An Approach to Stopping Problems of a Dynamic Fuzzy System

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Abstract

A stopping problem for a dynamic fuzzy system with fuzzy rewards is formulated, which is thought of as a natural fuzzification of non-fuzzy stopping problem for a determistic dynamic system. And, the validity of the approach by α-cuts of fuzzy sets will be discussed in constructing an optimal fuzzy stopping time. Also, a numerical example is given to illustrate the theoretical results.

Keywords: Fuzzy stopping problem; dynamic fuzzy system; α-cuts of fuzzy sets; optimal fuzzy stopping time.

1. Introduction and notations

In multistage decision-making models with fuzziness (Bellman and Zadeh [1]), the idea of a fuzzy stopping time has been introduced mainly by Kacprzyk [5, 6]. Recently an optimal stopping time which maximizes the measure of the truth of the fuzzy stopped system has been found by the method of dynamic programming in determistic or stochastic cases, whose theory and applications are surveyed by Kacprzyk and Esogbue [7].

In this paper, applying the idea of Kacprzyk [5, 6], we formulate a stopping problem for a dynamic fuzzy system ([9, 10]) with fuzzy rewards, which is thought of as a natural fuzzification of non-fuzzy stopping problems induced by determistic dynamic systems. And the validity of the approach by α-cuts of fuzzy sets will be discussed in constructing an optimal fuzzy stopping time. As a deeply related work, see Yoshida [15] in which Snell's optimal stopping for a Markov fuzzy process has been studied. In remainder of this section, we will give some notations, by which a fuzzy stopping problem is formulated in the following section.

Let E, E_1, E_2 be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. A fuzzy set $\tilde{u} : E \mapsto [0,1]$ is called convex if

$$\tilde{u}(\lambda x + (1-\lambda)y) \geq \tilde{u}(x) \wedge \tilde{u}(y), \quad x, y \in E,\ \lambda \in [0,1],$$

where $a \wedge b := \min\{a, b\}$ (c.f. Chen-wei Xu [2]). Also, a fuzzy relation $\tilde{h} : E_1 \times E_2 \mapsto [0,1]$ is called convex if

$$\tilde{h}(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \geq \tilde{h}(x_1, y_1) \wedge \tilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1$, $y_1, y_2 \in E_2$ and $\lambda \in [0,1]$. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{u}$ is defined by

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{u}_0 := \text{cl}\ \{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set.

Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x\in E} \tilde{u}(x) = 1$. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of E and by ρ_E the Hausdorff metric on $\mathcal{C}(E)$. Clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0,1]$.

Let $\mathbf{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbf{R})$ and $\mathcal{F}(\mathbf{R})$ are the set of all bounded closed intervals in $\mathbf{R}$ and all upper semi-continuous and convex fuzzy numbers on $\mathbf{R}$ with compact supports, respectively.

The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$,

$$(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbf{R}:\ x_1 + x_2 = x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\} \quad (x \in \mathbf{R}) \tag{1.1}$$

and

$$(\lambda \hat{m})(x) := \begin{cases} \hat{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbf{R}). \tag{1.2}$$

and, hence

$$(\tilde{m} + \tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \ (\alpha \in [0,1])$$

where $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A + \emptyset = \emptyset + A := A$ and $\lambda\emptyset := \emptyset$ for any non-empty closed intervals $A, B \in (\mathbf{R})$. We use the following lemma.

Lemma 1.1 (Chen-wei Xu [2]).

(i) *For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \tilde{n} \in \mathcal{F}(\mathbf{R})$.*

(ii) *For any $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$, it holds that $\sup_{x \in E_1} \{\tilde{u}(x) \wedge \tilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.*

We consider the fuzzy system([9, 10]) with fuzzy rewards, which is characterized by the elements $(S, \tilde{q}, \tilde{r})$ as follows:

Definition 1.

(i) The state space S is convex compact subsets of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by a element of $\mathcal{F}(S)$.

