A MONOTONE FUZZY STOPPING TIME IN DYNAMIC FUZZY SYSTEMS

Y. YOSHIDA[§], M. YASUDA[†], J. NAKAGAMI[†] and M. KURANO[‡]

[§]*Faculty of Economics and Business Administration, Kitakyushu University, Kitakyushu 802-8577, Japan.* [†]*Faculty of Science,* [‡]*Faculty of Education, Chiba University, Chiba 263-8522, Japan.*

Abstract

This paper is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times, which is called as a monotone fuzzy stopping time is introduced. This notion is well-known in a stochastic process. We have constructed it by subsets of α-sets of fuzzy states under appropriate assumptions. The aim is to consider the optimization of a stopping problem with an additive weighting function in the class of monotone fuzzy stopping times.

Keywords: fuzzy stopping time; monotone property; α-cuts of fuzzy sets; stopping problem; optimality.

1 Introduction and notations

The stopping time with fuzziness, which is called a 'fuzzy stopping time', is discussed by our previous paper [11] where we have considered the optimization of the corresponding fuzzy stopping problem by the constructive method. This kind of stopping times are first introduced by Kacprzyk [5, 6] for a restriction of stoppable times, and he gave dynamic programming under a time restriction for the multistage decision-making with fuzziness introduced by Bellman and Zadeh [1] (see [7]). In [11], we have discussed a stopping problem for a system with fuzzy states, which is called a 'dynamic fuzzy system' ([9], [15]), using fuzzy stopping times not as a restriction of stoppable times but as stopping strategies. Alternatively, it is well-known that a class of stopping times which has a monotone property is useful for various application problems. Because it is simple to understand and easy to calculate. Refer to Chow,Robbins and Siegmund [3] and Ross [14]. In this paper, we introduce a new class of fuzzy stopping times with a kind of monotone property and we apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards.

In the remainder of this section, a fuzzy dynamic system is defined and we prepare some notations for Section 2. In Section 2, a new class of fuzzy stopping times, which we call them 'monotone fuzzy stopping times', is introduced and their construction is also discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, a example is given to illustrate the results.

Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions defined on a convex compact subsets of some Banach space. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. The detail of its definition is omitted here.

Let E be a given convex compact subsets of some Banach space. $\mathcal{F}(E)$ denotes the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are assumed to be upper semi-continuous and have a compact support with the normality condition: $\sup_{x\in E}\tilde{u}(x) = 1$. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{u}$ is denoted as $\tilde{u}_\alpha$. $\mathcal{C}(E)$ means the collection of all compact convex subsets of E. Then clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0,1]$. Let $\mathbf{R}$ be the set of all real numbers and let $\mathcal{C}(\mathbf{R})$ be the set of all bounded closed intervals in $\mathbf{R}$. The elements of $\mathcal{F}(\mathbf{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are well-known. See Puri and Ralescu [13] for the details. The following results are known, so the proofs are omitted.

Lemma 1.1 (Chen-wei Xu [2], Kurano et al.[11]).

(i) *For any* $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ *and* $\lambda \geq 0$, *it holds that* $\tilde{m}+\tilde{n} \in \mathcal{F}(\mathbf{R})$ *and* $\lambda\tilde{m} \in \mathcal{F}(\mathbf{R})$.

(ii) *Let* E_1 *and* E_2 *be convex compact subsets. If* $\tilde{u} \in \mathcal{F}(E_1)$ *and* $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$ *satisfy* $\tilde{p}(x,\cdot) \in \mathcal{F}(E_2)$ *for* $x \in E_1$, *then* $\sup_{x\in E_1}\{\tilde{u}(x) \wedge \tilde{p}(x,\cdot)\} \in \mathcal{F}(E_2)$,

where $a \wedge b = \min\{a,b\}$ *for real numbers* a, b.

Now we will formulate the dynamic fuzzy system.

