A monotone fuzzy stopping time in dynamic fuzzy systems

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Abstract

This paper is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times, which is called as a monotone fuzzy stopping time is introduced. This notion is well known in a stochastic process. We have constructed it by subsets of α-sets of fuzzy states under appropriate assumptions. The aim is to consider the optimization of a stopping problem with an additive weighting function in the class of monotone fuzzy stopping times.

Keywords: fuzzy stopping time; monotone property; α-cuts of fuzzy sets; stopping problem; optimality.

1 Introduction and notations

The stopping time with fuzziness, which is called a fuzzy stopping time, is discussed by our previous paper[11] where we have considered the optimization of the corresponding fuzzy stopping problem by the constructive method. However this is not the first one and there are papers as Kacprzyk[5, 6] which are published at the beginning of the fuzzy theory([7]). Because it imposes Dynamic Programming method into the multistage decision-making, it is a straight line toward to consider a stopping problem which is incorporated with fuzziness([1]). Alternatively, it is well known that a class of stopping times which has a monotone property is useful for various application problems. Because it's simple to understand and easy to calculate. Refer to Chow, Robbins and Siegmund[3] and Ross[14]. In this paper, we introduce a new class of fuzzy stopping times which is called monotone and apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards. As related works, refer to [15].

In the remainder of this section, a fuzzy stopping time in a fuzzy dynamic system is defined and then a new class of fuzzy stopping times, which we call it as monotone fuzzy stopping times, is introduced in Section 2. Its construction is also discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, a example is given to illustrate the results.

Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions defined on a convex compact subsets of some Banach space. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. The detail of its definition is omitted here.

Let E be a given convex compact subsets of some Banach space, a convex fuzzy set can be defined. If $\mathcal{F}(E)$ is denoted the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are assumed to be a upper semi-continuous and it have compact supports with the normality condition: $\sup_{x\in E}\tilde{u}(x) = 1$. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{u}$ is denoted as $\tilde{u}_\alpha$. If $\mathcal{C}(E)$ means the collection of all compact convex subsets of E, then clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0,1]$. Let $\mathbf{R}$ be the set of all real numbers and $\mathcal{C}(\mathbf{R})$ be the set of all bounded closed intervals in $\mathbf{R}$. The elements of $\mathcal{F}(\mathbf{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are defined as will be seen in the next lemma. See Puri and Ralescu [13]) for the details. The following results are known so proofs are omitted.

Lemma 1.1 (Chen-wei Xu [2], Kurano et al.[11]).

(i) *For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda\tilde{m} \in \mathcal{F}(\mathbf{R})$.*

(ii) *Let E_1 and E_2 be convex compact subsets. If $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$ satisfy $\tilde{p}(x,\cdot) \in \mathcal{F}(E_2)$ for $x \in E_1$, then $\sup_{x\in E_1}\{\tilde{u}(x) \wedge \tilde{p}(x,\cdot)\} \in \mathcal{F}(E_2)$.*

Now we will formulate the dynamic fuzzy system.

Definition 1 (Kurano et al.[9]). The pair of $(S, \tilde{q})$ is called a dynamic fuzzy system if the following conditions (i) and (ii) are satisfied:

(i) The state space S is a convex compact subset of some Banach space. In generally, the state of the system is simply called as a fuzzy state and it belongs to an element of $\mathcal{F}(S)$.

(ii) The law of motion $\tilde{q}$ for the system is a time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$, and we assume $\tilde{q} \in \mathcal{F}(S \times S)$ and $\tilde{q}(x,\cdot) \in \mathcal{F}(S)$ for $x \in S$.

If the dynamic fuzzy system $(S, \tilde{q})$ is given, then we consider a sequential change of states as follows. Firstly, let a fuzzy state $\tilde{s} \in \mathcal{F}(S)$ the state is moved to a new fuzzy state $Q(\tilde{s})$ after unit time have passed, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined from the law of motion $\tilde{q}$ by

$$Q(\tilde{s})(y) := \sup_{x\in S}\{\tilde{s}(x) \wedge \tilde{q}(x,y)\} \quad (y \in S). \tag{1.1}$$

Note that the map Q is well-defined by Lemma 1.1.