(ii) The law of motion and the fuzzy reward for the system are denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$ and $\tilde{r} : S \times \mathbf{R} \mapsto [0,1]$ respectively. We assume that $\tilde{q} \in \mathcal{F}(S \times S)$ and $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a fuzzy reward $R(\tilde{s})$ is earned and the state is moved to a new fuzzy state $Q(\tilde{s})$, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ and $R : \mathcal{F}(S) \mapsto \mathcal{F}(\mathbf{R})$ are defined by

$$Q(\tilde{s})(y) := \sup_{x \in S}\{\tilde{s}(x) \wedge \tilde{q}(x,y)\} \quad (y \in S) \tag{1.3}$$

and

$$R(\tilde{s})(z) := \sup_{x \in S}\{\tilde{s}(x) \wedge \tilde{r}(x,z)\} \quad (z \in \mathbf{R}). \tag{1.4}$$

Note that by Lemma 1.1 the maps Q and R are well-defined.

For the dynamic fuzzy system $(S, \tilde{q}, \tilde{r})$, if we give an initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \tag{1.5}$$

In the following section, a fuzzy stopping problem for $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$ is formulated.

2. A fuzzy stopping problem

For the sake of brevity, denote $\mathcal{F} := \mathcal{F}(S)$. The metric ρ on $\mathcal{F}$ is given as $\rho(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \rho_S(\tilde{u}_\alpha, \tilde{v}_\alpha)$ for $\tilde{u}, \tilde{v} \in \mathcal{F}$ (see Nanda [11]). Let $\mathcal{B}(\mathcal{F})$ be the set of Borel measurable subsets of $\mathcal{F}$ with respect to ρ. Putting by $\Omega_t := \mathcal{F}^t$ the t times product of $\mathcal{F}$ and by $\mathcal{B}_t := \mathcal{B}(\mathcal{F}^t)$ the set of Borel measurable subsets of $\mathcal{F}^t$ with a metric ρ^t on $\mathcal{F}^t$ defined by

$$\rho^t(\{\tilde{s}_l\}_{l=1}^t, \{\tilde{s}'_l\}_{l=1}^t) := \sum_{l=1}^{t} 2^{-(l-1)} \rho(\tilde{s}_l, \tilde{s}'_l) \quad \text{for } \{\tilde{s}_l\}_{l=1}^t, \{\tilde{s}'_l\}_{l=1}^t \in \mathcal{F}^t \tag{2.1}$$

for $1 \leq t \leq \infty$, we can interpret $\{\tilde{s}_t\}_{t=1}^{\infty} \in \Omega_\infty$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.5) with any given initial fuzzy state $\tilde{s}_1 = \tilde{s} \in \mathcal{F}$. Here, applying the idea of fuzzy termination time in Kacprzyk [5, 6], we will define a fuzzy stopping time. Let $\mathbf{N}$ be the set of all natural numbers.

Definition 2. A fuzzy stopping time is a fuzzy relation $\tilde{\sigma} : \Omega_\infty \times \mathbf{N} \mapsto [0,1]$ such that

(i) for each $t \geq 1$, $\tilde{\sigma}(\cdot, t)$ is $\mathcal{B}_t$-measurable, and

(ii) for each $\overline{\omega} \in \Omega_\infty$, $\tilde{\sigma}(\overline{\omega}, \cdot)$ is non-increasing and there exists $t_{\overline{\omega}} \in \mathbf{N}$ with $\tilde{\sigma}(\overline{\omega}, t) = 0$ for all $t \geq t_{\overline{\omega}}$.

In the grade of membership of stopping times, '0' and '1' represent 'stop' and 'continue' respectively. We denote by Σ the set of all fuzzy stopping times.

Lemma 2.1. *Let any $\tilde{\sigma} \in \Sigma$. Define a map $\tilde{\sigma}_\alpha : \Omega_\infty \mapsto \mathbf{N}$ by*

$$\tilde{\sigma}_\alpha(\overline{\omega}) = \min\{t \geq 1 \mid \tilde{\sigma}(\overline{\omega}, t) < \alpha\} \quad (\overline{\omega} \in \Omega_\infty) \quad \textit{for } \alpha \in (0,1]. \tag{2.2}$$

Then, we have:

(i) $\{\tilde{\sigma}_\alpha \leq t\} \in \mathcal{B}_t \quad (t \geq 1)$;

(ii) $\tilde{\sigma}_\alpha(\varpi) \le \tilde{\sigma}_{\alpha'}(\varpi) \quad (\varpi \in \Omega_\infty) \quad$ *if* $\alpha \ge \alpha'$;

(iii) $\lim_{\alpha' \uparrow \alpha} \tilde{\sigma}_{\alpha'}(\varpi) = \tilde{\sigma}_\alpha(\varpi) \quad (\varpi \in \Omega_\infty) \quad$ *if* $\alpha > 0$.