Definition 1 (Kurano et al. [9]). The pair of $(S,\tilde{q})$ is called a dynamic fuzzy system if the following conditions (i) and (ii) are satisfied:

(i) The state space S is a convex compact subset of some Banach space. In generally, the state of the system is simply called as a fuzzy state and it is denoted by an element of $\mathcal{F}(S)$.

(ii) The law of motion for the system is based on a time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$, and we assume $\tilde{q} \in \mathcal{F}(S\times S)$ and $\tilde{q}(x,\cdot) \in \mathcal{F}(S)$ for $x \in S$.

If the dynamic fuzzy system $(S,\tilde{q})$ is given, then we consider a sequential transition of states as follows. Firstly, a fuzzy state $\tilde{s} \in \mathcal{F}(S)$ is moved to a new fuzzy state $Q(\tilde{s})$ after a unit time has passed, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined by

$$Q(\tilde{s})(y) := \sup_{x\in S}\{\tilde{s}(x) \wedge \tilde{q}(x,y)\} \quad \text{for } y \in S. \tag{1.1}$$

Note that the map Q is well-defined by Lemma 1.1. Explicitly, for the dynamic fuzzy system $(S,\tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \tag{1.2}$$

We need the following preliminaries to define fuzzy stopping times for this sequence $\{\tilde{s}_t\}_{t=1}^{\infty}$, which are given in the next section. Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ ($\alpha \in [0,1]$) are defined as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{if } \alpha > 0 \\ \mathrm{cl}\{y \in S \mid \tilde{q}(x,y) > 0 \text{ for some } x \in D\} & \text{if } \alpha = 0, \end{cases} \tag{1.3}$$

where cl means the closure of a set. From the assumption on $\tilde{q}$, the maps Q_α are well-defined. The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α-cuts of fuzzy state, $Q_\alpha(\tilde{s})$, are specified using the maps Q_α of (1.3).

Lemma 1.2 (Kurano et al. [9, 10]). *For any $\alpha \in [0,1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:*

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $\tilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha)$ $\ (t \geq 1)$,

where $\tilde{s}_\alpha$ and $\tilde{s}_{t,\alpha}$ are the α-cuts of fuzzy state $\tilde{s}$ and $\tilde{s}_t$ respectively and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.2) with the initial state $\tilde{s}_1 = \tilde{s}$.

2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here, and we introduce a new class of fuzzy stopping times, which is constructed through subsets of α-cuts of fuzzy states. For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$ and let $\mathcal{F}'$ be a subset of $\mathcal{F}$.

Definition 2 (Kurano et al. [11]). A *fuzzy stopping time* on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma}$: $\mathcal{F}' \times \mathbf{N} \mapsto [0,1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s},t)$ is non-increasing in t and there exists a natural number $t(\tilde{s}) \geq 1$ with $\tilde{\sigma}(\tilde{s},t) = 0$ for all $t \geq t(\tilde{s})$, where $\mathbf{N} := \{1, 2, \cdots\}$.

We note here that '$\tilde{\sigma} = 0$' represents 'stop' and '$\tilde{\sigma} = 1$' represents 'continuity' in the grade of membership (Kurano et al. [11]). Between the two decisions, the intermediate value '$0 < \tilde{\sigma} < 1$' is a notion of 'fuzzy stopping'. A fuzzy stopping time $\tilde{\sigma}(\tilde{s},t)$ means the degree of 'continuity' at time$t$ starting from a fuzzy state $\tilde{s}$. The set of all fuzzy stopping times on $\mathcal{F}'$ is denoted by $\Sigma(\mathcal{F}')$.

Definition 3. A fuzzy stopping time $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ is called *monotone* if there exists a map $\delta : \mathcal{F}' \mapsto [0,1]$ satisfying

(i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and

(ii) $\tilde{\sigma}(\tilde{s},t) = \bigwedge_{l=1}^t \delta(\tilde{s}_t)$ for all $\tilde{s} \in \mathcal{F}'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.2) with $\tilde{s}_1 = \tilde{s}$.