Explicitly, for the dynamic fuzzy system $(S, \tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a sequence of fuzzy states means $\{\tilde{s}_t\}_{t=1}^{\infty}$ defined by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \tag{1.2}$$

A fuzzy stopping time for this sequence $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in the next section. In order to define a fuzzy stopping time, we need the following preliminaries. Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ ($\alpha \in [0,1]$) are defined as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \text{cl}\{y \in S \mid \tilde{q}(x,y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0 \end{cases} \tag{1.3}$$

where cl means a closure of set. From the assumption on $\tilde{q}$, the maps Q_α is well-defined. The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α-cuts of fuzzy state, $Q_\alpha(\tilde{s})$, is specified using the maps Q_α of 1.3.

Lemma 1.2 (Kurano et al. [9, 10]). *For any $\alpha \in [0,1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:*

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $\tilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha) \quad (t \geq 1)$,

where $\tilde{s}_\alpha$ and $\tilde{s}_{t,\alpha}$ are the α-cuts of fuzzy state $\tilde{s}$, $(\tilde{s}_t)_\alpha$ respectively and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.2) with the initial state $\tilde{s}_1 = \tilde{s}$.

2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here. And a new class of fuzzy stopping times is introduced, which is constructed through subsets of α-cuts of fuzzy states. For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$. Let $\mathbf{N} = \{1, 2, \cdots\}$ and $\mathcal{F}'$ a subset of $\mathcal{F}$.

Definition 2 (Kurano et al. [11]). A *fuzzy stopping time* on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma}$: $\mathcal{F}' \times \mathbf{N} \mapsto [0,1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s}, t)$ is non-increasing in t and there exists a natural number $t(\tilde{s}) \in \mathbf{N}$ with $\tilde{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t(\tilde{s})$.

We note here that '$\tilde{\sigma} = 0$' represents 'stop' and '$\tilde{\sigma} = 1$' represents 'continue' in the grade of membership (Kurano et al. [11]). Between the two decisions, the intermediate value '$0 < \tilde{\sigma} < 1$' is a notion of 'fuzzy stopping'. A fuzzy stopping time $\tilde{\sigma}(\tilde{s}, t)$ means the degree of 'continue' at time t starting from a fuzzy state $\tilde{s}$. The set of all fuzzy stopping times on $\mathcal{F}'$ is denoted by $\Sigma(\mathcal{F}')$.

Definition 3. A fuzzy stopping time $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ is called *monotone* if there exist a mapping $\delta : \mathcal{F}' \mapsto [0,1]$ satisfying

(i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and

(ii) $\tilde{\sigma}(\tilde{s}, t) = \bigwedge_{l=1}^t \delta(\tilde{s}_t)$ for all $\tilde{s} \in \mathcal{F}'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.2) with $\tilde{s}_1 = \tilde{s}$.

The real valued mapping on a fuzzy state δ in the above is called a *support* of $\tilde{\sigma}$. The definition is motivated from the monotone stopping time in the usual sense([3]). It is known that it has a natural and good property, because it is simple and easy to calculate the optimal stopping time in actual optimization problems. The degree of monotone fuzzy stopping times is given by only the fuzzy state at present time t. Therefore, in stopping problems, the criterion is reduced whether the fuzzy state at present time t belongs to the optimal stopping region or not.

We now construct a monotone fuzzy stopping times which is to be the subject of this paper. For this purpose of the construction, we assume the following conditions.

Condition A1. For each $\alpha \in [0,1]$, there exists a non-empty subset $\mathcal{K}_\alpha$ of $\mathcal{C}(S)$ satisfying

$$Q_\alpha(\mathcal{K}_\alpha) \subset \mathcal{K}_\alpha. \tag{2.1}$$

Using this subset $\mathcal{K}_\alpha$, we define a sequence of subsets $\{\mathcal{K}_\alpha^t\}_{t=1}^\infty$ inductively by

$$\begin{aligned} \mathcal{K}_\alpha^1 &:= \mathcal{K}_\alpha \\ \mathcal{K}_\alpha^t &:= \{c \in \mathcal{C}(S) \mid Q_\alpha(c) \in \mathcal{K}_\alpha^{t-1}\} \quad (t \geq 2). \end{aligned} \tag{2.2}$$

Clearly, $\mathcal{K}_\alpha^t = Q_\alpha^{-1}(\mathcal{K}_\alpha^{t-1}) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha)$. Also, it holds from 2.1 that $\mathcal{K}_\alpha^t \subset \mathcal{K}_\alpha^{t+1}\,(t \geq 1)$.