Proof. (i) is from $\{\tilde{\sigma}_\alpha > t\} = \{\varpi \in \Omega_\infty \mid \tilde{\sigma}(\varpi, t) \ge \alpha\} \in \mathcal{B}_t$. Also, (ii) and (iii) follow clearly. $\square$

In order to complete the description of an optimal fuzzy stopping problem, we will specify a function which measures the system's performance when a fuzzy stopping time $\tilde{\sigma} \in \Sigma$ and an initial fuzzy state $\tilde{s} \in \mathcal{F}$ are given. We define $\omega_\infty(\cdot) : \mathcal{F} \mapsto \Omega_\infty$ by

$$\omega_\infty(\tilde{s}) := \{\tilde{s}_t\}_{t=1}^\infty, \tag{2.3}$$

and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\tilde{s}_1 = \tilde{s}$. Let $g : \mathcal{C}(\mathbf{R}) \mapsto \mathbf{R}$ be a continuous and monotone function. Using this g as a weighting function (see Fortemps and Roubens [4]), the scalarization of the total fuzzy reward will be done by

$$G(\tilde{s}, \tilde{\sigma}) := \int_0^1 g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha)\, d\alpha = \int_0^1 g\left(\sum_{t=1}^{\tilde{\sigma}_\alpha - 1} R(\tilde{s}_t)_\alpha\right) d\alpha, \tag{2.4}$$

where $\tilde{\sigma}_\alpha := \tilde{\sigma}_\alpha(\omega_\infty(\tilde{s}))$ and $\varphi(\tilde{s}, \tilde{\sigma})_\alpha := \sum_{t=1}^{\tilde{\sigma}_\alpha - 1} R(\tilde{s}_t)_\alpha$ (We define $\sum_1^0 := \{0\}$). Note that $\varphi(\tilde{s}, \tilde{\sigma})_\alpha \in \mathcal{C}(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha)$ is left-continuous on $(0, 1]$, so that the right-hand integral of (2.4) is well-defined. Now, our objective is to maximize (2.4) over all fuzzy stopping times $\tilde{\sigma} \in \Sigma$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}$.

Definition 3. For $\tilde{s} \in \mathcal{F}$, a fuzzy stopping time $\tilde{\sigma}^*$ is called $\tilde{s}$-optimal if $G(\tilde{s}, \tilde{\sigma}) \le G(\tilde{s}, \tilde{\sigma}^*)$ for all $\tilde{\sigma} \in \Sigma$. If $\tilde{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}$, $\tilde{\sigma}^*$ is called optimal.

In the following section, the α-cuts of fuzzy stopping time will be investigated, whose results are used to construct an optimal fuzzy stopping time in Section 4.

3. α-cut of fuzzy stopping times

First, we establish several notations that will be used in the sequel. Associated with the fuzzy relations $\tilde{q}$ and $\tilde{r}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ and $R_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(\mathbf{R})$ $(\alpha \in [0, 1])$ are defined, respectively, as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x, y) \ge \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{y \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \tag{3.1}$$

and

$$R_\alpha(D) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x, z) \ge \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{z \in \mathbf{R} \mid \tilde{r}(x, z) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0. \end{cases} \tag{3.2}$$

By $\tilde{q} \in \mathcal{F}(S \times S)$ and $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$, the maps Q_α and R_α $(\alpha \in [0, 1])$ are well-defined. The iterates Q_α^t $(t \ge 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \ge 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α-cuts of $Q(\tilde{s})$ and $R(\tilde{s})$ defined by (1.3) and (1.4) are specified using the maps Q_α and R_α.