The real-valued map δ on fuzzy states in the above definition is called a *support* of $\tilde{\sigma}$. The definition means a natural and good property for fuzzy stopping times, which is simple and easy to calculate optimal stopping times in actual optimization problems. The degree of monotone fuzzy stopping times is given by only the fuzzy state atcurrent time t. Therefore, in stopping problems, the criterion is reduced whether the fuzzy state at current time t belongs to the optimal stopping region or not.

We now construct a monotone fuzzy stopping times to be the subject of this paper. For the purpose of this construction, we assume the following condition.

Condition 1. For each $\alpha \in [0,1]$, there exists a non-empty subset $\mathcal{K}_\alpha$ of $\mathcal{C}(S)$ satisfying

$$Q_\alpha(\mathcal{K}_\alpha) \subset \mathcal{K}_\alpha. \tag{2.1}$$

Using this subset $\mathcal{K}_\alpha$, we define a sequence of subsets $\{\mathcal{K}_\alpha^t\}_{t=1}^\infty$ inductively by

$$\begin{aligned} \mathcal{K}_\alpha^1 &:= \mathcal{K}_\alpha; \\ \mathcal{K}_\alpha^t &:= \{c \in \mathcal{C}(S) \mid Q_\alpha(c) \in \mathcal{K}_\alpha^{t-1}\} \quad (t \geq 2). \end{aligned} \tag{2.2}$$

Clearly, $\mathcal{K}_\alpha^t = Q_\alpha^{-1}(\mathcal{K}_\alpha^{t-1}) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha)$. Also, it holds from (2.4) that $\mathcal{K}_\alpha^t \subset \mathcal{K}_\alpha^{t+1}$ $(t \geq 1)$. To simplify our discussion, we assume the following condition holds henceforth.

Condition 2. For all $\alpha \in [0,1]$, it holds that

$$\mathcal{C}(S) = \bigcup_{t=1}^{\infty} \mathcal{K}_\alpha^t. \tag{2.3}$$

For $c \in \mathcal{C}(S)$ and $\alpha \in [0,1]$, we define a stopping time $\hat{\sigma}_\alpha(c)$ by

$$\hat{\sigma}_\alpha(c) := \min\{t \geq 1 \mid c \in \mathcal{K}_\alpha^t\}. \tag{2.4}$$

This is the first entry time of a closed interval $c(\in \mathcal{C}(S))$ with the grade α. We define a restricted class $\hat{\mathcal{F}}(\subset \mathcal{F})$ by

$$\hat{\mathcal{F}} := \{\tilde{s} \in \mathcal{F} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0,1]\}. \tag{2.5}$$

Using the class $\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0,1]\}$, for the restricted element $\tilde{s} \in \hat{\mathcal{F}}$, we define

$$\hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(t)\} \quad (t \geq 1), \tag{2.6}$$

where 1_{D_α} is the indicator of a set $D_\alpha := \{t \geq 1 \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique to construct a fuzzy number from a family of level sets. Then we obtain the following theorem.

Theorem 2.1 *Under Conditions 1 and 2, the following (i) and (ii) hold.*

(i) $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ *for* $\tilde{s} \in \hat{\mathcal{F}}$ *and* $\alpha \in [0,1]$.

(ii) $\hat{\sigma}$ *of (2.9) is a fuzzy stopping time on* $\hat{\mathcal{F}}$.

Proof. By the definition, $\hat{\sigma}(\tilde{s},t) < \alpha$ is equivalent to $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le t$. This fact shows (i). From Condition 2, there exists $t^* \ge 1$ with $\tilde{s}_0 \in \mathcal{K}_0^{t^*}$. So, $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le \hat{\sigma}_0(\tilde{s}_0) \le t^*$ for all $\alpha \in [0,1]$, which implies that $\hat{\sigma}(\tilde{s},t) = 0$ for all $t \ge t^*$ by the definition of $\hat{\mathcal{F}}$. Since $\hat{\sigma}(\tilde{s},t+1) \le \hat{\sigma}(\tilde{s},t)$ holds clearly for $t \ge 1$ from the definition (2.9), we also obtain (ii). □

In order to show the monotone property of $\hat{\sigma}$, we need the following lemma.