To simplify our discussion, we assume the following condition holds henceforth.

Condition A2. For all $\alpha \in [0,1]$, it is assumed that

$$\mathcal{C}(S) = \bigcup_{t=1}^{\infty} \mathcal{K}_\alpha^t. \tag{2.3}$$

For $c \in \mathcal{C}(S)$ and $\alpha \in [0,1]$, define $\hat{\sigma}_\alpha(c)$ by

$$\hat{\sigma}_\alpha(c) := \min\{t \geq 1 \mid c \in \mathcal{K}_\alpha^t\}. \tag{2.4}$$

That is, it is the first entry time of $c \in \mathcal{C}(S)$ with the grade α. We define a restricted class $\hat{\mathcal{F}} \subset \mathcal{F}$ by

$$\hat{\mathcal{F}} := \{\tilde{s} \in \mathcal{F} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0,1]\}.$$

Using the class $\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0,1]\}$, for the restricted element $\tilde{s} \in \hat{\mathcal{F}}$, let us construct

$$\hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(t)\} \quad (t \geq 1), \tag{2.5}$$

where 1_{D_α} is the indicator of a set $D_\alpha = \{t \in \mathbf{N} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique of constructing a fuzzy number from a family of level sets. Then we obtain the following theorem.

Theorem 2.1 *Under Condition A1 and A2, it holds that*

(i) $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \in \mathbf{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ *for* $\tilde{s} \in \hat{\mathcal{F}}, \alpha \in [0,1]$, *and*

(ii) $\hat{\sigma}$ *of (2.5) is a fuzzy stopping time on* $\hat{\mathcal{F}}$.

Proof. By the definition, we have that $\hat{\sigma}(\tilde{s},t) < \alpha$ is equivalent to $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le t$ for all $t \ge 1$. This fact shows (i). From Condition A2, there exists $t^* \in \mathbf{N}$ with $\tilde{s}_0 \in \mathcal{K}_0^{t^*}$. So, $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le \hat{\sigma}_0(\tilde{s}_0) \le t^*$ for all $\alpha \in [0,1]$, which implies that $\hat{\sigma}(\tilde{s},t) = 0$ for all $t \ge t^*$ by the definition of $\hat{\mathcal{F}}$. Since $\hat{\sigma}(\tilde{s},t+1) \le \hat{\sigma}(\tilde{s},t)$ holds clearly for $t \ge 1$ from the definition (2.5), we also obtain (ii). □

In order to show the monotone property of $\hat{\sigma}$, we need the following lemma.

Lemma 2.1 *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then*

(i) $\hat{\sigma}(\tilde{s},t) = \alpha$ *if and only if, for any* $\epsilon > 0$,

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^t \text{ and } \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^t;$$

(ii) $\tilde{s}_t \in \hat{\mathcal{F}}$ *$(t \ge 1)$.*

Proof. By (2.5), $\hat{\sigma}(\tilde{s},t) = \sup\{\alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. So, (i) follows from (2.4). From 1.2(ii), for $l \ge 1$, $\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_{l,\alpha}) = \hat{\sigma}_\alpha(Q_\alpha^{l-1}(\tilde{s}_\alpha))$. By (2.2) and (2.4),

$$\begin{aligned} \hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) &= \min\{t \ge 1 \mid Q_\alpha^{l-1}(\tilde{s}_\alpha) \in \mathcal{K}_\alpha^t\} \\ &= \min\{t \ge 1 \mid \tilde{s}_\alpha \in \mathcal{K}_\alpha^{t+l-1}\} \\ &= \max\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l-1), 1\}, \end{aligned}$$

and it is non-increasing in $\alpha \in [0,1]$ since $\tilde{s} \in \hat{\mathcal{F}}$. Therefore we obtain (ii). □

Theorem 2.2 *Let $\tilde{s} \in \hat{\mathcal{F}}$ be given and Condition A1 and A2 are assumed. Then, $\hat{\sigma} = \hat{\sigma}(\tilde{s},t)$, $t \ge 1$, is a monotone fuzzy stopping time with the initial state $\tilde{s}$.*