Lemma 3.1 ([9, 10]). *For any $\alpha \in [0,1]$ and $\tilde{s} \in \mathcal{F}$, we have:*

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $R(\tilde{s})_\alpha = R_\alpha(\tilde{s}_\alpha)$;

(iii) $\tilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha) \quad (t \geq 1)$,

where $\tilde{s}_{t,\alpha} := (\tilde{s}_t)_\alpha$ and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\tilde{s}_1 = \tilde{s}$.

Here we need the following assumption which is assumed to hold henceforth.

Assumption A (Lipschitz condition). There exists a constant $K > 0$ such that

$$\rho_S(Q_\alpha(D_1), Q_\alpha(D_2)) \leq K\rho_S(D_1, D_2) \tag{3.3}$$

for all $\alpha \in [0,1]$ and $D_1, D_2 \in \mathcal{C}(S)$.

Theorem 3.1. *Let a fuzzy stopping time $\tilde{\sigma} \in \Sigma$. Then, the map $\tilde{\sigma}'(\cdot,\cdot) : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ defined by $\tilde{\sigma}'(\tilde{s},t) := \tilde{\sigma}(\omega_\infty(\tilde{s}),t)$ $(\tilde{s} \in \mathcal{F}, t \in \mathbf{N})$ has the following properties (i) and (ii):*

(i) *$\tilde{\sigma}'(\cdot,t)$ is $\mathcal{B}(\mathcal{F})$-measurable for each $t \geq 1$.*

(ii) *For each $\tilde{s} \in \mathcal{F}$, $\tilde{\sigma}'(\tilde{s},\cdot)$ is non-increasing and there exists $t_{\tilde{s}} \in \mathbf{N}$ such that $\tilde{\sigma}'(\tilde{s},t) = 0$ for all $t \geq t_{\tilde{s}}$.*

Proof. For $t \geq 1$, we define a map $\omega_t : \mathcal{F} \mapsto \mathcal{F}^t$ by $\omega_t(\tilde{s}) := \{\tilde{s}_l\}_{l=1}^t$, where $\{\tilde{s}_l\}_{l=1}^\infty$ is defined by (1.5) with $\tilde{s}_1 = \tilde{s}$. For (i), it suffices to prove that ω_T is continuous for each $t \geq 1$, together with the measurability of $\tilde{\sigma}$. We will show only the case of $t = 2$, since the case of $t \geq 3$ is proved in the same manner. For $\tilde{s}, \tilde{s}' \in \mathcal{F}$, we have

$$\rho^2(\omega_2(\tilde{s}), \omega_2(\tilde{s}')) \leq \rho(\tilde{s}, \tilde{s}') + 2^{-1}\rho(Q(\tilde{s}), Q(\tilde{s}')) \leq (1 + K/2)\rho(\tilde{s}, \tilde{s}'),$$

from Lemma 3.1 and Assumption A. This shows the continuity of $\omega_2(\cdot)$. Also, (ii) follows from the definition of a fuzzy stopping time. $\square$

Observing (2.4) and the form of the objective function $G(\tilde{s}, \tilde{\sigma})$ for our stopping problem, we can confine ourselves to the class of fuzzy stopping times $\tilde{\sigma}'(\cdot,\cdot) : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ satisfying (i) and (ii) in Theorem 3.1, and so the class of such fuzzy stopping times will be denoted by Σ'. The following theorem is useful in constructing an optimal fuzzy time which is done in Section 4.

Theorem 3.2. *Suppose that, for each $\alpha \in [0,1]$, there exists a $\mathcal{B}(\mathcal{C}(S))$-measurable map $\sigma_\alpha : \mathcal{C}(S) \mapsto \mathbf{N}$. Using this family $\{\sigma_\alpha\}_{\alpha \in [0,1]}$, define the map $\tilde{\sigma} : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ by*

$$\tilde{\sigma}(\tilde{s},t) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{\{\sigma_\alpha(\tilde{s}_\alpha) > t\}}\}, \quad \tilde{s} \in \mathcal{F},\ t \geq 1. \tag{3.4}$$

Then, if for each $\tilde{s} \in \mathcal{F}$, $\sigma_\alpha(\tilde{s}_\alpha)$ is non-increasing and left-continuous in $\alpha \in [0,1]$, it holds that

(i) $\tilde{\sigma} \in \Sigma'$, *and*

(ii) $\sigma_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid \tilde{\sigma}(\tilde{s}, t) < \alpha\} \quad (\alpha \in (0,1])$.