Lemma 2.1 *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then*

(i) $\hat{\sigma}(\tilde{s},t) = \alpha$ *if and only if, for any* $\epsilon > 0$,

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}^t_{\alpha+\epsilon} \text{ and } \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}^t_{\alpha-\epsilon};$$

(ii) $\tilde{s}_t \in \hat{\mathcal{F}}$ $(t \ge 1)$.

Proof. By (2.9), $\hat{\sigma}(\tilde{s},t) = \sup\{\alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. So, (i) follows from (2.7). From Lemma 1.2(ii), for $l \ge 1$, $\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_{l,\alpha}) = \hat{\sigma}_\alpha(Q_\alpha^{l-1}(\tilde{s}_\alpha))$. By (2.5) and (2.7),

$$\begin{aligned}
\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) &= \min\{t \ge 1 \mid Q_\alpha^{l-1}(\tilde{s}_\alpha) \in \mathcal{K}^t_\alpha\} \\
&= \min\{t \ge 1 \mid \tilde{s}_\alpha \in \mathcal{K}^{t+l-1}_\alpha\} \\
&= \max\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l-1), 1\},
\end{aligned}$$

and it is non-increasing in $\alpha \in [0,1]$ since $\tilde{s} \in \hat{\mathcal{F}}$. Therefore we obtain (ii). □

Theorem 2.2 *Let $\tilde{s} \in \hat{\mathcal{F}}$ be given and assume that Conditions 1 and 2 hold. Then, $\hat{\sigma} = \hat{\sigma}(\tilde{s},t)$, $t \ge 1$, is a monotone fuzzy stopping time with the initial state $\tilde{s}$.*

Proof. Let $\{\tilde{s}_t\}_{t=1}^\infty$ be defined by (1.2) with $\tilde{s}_1 = \tilde{s}$. First, we will prove that

$$\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) \quad \text{for } t,r \ge 1. \tag{2.7}$$

Note that $\hat{\sigma}(\tilde{s}_{t+1},r)$ is well-defined from Lemma 2.1(ii). Let $\alpha = \hat{\sigma}(\tilde{s},t+r)$. From Lemma 2.1(i), we have

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}^{t+r}_{\alpha+\epsilon} \quad \text{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}^{t+r}_{\alpha-\epsilon} \quad \text{for any } \epsilon > 0.$$

Noting $Q^t_\alpha(\mathcal{K}^l_\alpha) = \mathcal{K}^{l-t}_\alpha$ $(1 \le t < l)$ and Lemma 1.2(ii), we obtain

$$\tilde{s}_{t+1,\alpha+\epsilon} = Q^t_{\alpha+\epsilon}(\tilde{s}_{\alpha+\epsilon}) \in Q^t_{\alpha+\epsilon}(\mathcal{K}^{t+r}_{\alpha+\epsilon}) = \mathcal{K}^r_{\alpha+\epsilon}$$

and

$$\tilde{s}_{t+1,\alpha-\epsilon} = Q^t_{\alpha-\epsilon}(\tilde{s}_{\alpha-\epsilon}) \notin Q^t_{\alpha-\epsilon}(\mathcal{K}^{t+r}_{\alpha-\epsilon}) = \mathcal{K}^r_{\alpha-\epsilon}.$$

Therefore, we get $\hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$ from Lemma 2.1(i). Namely, $\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s}_{t+1},r)$. Since $\hat{\sigma}(\tilde{s},t+r) \le \hat{\sigma}(\tilde{s},t)$ from Theorem 2.1(ii), we obtain $\hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$, and so