Proof. Let $\{\tilde{s}_t\}_{t=1}^\infty$ be defined by (1.2) with $\tilde{s}_1 = \tilde{s}$. First, we will prove that

$$\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) \quad \text{for } t,r \in \mathbf{N}. \tag{2.6}$$

Note that $\hat{\sigma}(\tilde{s}_{t+1},r)$ is well-defined from Lemma 2.1(ii). Let $\alpha = \hat{\sigma}(\tilde{s},t+r)$. From Lemma 2.1(i), we have

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^{t+r} \quad \text{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^{t+r} \quad \text{for any } \epsilon > 0.$$

Noting $Q_\alpha^t(\mathcal{K}_\alpha^l) = \mathcal{K}_\alpha^{l-t}$ $(1 \le t < l)$ and Lemma 1.2(ii), we obtain

$$\tilde{s}_{t+1,\alpha+\epsilon} = Q_{\alpha+\epsilon}^t(\tilde{s}_{\alpha+\epsilon}) \in Q_{\alpha+\epsilon}^t(\mathcal{K}_{\alpha+\epsilon}^{t+r}) = \mathcal{K}_{\alpha+\epsilon}^r$$

and

$$\tilde{s}_{t+1,\alpha-\epsilon} = Q_{\alpha-\epsilon}^t(\tilde{s}_{\alpha-\epsilon}) \notin Q_{\alpha-\epsilon}^t(\mathcal{K}_{\alpha-\epsilon}^{t+r}) = \mathcal{K}_{\alpha-\epsilon}^r.$$

Therefore, we get $\hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$ from Lemma 2.1(i). Namely, $\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s}_{t+1},r)$. Since $\hat{\sigma}(\tilde{s},t+r) \le \hat{\sigma}(\tilde{s},t)$ from Theorem 2.1(ii), we obtain $\hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$, and so (2.6) holds.

Next, we put $\delta(\tilde{s}) = \hat{\sigma}(\tilde{s}, 1)$ for $\tilde{s} \in \hat{\mathcal{F}}$. From (2.6), we get

$$\begin{aligned}
\hat{\sigma}(\tilde{s}, t) &= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, t-1) \\
&= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, 1) \wedge \hat{\sigma}(\tilde{s}_3, t-2) \\
&= \cdots \\
&= \bigwedge_{l=1}^{t} \hat{\sigma}(\tilde{s}_l, 1) \\
&= \bigwedge_{l=1}^{t} \delta(\tilde{s}_l) \quad \text{for } t \in \mathbf{N}.
\end{aligned}$$

Since we also have $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$ from Theorem 2.1(ii), $\hat{\sigma}$ is a monotone fuzzy stopping time with $\tilde{s}$. □

3 Fuzzy stopping problems

In this section, applying the results in the previous section, we obtain the optimal fuzzy stopping time for a fuzzy dynamic system with fuzzy rewards([10]) when the weighting function is additive.

Firstly, we will formulate the stopping problem to be considered here. Let $\tilde{r} : S \times \mathbf{R} \mapsto [0, 1]$ be a fuzzy relation satisfying $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$ and $\tilde{r}(x, \cdot) \in \mathcal{F}(\mathbf{R})$ for $x \in S$. If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}$, a fuzzy reward is earned:

$$R(\tilde{s})(z) := \sup_{x \in S} \{\tilde{s}(x) \wedge \tilde{r}(x, z)\}, \quad z \in \mathbf{R}.$$

Then we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in (1.2) with the initial fuzzy state $\tilde{s}_1 = \tilde{s}$. Because of time-parameter index, let the the sum of fuzzy rewards from $l = 1$ to $t - 1$ as

$$\varphi(\tilde{s}, t) := \sum_{l=1}^{t-1} R(\tilde{s}_l) \quad \text{for } t \in \mathbf{N}. \tag{3.1}$$

Note that $\sum$ of (3.1) designates the summation between fuzzy numbers. Details refer to Puri and Ralescu([13]) and Kurano et al.([10]). We need the following lemma, which is proved in Kurano et al.([9, 10]).