Proof. If $\sigma_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$, the inequalities $\tilde{\sigma}(\tilde{s}, t) \geq \tilde{\sigma}(\tilde{s}, t+1)$ $(t \geq 1)$ follow from (3.4). Also, (3.4) implies that, for each $t \geq 1$ and $\alpha \in [0,1]$,

$$\{\tilde{s} \in \mathcal{F} \mid \tilde{\sigma}(\tilde{s}, t) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{\tilde{s} \in \mathcal{F} \mid \sigma_{\alpha-1/n}(\tilde{s}_{\alpha-1/n}) > t\}. \tag{3.5}$$

For a continuous map $\eta_\alpha : \mathcal{F} \mapsto \mathcal{C}(S)$ defined by $\eta_\alpha(\tilde{s}) = \tilde{s}_\alpha$ $(\tilde{s} \in \mathcal{F})$, we have

$$\{\tilde{s} \in \mathcal{F} \mid \sigma_\alpha(\tilde{s}_\alpha) > t\} = \eta_\alpha^{-1}(\{D \in \mathcal{C}(S) \mid \sigma_\alpha(D) \geq t+1\}),$$

so that $\{\tilde{s} \in \mathcal{F} \mid \tilde{\sigma}(\tilde{s}, t) \geq \alpha\} \in \mathcal{B}(\mathcal{F})$ follows from (3.5) and $\mathcal{B}(\mathcal{C}(S))$-measurability of σ_α. The above facts imply $\tilde{\sigma} \in \Sigma'$. Also, (ii) holds obviously. $\square$

4. Optimal fuzzy stopping times

In this section, we try to construct an optimal fuzzy stopping time, by applying an approach by α-cuts. Now, we define a non-fuzzy stopping problem specified by $\mathcal{C}(S)$, Q_α and R_α $(\alpha \in [0,1])$, associated with the fuzzy stopping problem considered in the preceding section. For each $\alpha \in [0,1]$ and any initial subset $c \in \mathcal{C}(S)$, a sequence $\{c_t\}_{t=1}^{\infty} \subset \mathcal{C}(S)$ is defined by

$$c_1 := c \quad \text{and} \quad c_{t+1} := Q_\alpha(c_t) \quad (t \geq 1). \tag{4.1}$$

Let

$$\Sigma_1 := \{\sigma : \mathcal{C}(S) \mapsto \mathbf{N} \mid \{\sigma = t\} \in \mathcal{B}(\mathcal{C}(S)) \text{ for each } t \geq 1\}. \tag{4.2}$$

Using this sequence $\{c_t\}_{t=1}^{\infty}$ given by (4.1) with $c_1 := c$, let

$$\varphi^\alpha(c, t) := \sum_{l=1}^{t-1} R_\alpha(c_l) \quad \text{for } c \in \mathcal{C}(S). \tag{4.3}$$

Note that $\varphi^\alpha(c, \sigma(c)) = \sum_{l=1}^{\sigma(c)-1} R_\alpha(Q_\alpha^{t-1}(c)) \in \mathcal{C}(\mathbf{R})$ for all $\sigma \in \Sigma_1$. The non-fuzzy stopping problem considered here is to maximize $g(\varphi^\alpha(c, \sigma(c)))$ over all $\sigma \in \Sigma_1$, where g is the weighting function given in Section 2. A map $\tau_\alpha \in \Sigma_1$ is called an α-optimal stopping time if

$$g(\varphi^\alpha(c, \tau_\alpha(c))) \geq g(\varphi^\alpha(c, \sigma(c))) \quad \text{for all } \sigma \in \Sigma_1.$$

In order to characterize α-optimal stopping times, let

$$\gamma_t^\alpha(c) := \sup_{\sigma \in \Sigma_t} g(\varphi^\alpha(c, \sigma(c))) \quad \text{for } t \geq 1 \text{ and } c \in \mathcal{C}(S), \tag{4.4}$$

where $\Sigma_t := \{\sigma \vee t \mid \sigma \in \Sigma_1\}$ $(t \geq 1)$. Then, the next lemma is given as deterministic versions of the results for stochastic stopping problems in Chow et al. [3].