(2.10) holds. Next, we put $\delta(\tilde{s}) = \hat{\sigma}(\tilde{s}, 1)$ for $\tilde{s} \in \hat{\mathcal{F}}$. From (2.10), we get

$$\begin{aligned}
\hat{\sigma}(\tilde{s}, t) &= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, t-1) \\
&= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, 1) \wedge \hat{\sigma}(\tilde{s}_3, t-2) \\
&= \cdots \\
&= \bigwedge_{l=1}^{t} \hat{\sigma}(\tilde{s}_l, 1) \\
&= \bigwedge_{l=1}^{t} \delta(\tilde{s}_l) \quad \text{for } t \geq 1.
\end{aligned}$$

Since we also have $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$ from Theorem 2.1(ii), $\hat{\sigma}$ is a monotone fuzzy stopping time with $\tilde{s}$. The proof of this theorem is completed. □

3 Fuzzy stopping problems

In this section, applying the results in the previous section, we obtain the optimal fuzzy stopping time for a fuzzy dynamic system with fuzzy rewards (see Kurano et al. [10]) when the weighting function is additive.

Firstly, we formulate the stopping problem to be considered here. Let $\tilde{r} : S \times \mathbf{R} \mapsto [0, 1]$ be a fuzzy relation satisfying $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$ and $\tilde{r}(x, \cdot) \in \mathcal{F}(\mathbf{R})$ for $x \in S$. If the system is in a current fuzzy state $\tilde{s} \in \mathcal{F}$, a fuzzy reward is earned:

$$R(\tilde{s})(z) := \sup_{x \in S} \{\tilde{s}(x) \wedge \tilde{r}(x, z)\}, \quad z \in \mathbf{R}.$$

Then we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in (1.2) with the initial fuzzy state $\tilde{s}_1 = \tilde{s}$. Let

$$\varphi(\tilde{s}, t) := \sum_{l=1}^{t} R(\tilde{s}_l) \quad \text{for } t \geq 1. \tag{3.1}$$

Note that (3.11) designates the summation of fuzzy numbers. For details, refer to Puri and Ralescu [13] and Kurano et al. [10]. We need the following lemma, which is proved in Kurano et al. [9, 10].

Lemma 3.1 (Kurano et al.[9, 10]). *For $t \geq 1$ and $\alpha \geq 0$,*

$$\varphi(\tilde{s}, t)_\alpha = \sum_{l=1}^{t} R_\alpha(\tilde{s}_{l,\alpha})$$

holds, where

$$R_\alpha(\tilde{s}_{l,\alpha}) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x, z) \geq \alpha \textit{ for some } z \in \tilde{s}_{l,\alpha}\} & \textit{if } \alpha > 0 \\ \mathrm{cl}\{z \in \mathbf{R} \mid \tilde{r}(x, z) > 0 \textit{ for some } z \in \tilde{s}_{l,\alpha}\} & \textit{if } \alpha = 0. \end{cases} \tag{3.2}$$

Let $g : C(\mathbf{R}) \mapsto \mathbf{R}$ be any additive map, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S). \tag{3.3}$$

Adapting this map g for a weighting function (see Fortemps and Roubens [4]), for a fuzzy stopping time $\hat{\sigma} \in \Sigma(\hat{\mathcal{F}})$ and an initial fuzzy state $\tilde{s} \in \hat{\mathcal{F}}$, the scalarization of the total fuzzy reward is given by

$$\begin{aligned} G(\tilde{s}, \hat{\sigma}) \; &:= \int_0^1 g\left(\varphi(\tilde{s}, \hat{\sigma}_\alpha)\right) \, d\alpha \\ &= \int_0^1 g\left(\sum_{t=1}^{\hat{\sigma}_\alpha} R_\alpha(\tilde{s}_{t,\alpha}) \right) d\alpha, \end{aligned} \tag{3.4}$$

where $\hat{\sigma}_\alpha$ means $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \min\{t \geq 1 \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ for simplicity. Since $\varphi(\tilde{s}, \hat{\sigma}_\alpha) \in C(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \hat{\sigma}_\alpha)_\alpha)$ is left-continuous in $\alpha \in (0, 1]$, therefore the right-hand integral of (3.14) is well-defined. For a given $\mathcal{F}'(\subset \mathcal{F})$, our objective is to maximize (3.14) over all fuzzy stopping times $\hat{\sigma} \in \Sigma(\mathcal{F}')$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}'$.