Lemma 3.1 (Kurano et al.[9, 10]). *For $t \in \mathbf{N}$ and $\alpha \geq 0$,*

$$\varphi(\tilde{s}, t)_\alpha = \sum_{l=1}^{t-1} R_\alpha(\tilde{s}_{l,\alpha})$$

holds, where

$$R_\alpha(\tilde{s}_{l,\alpha}) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x, z) \geq \alpha \textit{ for some } z \in \tilde{s}_{l,\alpha}\} & \textit{for } \alpha > 0 \\ \mathrm{cl}\{z \in \mathbf{R} \mid \tilde{r}(x, z) > 0 \textit{ for some } z \in \tilde{s}_{l,\alpha}\} & \textit{for } \alpha = 0. \end{cases} \tag{3.2}$$

Let $g : C(\mathbf{R}) \mapsto \mathbf{R}$ be any additive map, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S). \tag{3.3}$$

Adapting this g for a weighting function(see [4]), when a fuzzy stopping time $\hat{\sigma} \in \Sigma(\hat{\mathcal{F}})$ and an initial fuzzy state $\tilde{s} \in \hat{\mathcal{F}}$, the scalarization of the total fuzzy reward is given by

$$\begin{aligned} G(\tilde{s}, \hat{\sigma}) &= \int_0^1 g\left(\varphi(\tilde{s}, \hat{\sigma}_\alpha)\right) d\alpha \\ &= \int_0^1 g\left(\sum_{t=1}^{\hat{\sigma}_\alpha - 1} R_\alpha(\tilde{s}_{t,\alpha})\right) d\alpha, \end{aligned} \tag{3.4}$$

where $\sum_{t=1}^{0} R_\alpha(\tilde{s}_{t,\alpha}) = \emptyset$ and $\hat{\sigma}_\alpha$ means $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \min\{t \in \mathbf{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ for simplicity. Since $\varphi(\tilde{s}, \hat{\sigma}_\alpha) \in C(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \hat{\sigma}_\alpha)_\alpha)$ is left-continuous in $\alpha \in (0, 1]$, therefore the right-hand integral of (3.4) is well-defined. For a given $\mathcal{F}' \subset \mathcal{F}$, our objective is to maximize (3.4) over all fuzzy stopping times $\hat{\sigma} \in \Sigma(\mathcal{F}')$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}'$.

Definition 4. A fuzzy stopping time $\hat{\sigma}^*$ with $\tilde{s} \in \mathcal{F}'$ is called an $\tilde{s}$-*optimal* if

$$G(\tilde{s}, \hat{\sigma}) \le G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}').$$

If $\hat{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}'$, $\hat{\sigma}^*$ is called *optimal* in $\mathcal{F}'$.

Now we will seek an $\tilde{s}$-optimal or an optimal fuzzy stopping time by using the results in the previous sections. For each $\alpha \in [0, 1]$, let

$$\mathcal{K}_\alpha(g) := \{c \in C(S) \mid g(R_\alpha(Q_\alpha(c))) \le 0\}. \tag{3.5}$$

Here we need the following Assumptions B1 and B2, which are assumed to hold henceforth.

Assumption B1 (Closedness). For all $\alpha \in [0, 1]$, $Q_\alpha(\mathcal{K}_\alpha(g)) \subset \mathcal{K}_\alpha(g) \quad$ for all $\alpha \in [0, 1]$

By (2.2), we define the sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$, that is,

$$\mathcal{K}_\alpha^t(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) \quad \text{for } t \ge 1. \tag{3.6}$$

Assumption B2. For all $\alpha \in [0, 1]$, $C(S) = \bigcup_{t=1}^\infty \mathcal{K}_\alpha^t(g)$.

Using the sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$ given in (3.6), we define $\hat{\sigma}_\alpha$, $\hat{\mathcal{F}}$, $\hat{\sigma}$ and $\hat{\sigma}(\tilde{s}, \cdot)_\alpha$, respectively, by (2.4) and (2.5). Then, from Theorems 2.1 and 2.2, $\hat{\sigma}$ is a monotone fuzzy stopping time on $\hat{\mathcal{F}}$.

The following theorem will be proved by applying the idea of the monotone policy([3, 8, 14]) for stochastic stopping problems.