Lemma 4.1 (c.f.[3,Theorems 4.1]) *It holds that*

$$\gamma_t^\alpha(c) = \max\{g(\varphi^\alpha(c,t)), \gamma_{t+1}^\alpha(c)\} \quad (t \geq 1, c \in \mathcal{C}(S)).$$

For $c \in \mathcal{C}(S)$, let $\tau_\alpha^*(c)$ be the first time $t \geq 1$ such that $g(\varphi^\alpha(c,t)) \geq \gamma_{t+1}^\alpha(c)$.

Lemma 4.2 (c.f.[3,Theorems 4.5]) *Let $\alpha \in [0,1]$. Suppose that $\lim_{t\to\infty} g(\varphi^\alpha(c,t)) = -\infty$ and $\sup_{t\geq 1} g(\varphi^\alpha(c,t)) < \infty$ for each $c \in \mathcal{C}(S)$. Then, τ_α^* is α-optimal and $\gamma_1^\alpha(\cdot) = g(\varphi^\alpha(\cdot, \tau_\alpha^*(\cdot)))$.*

Recently, Kadota et al.[8] discusses the validity of the one-step look ahead(OLA) policy for Markov chains with general utility, whose results are applicable to our problems. For the OLA stopping time, refer to Ross[14]

Assumption B (Closedness). For any $\alpha \in [0,1]$, if $(\varphi^\alpha(\tilde{s}_\alpha, t), \tilde{s}_{t,\alpha}) \in K^\alpha(g)$ for some t, then $(\varphi^\alpha(\tilde{s}_\alpha, t'), \tilde{s}_{t',\alpha}) \in K^\alpha(g)$ for all $t' > t$.

Lemma 4.3 *Suppose that Assumption B holds. Then, under Assumptions in (ii) of Lemma 4.2, α-optimal τ_α^* is represented as*

$$\tau_\alpha^*(c) = \min\{t \in \mathbf{N} \mid (\varphi^\alpha(c,t), c_t) \in K^\alpha(g)\}, \tag{4.5}$$

where $K^\alpha(g) := \{(h,c) \in \mathcal{C}(\mathbf{R}) \times \mathcal{C}(S) \mid g(h) \geq g(h + R_\alpha(c))\}$.
Proof. By deterministic version of Lemma 3.2 in Kadota et al.[8], we observe that if $(\varphi^\alpha(\tilde{s}_\alpha, t), \tilde{s}_{t,\alpha}) \in K^\alpha(g)$, then $\gamma_t^\alpha(\tilde{s}_\alpha) = g(\varphi^\alpha(\tilde{s}_\alpha, t))$. Thus from the definition of τ_α^* and Lemma 4.1 and 4.2, the above assersion (4.5) follows. □

For each $\alpha \in [0,1]$, applying the above lemmas, we can find an α-optimal stopping time τ_α^*. Assuming the existence of α-optimal stopping times for each $\alpha \in [0,1]$, let $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$ be the family of such stopping times. Here, we try to construct an optimal fuzzy stopping time from $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$. For this purpose, we need a regularity condition. In order to derive our main result, we introduce the following Assumption C.

Assumption C (Regularity). $\tau_\alpha^*(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$.

We can assume the left-continuity of the map $\alpha \mapsto \tau_\alpha^*(\tilde{s}_\alpha)$, by considering $\lim_{\alpha'\uparrow\alpha} \tau_{\alpha'}^*(\tilde{s}_{\alpha'})$ instead of $\tau_\alpha^*(\tilde{s}_\alpha)$. Define a map $\tilde{\tau}^* : \mathcal{F} \times \mathbf{N} \mapsto [0,1]$ by

$$\tilde{\tau}^*(\tilde{s}, t) := \sup_{\alpha\in[0,1]} \left\{\alpha \wedge 1_{\{\tau_\alpha^*(\tilde{s}_\alpha) > t\}}\right\} \tag{4.6}$$

for all $\tilde{s} \in \mathcal{F}$ and $t \in \mathbf{N}$.

