Optimal Fuzzy Stopping of a Sequence of Fuzzy Random Variables

Y. YOSHIDA[§], M. YASUDA[†], J. NAKAGAMI[†] and M. KURANO[‡]

[§]*Faculty of Economics and Business Administration, Kitakyushu University, Kokuraminami, Kitakyushu 802, Japan.* [†]*Faculty of Science, Chiba University,* [‡]*Faculty of Education, Yayoi-cho,Inage-ku,Chiba 263-8522, Japan.*

Abstract

The aim of this paper is to apply the technique of fuzzy stopping times to an optimal stopping problem of a sequence of fuzzy random variables.

1. Introduction

Fuzzy random variables was first studied by Puri and Ralescu [7] and have been studied by many authors. Stojaković [9] discussed fuzzy conditional expectation and Puri and Ralescu [8] studied fuzzy martingales. On the other hand, Neveu [5], Chow et al. [1] and many authors studied optimal stopping problems for a sequence of real-valued random variables. In this paper, we extend their results of the stochastic processes to the case of fuzzy-valued random variables. As an extension of stopping rules, we also use fuzzy stopping times by the idea of Kurano et al. [4]. In this paper, we will give a formulation of sequences of fuzzy-valued random variables and we give an optimal fuzzy stopping times for sequences under the assumptions of monotonicity and regularity for stopping rules. A numerical example is given to illustrate our idea.

2. Fuzzy random variables

Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a probability measure. Let $\mathbf{R}$ and $\mathbf{N}$ be the set of all real numbers and the set of all nonnegative integers respectively. $\mathcal{B}$ denotes the Borel σ-field of $\mathbf{R}$ and $\mathcal{I}$ denotes the set of all bounded closed sub-intervals of $\mathbf{R}$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbf{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [11] and Novák [6] for the theory of fuzzy sets. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is written by closed intervals

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1]. \tag{2.1}$$

The condition (2.1) is also written as

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid x \in \tilde{X}_\alpha(\omega)\} \in \mathcal{M} \times \mathcal{B} \neq \emptyset\} \in \mathcal{M} \quad \text{for all } \alpha \in [0,1], \tag{2.2}$$

where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$. We can find some equivalent conditions (see [8, 9]). However, in this paper, we adopt a simple equivalent condition in the following lemma.

Lemma 2.1 (Wang and Zhang [10, Theorems 2.1 and 2.2]). *For a map $\tilde{X} : \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:*

(i) *$\tilde{X}$ is a fuzzy random variable.*

(ii) *The maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$.*

Next we introduce expectations and conditional expectations of fuzzy random variables for the description and calculation of an optimal stopping problem for a sequence of fuzzy random variables. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. We put closed intervals

$$E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)\right], \quad \alpha \in [0,1]. \tag{2.3}$$

Since the map $\alpha \mapsto E(\tilde{X})_\alpha$ is left-continuous by the monotone convergence theorem, the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number ([3, Lemma 3])

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\right\} \quad \text{for } x \in \mathbf{R}, \tag{2.4}$$

where 1_D is the classical indicator function of a set D. For a sub-σ-field $\mathcal{N}$ of $\mathcal{M}$, the conditional expectation $E(\tilde{X}|\mathcal{N})$ is defined as follows: For $\alpha \in [0,1]$, there exist unique classical conditional expectations

$E(\tilde{X}_\alpha^-|\mathcal{N})$ and $E(\tilde{X}_\alpha^+|\mathcal{N})$ such that

$$\int_\Lambda E(\tilde{X}_\alpha^-|\mathcal{N})(\omega)\,\mathrm{d}P(\omega) = \int_\Lambda \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } \Lambda \in \mathcal{N}, \tag{2.5}$$

and

$$\int_\Lambda E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)\,\mathrm{d}P(\omega) = \int_\Lambda \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } \Lambda \in \mathcal{N}. \tag{2.6}$$

