\mathbf{Var}T &= \mathbf{Var}(U + T_m) \\ &= \mathbf{Var}(U) + \mathbf{Var}(T_m) + 2\mathbf{Cov}(U, T_m) \\ &= \mathbf{Var}(U) + \mathbf{Var}(T_m) \geqslant \mathbf{Var}(T_m).\end{aligned}$$

which means T_m is a UMVUE of θ.
(4). By the assumption of (4), it is not difficult to obtain that

$$\mathbf{Var}(T_{m1}) = \mathbf{Var}(T_{m2}),$$

which means that

$$\mathbb{E}|\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta)| = 0,$$

thus, we have

$$\begin{aligned}&\mathbf{Var}(\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta)) \\ &\qquad = \mathbb{E}(\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta) \\ &\qquad\quad - \mathbb{E}(\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta)))^2 \\ &\qquad = \mathbb{E}|\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta)|^2 = 0.\end{aligned}$$

By Tchebyshev inequlity we obtain

$$P(\{|\rho^2(T_{m1},\, \theta) - \rho^2(T_{m2},\, \theta)| > \epsilon\}) = 0$$

for each $\epsilon > 0$. Thus the assertion (4) is followed.

Corollary 3.2. *Let the assumption of Theorem 3.5. hold. Then*

(1) $\overline{X_m}$ *and* S_m^{*2} *are uniform mean square error estimates of* $\mathbb{E}X_1$ *and* $\mathbf{Var}X_1$ *respectively.*

(2) $\overline{X_m^r}$ *and* S_m^{r*2} *are also uniform mean square error estimates of* $\mathbb{E}X_1^r$ *and* $\mathbf{Var}X_1^r$ *respectively.*

(3) S_m^{*2} *and* S_m^{r*2} *are more efficient than* S_m^2 *and* S_m^{r2} *respectively.*

Proof. To prove (1), it holds that $\{\overline{X_m}\}$ is a sequence of unbiased estimate for $\mathbb{E}X_1$ by Lemma 3, and $\mathbb{E}\langle X_i, X_j\rangle = \langle \mathbb{E}X_i, \mathbb{E}X_j\rangle$, thus

$$\begin{aligned} & \lim_{m\to\infty} \mathbb{E}(\rho^2(\overline{X_m}, \mathbb{E}X_1)) \\ = & \lim_{m\to\infty} \left[\frac{1}{m}\mathbb{E}\|X_1\|_\rho^2 + \|\mathbb{E}_1\|_\rho^2 + \frac{m(m-1)}{m^2}\|\mathbb{E}X_1\|_\rho^2 - 2\|\mathbb{E}X_1\|_\rho^2\right] \\ = & \lim_{m\to\infty} \frac{1}{m}\mathbf{Var}(X_1) = 0 \end{aligned}$$

By definition 9, we obtain the resuls of (1). For the proof of (2), it holds that $\{S_m^{*2}\}$ is a seqence of unbiased estimates for $Var(X_1)$ by Lemma 2.4, and $\|\cdot\|_\rho^2$ is continuous (see Näther (2000)), therefore $\|S_m^{*2}\|_\rho^2 \xrightarrow{a.s.} \|\mathbf{Var}(X_1)\|_\rho^2$, and $\mathbb{E}[\sup_{n\geq 1}\|S_m^{*2}\|_\rho^2] < \infty$. By Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E}\|S_m^{*2}\|_\rho^2 = \mathbb{E}\|\mathbf{Var}(X_1)\|_\rho^2$$

whence

$$\begin{aligned} & \lim_{m\to\infty} \mathbb{E}\left[\rho^2(S_m^{*2}, \mathbf{Var}(X_1))\right] \\ & = \lim_{m\to\infty} \mathbb{E}[\langle S_m^{*2}, S_m^{*2}\rangle - 2\langle S_m^{*2}, \mathbf{Var}(X_1)\rangle + \|\mathbf{Var}(X_1)\|_\rho^2] \\ & = \lim_{m\to\infty} (\mathbb{E}\|S_m^{*2}\|_\rho^2 - 2\langle \mathbb{E}S_m^{*2}, \mathbf{Var}(X_1)\rangle + \|\mathbf{Var}(X_1)\|_\rho^2) \\ & = \lim_{m\to\infty} (\mathbb{E}\|S_m^{*2}\|_\rho^2 - 2\langle \mathbf{Var}(X_1), \mathbf{Var}(X_1)\rangle + \|\mathbf{Var}(X_1)\|_\rho^2) \\ & = \mathbb{E}\|\mathbf{Var}(X_1)\|_\rho^2 - \|\mathbf{Var}(X_1)\|_\rho^2 = 0. \end{aligned}$$

By definition 9, we obtain the result of (2). The assertion (3) is proved trivially. □

Conclusion In this paper, we have presented several results concerning with the statistical studies of fuzzy random variables with respect to the metric ρ, which reveals some basic properties of the fuzzy statistic in a multidimensional space with respect to the considered metric. These rather general results can be used for further investigating statistical properties of fuzzy random variables with respect to some metric which is in special case of the metric ρ.

For obtaining an extensive and valuable statistical result for fuzzy random variables, the next interesting but complicated problem is assumed to be that find out a suitable distribution for fuzzy random variable and the structure of the unknown parameters.

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