Theorem 4.1. *Suppose Assumptions B and C hold. Then, $\tilde{\tau}^*$ defined by (4.6) is an $\tilde{s}$-optimal fuzzy stopping time. Further, if g is additive, i.e.*

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in \mathcal{C}(S), \tag{4.7}$$

then it holds that

$$\tilde{\tau}^*(\tilde{s}, t+r) = \tilde{\tau}^*(\tilde{s}, t) \wedge \tilde{\tau}^*(\tilde{s}_{t+1}, r) \quad \textit{for each } \tilde{s} \in \mathcal{F} \textit{ and } t, r \in \mathbf{N}. \tag{4.8}$$

Proof. From Assumption C, $\tau^*_\alpha(\tilde{s}_\alpha) \leq \tau^*_{\alpha'}(\tilde{s}_{\alpha'})$ if $\alpha \geq \alpha'$, so that $\tilde{\tau}^* \in \Sigma'$ follows from Theorem 3.2. For any $\tilde{s} \in \mathcal{F}$ and $\tilde{\sigma} \in \Sigma'$, from Lemmas 2.1 and 3.1 we have

$$\varphi(\tilde{s}, \tilde{\sigma})_\alpha = \sum_{t=1}^{\tilde{\sigma}_\alpha(\tilde{s}_\alpha)-1} R_\alpha(\tilde{s}_{t,\alpha}) = \sum_{t=1}^{\tilde{\sigma}_\alpha(\tilde{s}_\alpha)-1} R_\alpha(Q_\alpha^{t-1}(\tilde{s}_\alpha)). \tag{4.9}$$

Also, since $\sigma_\alpha \in \Sigma_1$, the optimality of τ^*_α implies by (4.9) that, for all $\alpha \in [0,1]$,

$$g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha) = g(\varphi^\alpha(\tilde{s}_\alpha, \sigma_\alpha(\tilde{s}_\alpha))) \leq g(\varphi^\alpha(\tilde{s}_\alpha, \tau^*_\alpha(\tilde{s}_\alpha))) = g(\varphi(\tilde{s}, \tilde{\tau}^*)_\alpha).$$

Therefore, we have

$$G(\tilde{s}, \tilde{\sigma}) = \int_0^1 g(\varphi(\tilde{s}, \tilde{\sigma})_\alpha)\, d\alpha \leq \int_0^1 g(\varphi(\tilde{s}, \tilde{\tau}^*)_\alpha)\, d\alpha = G(\tilde{s}, \tilde{\tau}^*).$$

This means that $\tilde{\tau}^*$ is $\tilde{s}$-optimal, as required.

If g is additive, we can easily check that, for all $t \geq 1$ and $\alpha \in [0,1]$, $t < \tau^*_\alpha(c)$ is equivalent to $g(R_\alpha(c_t)) > 0$. Therefore, for $t, r \in \mathbf{N}$ and $\alpha \in [0,1]$, we get that $t + r < \tau^*_\alpha(c)$ is equivalent to $t < \tau^*_\alpha(c)$ and $r < \tau^*_\alpha(\theta_t(c))$. We obtain this theorem from (4.6) and this fact. □

If the regularity does not hold for some $\tilde{s} \in \mathcal{F}$, the $\tilde{s}$-optimality of $\tilde{\tau}^*$ does not follow. But, $\tilde{\tau}^*$ defined by (4.6) is thought of as a good fuzzy stopping time. We note that weighting functions g are usually additive in the sense of (4.7) (see Fortemps and Roubens [4]), and (4.8) gives a concrete representation of the non-increase in Definition 2(ii).