Definition 4. A fuzzy stopping time $\hat{\sigma}^*$ with $\tilde{s} \in \mathcal{F}'$ is called an $\tilde{s}$-*optimal* if

$$G(\tilde{s}, \hat{\sigma}) \leq G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}').$$

If $\hat{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}'$, then $\hat{\sigma}^*$ is called *optimal* in $\mathcal{F}'$.

Now we will seek an $\tilde{s}$-optimal or an optimal fuzzy stopping time by using the results in the previous sections. For each $\alpha \in [0, 1]$, let

$$\mathcal{K}_\alpha(g) := \{c \in C(S) \mid g(R_\alpha(Q_\alpha(c))) \leq 0\}. \tag{3.5}$$

Hence we need the following Assumptions 1 and 2, which are assumed to hold henceforth.

Assumption 1 (Closedness). For all $\alpha \in [0, 1]$, $Q_\alpha(\mathcal{K}_\alpha(g)) \subset \mathcal{K}_\alpha(g)$.

By (2.5), we define a sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$ by

$$\mathcal{K}_\alpha^t(g) := Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) \quad \text{for } t \geq 1. \tag{3.6}$$

Assumption 2. For all $\alpha \in [0, 1]$, $C(S) = \bigcup_{t=1}^\infty \mathcal{K}_\alpha^t(g)$.

Using the sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$ given in (3.16), we define $\hat{\sigma}_\alpha$, $\hat{\mathcal{F}}$, $\hat{\sigma}$ and $\hat{\sigma}(\tilde{s}, \cdot)_\alpha$ by (2.7) – (2.9). Then, from Theorems 2.1 and 2.2, $\hat{\sigma}$ is a monotone fuzzy stopping time on $\hat{\mathcal{F}}$. The following theorem will be proved by applying the idea of the monotone policy ([3, 8, 14]) for stochastic stopping problems.

Theorem 3.1 *Under Assumptions 1 and 2, $\hat{\sigma}$ is an optimal monotone fuzzy stopping timein $\hat{\mathcal{F}}$.*

Proof. Firstly, we will consider a deterministic stopping problem which maximizes the reward of a weighting function $g(\varphi(\tilde{s},t)_\alpha)$ over $t \geq 1$. Since g is additive, $g(\varphi(\tilde{s},t)_\alpha) = \sum_{l=1}^{t} g(R_\alpha(\tilde{s}_{l,\alpha}))$ holds. Therefore $g(\varphi(\tilde{s},t)_\alpha) \geq g(\varphi(\tilde{s},t+1)_\alpha)$ if and only if $\tilde{s}_{t,\alpha} \in K_\alpha(g)$. By Assumption 1, $\tilde{s}_{t,\alpha} \in K_\alpha(g)$ implies $g(\varphi(\tilde{s},t)_\alpha) \geq g(\varphi(\tilde{s},l)_\alpha)$ for all $l \geq t$. Together with (3.5), we obtain

$$g(\varphi(\tilde{s},\hat{\sigma}(\tilde{s},\cdot)_\alpha))) \geq g(\varphi(\tilde{s},\tilde{\sigma}(\tilde{s},\cdot)_\alpha)))$$

for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ and $\alpha \in [0,1]$. This implies that $G(\tilde{s},\hat{\sigma}) \geq G(\tilde{s},\tilde{\sigma})$ for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ by using (3.14). This completes the proof. □

4 A numerical example

An example is given to illustrate the previous results of fuzzy stopping problem in this section.

Let $S := [0,1]$. The fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x,y) = \begin{cases} 1 & \text{if } y = \beta x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{r}(x,z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in [0,1]$ and $z \in \mathbf{R}$, where $\lambda > 0$ is an observation cost and $0 < \beta < 1$. Then, Q_α and R_α defined by (1.3) and (3.12) are independent of α and are calculated as follows:

$$Q_\alpha([a,b]) = [\beta a, \beta b] \quad \text{and} \quad R_\alpha([a,b]) = [a-\lambda, b-\lambda]$$

for $0 \leq a \leq b \leq 1$.