Theorem 3.1 *Under Assumptions B1 and B2, $\hat{\sigma}$ is optimal in $\hat{\mathcal{F}}$.*

Proof. Firstly, we will consider a deterministic stopping problem which maximizes the reward of a weighting function $g(\varphi(\tilde{s},t)_\alpha)$ over $t \ge 1$. Since g is additive, $g(\varphi(\tilde{s},t)_\alpha) = \sum_{l=1}^{t-1} g(R_\alpha(\tilde{s}_{l,\alpha}))$ holds. Therefore $g(\varphi(\tilde{s},t)_\alpha) \ge g(\varphi(\tilde{s},t+1)_\alpha)$ if and only if $\tilde{s}_{t,\alpha} \in K_\alpha(g)$. By Assumption B1, $\tilde{s}_{t,\alpha} \in K_\alpha(g)$ implies $g(\varphi(\tilde{s},t)_\alpha) \ge g(\varphi(\tilde{s},l)_\alpha)$ for all $l \ge t$. Thus, since $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \hat{\sigma}(\tilde{s},\cdot)_\alpha$ by Theorem 2.1, we can show

$$g(\varphi(\tilde{s},\hat{\sigma}(\tilde{s},\cdot)_\alpha))) \ge g(\varphi(\tilde{s},\tilde{\sigma}(\tilde{s},\cdot)_\alpha)))$$

for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ and $\alpha \in [0,1]$. This implies that $G(\tilde{s},\hat{\sigma}) \ge G(\tilde{s},\tilde{\sigma})$ for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ by using (3.4). This completes the proof. □

4 A numerical example

An example is given to illustrate the previous results of fuzzy stopping problem in this section.

Let $S := [0,1]$. The fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x,y) = \begin{cases} 1 & \text{if } y = \beta x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{r}(x,z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is an observation cost and $0 < \beta < 1$ for $x, y, z \in [0,1]$ and $z \in \mathbf{R}$. Then, Q_α and R_α defined by (1.3) and (3.2) are independent of α and are calculated as follows:

$$Q_\alpha([a,b]) = [\beta a,\, \beta b] \quad \text{and} \quad R_\alpha([a,b]) = [a-\lambda, b-\lambda]$$

for $0 \le a \le b \le 1$.

Let $g([a,b]) := (a+2b)/3$ for $0 \le a \le b \le 1$, which is additive. Then, $\mathcal{K}_{al} := \mathcal{K}_\alpha(g)$ is given as

$$\mathcal{K}_{al} := \mathcal{K}_\alpha(g) = \{[a,b] \in C(S) \mid a + 2b \le 3\lambda/\beta\},$$

So $\mathcal{K}_\alpha^t = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha) = \{[a,b] \in C(S) \mid a + 2b \le 3\lambda/\beta^t\}$. Since $\mathcal{K}_\alpha^t$ is independent of α, we see that $Q_\alpha(\mathcal{K}_\alpha) = \{\beta[a,b] \mid [a,b] \in \mathcal{K}_\alpha\}$ and $\bigcup_{t=1}^\infty \mathcal{K}_\alpha^t = C(S)$. Thus Assumptions B1 and B2 in Section 3 are satisfied in this example.

Let the initial fuzzy state be

$$\tilde{s}(x) := \max\{1 - |8x-4|, 0\} \quad \text{for } x \in [0,1].$$

For the stopping time $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ given in (2.4), we easily obtain that $\tilde{s}_\alpha = [(3+\alpha)/8, (5-\alpha)/8]$ and $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \ge 1 \mid 13 - \alpha \le 24\lambda\beta^{-t}\}$. Thus, as $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$, we have $\tilde{s} \in \hat{\mathcal{F}}$.

Since $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \in \mathcal{K}^t(g)$ means $13 - \alpha \le 24\lambda\beta^{-t}$, then

$$\hat{\sigma}(\tilde{s},t) = \min\{1, \max\{(13 - 24\lambda)\beta^{-t}, 0\}\}.$$

The numerical value of $\hat{\sigma}$ is given in Table 1.

Table 1. An $\tilde{s}$-optimal fuzzy stopping time (λ = 0.48, β =0.98).

t	1	2	3	4	5	6	7	$\cdots$
$\hat{\sigma}(\tilde{s}, t)$	1	1	.7603	.5108	.2552	.00	.00	$\cdots$

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