5. A numerical example

In this section, an example is given to illustrate the theoretical results. Let $S := [0,1]$ and $0 < \beta < 0.98$. the fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x,y) = (1 - |y - \beta x|/100) \vee 0, \quad x, y \in [0,1]$$

and

$$\tilde{r}(x,z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x \in [0,1],\ z \in \mathbf{R},$$

where λ is an observation cost satisfying $\lambda > 1/100(1-\beta)$. Then, Q_α and R_α defined by (3.1) and (3.2) are as follows:

$$Q_\alpha([a,b]) = [\beta a - (1-\alpha), \beta b + (1-\alpha)] \quad \text{and} \quad R_\alpha([a,b]) = [a-\lambda, b-\lambda] \quad \text{for } 0 \leq a \leq b \leq 1.$$

Now, let $c = [a,b]$ $(0 \leq a \leq b \leq 1)$ and $g(c) = b$. Then

$$g(\varphi^\alpha(c,t)) = g\left(\sum_{l=1}^{t-1} R_\alpha(c_l)\right) = \frac{(1-\beta^{t-1})b_\alpha}{1-\beta} - \lambda_\alpha(t-1)$$

and

$$\gamma_t^\alpha(c) = \sup_{\sigma\in\Sigma_t} g(\varphi^\alpha(c,\sigma(c))) = \sup_{n\geq t}\left\{\frac{(1-\beta^{n-1})b_\alpha}{1-\beta} - \lambda_\alpha(n-1)\right\},$$

where $b_\alpha := b-(1-\alpha)/100(1-\beta)$ and $\lambda_\alpha := \lambda-(1-\alpha)/100(1-\beta)$ for $\alpha\in[0,1]$. Applying Lemma 4.1, for each $\alpha\in[0,1]$, the α-optimal stopping time τ_α^* is given by

$$\begin{aligned}\tau_\alpha^*([a,b]) &= \min\left\{t\geq 1 | (\varphi^\alpha([a,b],t),\beta^{t-1}[a,b])\in K^\alpha(g)\right\}\\ &= \min\left\{t\geq 1 \left| \frac{(1-\beta^{t-1})b_\alpha}{1-\beta} - \lambda_\alpha(t-1) \geq \frac{(1-\beta^t)b_\alpha}{1-\beta} - \lambda_\alpha t\right.\right\}.\end{aligned}$$

Let

$$\tilde{s}(x) = (1-|8x-4|)\vee 0 \quad \text{for } x\in[0.1].$$

Then we see

$$\tilde{s}_\alpha = [(3+\alpha)/8,(5-\alpha)/8]\,.$$

Therefore

$$\tau_\alpha^*(\tilde{s}_\alpha) = \left[\log\frac{\lambda_\alpha(1-\beta)}{-b_\alpha\log\beta}\Big/\log\beta\right]+1,$$

where for a real number z, $[z]$ is the largest integer equal to or less than z. Since $\tilde{s}$ is regular with respect to $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$, Theorem 4.1 implies that the $\tilde{s}$-optimal fuzzy stopping time $\tilde{\tau}^*$ is given by

$$\begin{aligned}\tilde{\tau}^*(\tilde{s},t) &= \sup_{\alpha\in[0,1]}\{\alpha\wedge 1_{\{\tau_\alpha^*(\tilde{s}_\alpha)>t\}}\}\\ &= \sup\left\{\alpha\in[0,1]\left|\left[\log\frac{\lambda_\alpha(1-\beta)}{-b_\alpha\log\beta}\Big/\log\beta\right]\geq t\right.\right\}\\ &= 0\vee\left(\frac{8(1-\beta+\beta^t\log\beta)+500\beta^t\log\beta+800(1-\beta)\lambda}{8(1-\beta+\beta^t\log\beta)+100\beta^t\log\beta}\right)\wedge 1,\end{aligned}$$

where the values are given in Table 1.

t	1	2	3	4	5	6	7	8	9	$\cdots$
$\tilde{\tau}^*(\tilde{s},t)$	0.938	0.812	0.681	0.546	0.405	0.260	0.108	0.0	0.0	$\cdots$

Table 1. $\tilde{s}$-optimal fuzzy stopping time $\tilde{\tau}^*(\tilde{s},\cdot)$ when $\lambda=0.5$ and $\beta=0.97$.

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