Let $g([a,b]) := (a+2b)/3$ for $0 \leq a \leq b \leq 1$, which is additive. Then, $\mathcal{K}_\alpha := \mathcal{K}_\alpha(g)$ is given as

$$\mathcal{K}_\alpha = \mathcal{K}_\alpha(g) = \{[a,b] \in C(S) \mid a + 2b \leq 3\lambda/\beta\},$$

So $\mathcal{K}_\alpha^t = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha) = \{[a,b] \in C(S) \mid a + 2b \leq 3\lambda/\beta^t\}$. Since $\mathcal{K}_\alpha^t$ is independent of α, we see that $Q_\alpha(\mathcal{K}_\alpha) = \{\beta[a,b] \mid [a,b] \in \mathcal{K}_\alpha\}$ and $\bigcup_{t=1}^{\infty} \mathcal{K}_\alpha^t = C(S)$. Thus Assumptions 1 and 2 in Section 3 are fulfilled in this example.

Let the initial fuzzy state be

$$\tilde{s}(x) := \max\{1 - |8x-4|, 0\} \quad \text{for } x \in [0,1].$$

For the stopping time $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ given in (2.7), we easily obtain that $\tilde{s}_\alpha = [(3+\alpha)/8, (5-\alpha)/8]$ and $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid 13 - \alpha \leq 24\lambda\beta^{-t}\}$. Thus, as $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$, we have $\tilde{s} \in \hat{\mathcal{F}}$. Since $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \in \mathcal{K}^t(g)$ means $13 - \alpha \leq 24\lambda\beta^{-t}$, we obtain

$$\hat{\sigma}(\tilde{s},t) = \min\{1, \max\{(13-24\lambda)\beta^{-t}, 0\}\}.$$

The numerical value of $\hat{\sigma}$ is given in Table 1.

Table 1. An $\tilde{s}$-optimal fuzzy stopping time (λ = 0.48, β =0.98).

t	1	2	3	4	5	6	7	$\cdots$
$\hat{\sigma}(\tilde{s},t)$	1	1	.7603	.5108	.2552	.00	.00	$\cdots$

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] Chen-wei Xu, On convexity of fuzzy sets and fuzzy relations, *Information Science* **59** (1992) 92-102.

[3] Y.S.Chow, H.Robbins and D.Siegmund, *The theory of optimal stopping: Great expectations*, Houghton Mifflin Company, New York, 1971.

[4] P.Fortemps and M.Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[5] J.Kacprzyk, Control of a non-fuzzy system in a fuzzy environment with fuzzy termination time, *System Sciences* **3** (1977) 325-341.

[6] J.Kacprzyk, Decision making in a fuzzy environment with fuzzy termination time, *Fuzzy Sets and Systems* **1** (1978) 169-179.

[7] J.Kacprzyk and A.O.Esogbue, Fuzzy dynamic programming: Main developments and applications, *Fuzzy Sets and Systems* **81** (1996) 31-48.

[8] Y.Kadota, M.Kurano and M.Yasuda, Utility-Optimal Stopping in a Denumerable Markov Chain, *Bull. Infor. Cyber. Res. Ass. Stat. Sci., Kyushu University* **28** (1996) 15-21.

[9] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[10] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Markov-type fuzzy decision processes with a discounted reward on a closed interval, *European Journal of Operational Research* **92** (1996) 649-662.

[11] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, An approach to stopping problems of a dynamic fuzzy system, *preprint.*

[12] V.Novák, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[13] M.L.Puri and D.A.Ralescu, The concept of normality for fuzzy random variables, *Ann. Prob.* **13** (1985) 1373-1379.

[14] S.M.Ross, *Applied Probability Models with Optimization Applications*, Holden-Day, 1970.

[15] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.

[16] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.