Then we can easily check the maps $\alpha \mapsto E(\tilde{X}_\alpha^-|\mathcal{N})(\omega)$ and $\alpha \mapsto E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)$ are left-continuous by the monotone convergence theorem. Therefore we can define the conditional expectation of the fuzzy random variable $\tilde{X}$ by a fuzzy random variable

$$E(\tilde{X}|\mathcal{N})(\omega)(x) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{E(\tilde{X}_\alpha|\mathcal{N})(\omega)}(x)\right\} \quad \text{for } \omega \in \Omega \text{ and } x \in \mathbf{R}, \tag{2.7}$$

where

$$E(\tilde{X}_\alpha|\mathcal{N})(\omega) := [E(\tilde{X}_\alpha^-|\mathcal{N})(\omega), E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)] \quad \text{for } \omega \in \Omega. \tag{2.8}$$

3. Fuzzy stopping times

In this section, we deal with fuzzy stopping times for a sequence of fuzzy random variables. Let $\{\tilde{X}_n\}_{n\in\mathbf{N}}$ be a sequence of integrably bounded fuzzy random variables such that $E(\sup_{n\in\mathbf{N}} \tilde{X}^+_{n,0}) < \infty$, where $\tilde{X}^+_{n,0}(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_n(\omega)$ for $n \in \mathbf{N}$. For $n \in \mathbf{N}$, $\mathcal{M}_n$ denotes the smallest σ-field on Ω generated by all random variables $\tilde{X}^-_{k,\alpha}$ and $\tilde{X}^+_{k,\alpha}$ $(k = 0, 1, 2, \cdots, n; \alpha \in [0, 1])$, and $\mathcal{M}_\infty$ denotes the smallest σ-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$. A map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ is said a stopping time if

$$\{\omega \mid \tau(\omega) = n\} \in \mathcal{M}_n \quad \text{for all } n \in \mathbf{N}. \tag{3.1}$$

Then we have the following lemma.

Lemma 3.1. *For a finite stopping time τ, we define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_n(\omega), \quad \omega \in \{\tau = n\} \quad \textit{for } n \in \mathbf{N}. \tag{3.2}$$

Then, $\tilde{X}_\tau$ is a fuzzy random variable.

Proof. This lemma follows from

$$\begin{aligned} &\{(\omega, x) \mid \tilde{X}_\tau(\omega)(x) \geq \alpha\} \\ = \; &\bigcup_{n\in\mathbf{N}} (\{(\omega, x) \mid \tilde{X}_n(\omega)(x) \geq \alpha\} \cap \{(\omega, x) \mid \tau(\omega) = n\}) \\ \in \; &\mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0, 1]. \quad \square \end{aligned}$$

By extending the stopping times, we introduce fuzzy stopping times which we studied fuzzy stopping times in dynamic fuzzy systems (see Kurano et al. [4]).

Definition 3.1. A map $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0, 1]$ is called a fuzzy stopping time if it satisfies the following (i) – (iii):

(i) For each $n \in \mathbf{N}$, $\tilde{\tau}(n, \cdot)$ is $\mathcal{M}_n$-measurable.

(ii) For almost all $\omega \in \Omega$, the map $n \mapsto \tilde{\tau}(n, \omega)$ is non-increasing.

(iii) For almost all $\omega \in \Omega$, there exists an integer n_0 such that $\tilde{\tau}(n, \omega) = 0$ for all $n \geq n_0$.

Regarding the membership grade of fuzzy stopping times, $\tilde{\tau}(n, \omega) = 0$ means 'to stop at time n' and $\tilde{\tau}(n, \omega) = 1$ means 'to continue at time n' respectively. We can easily check the following lemma regarding the properties of fuzzy stopping times (see [4]).

Lemma 3.2.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{n \in \mathbf{N} \mid \tilde{\tau}(n, \omega) < \alpha\}, \quad \omega \in \Omega \quad \text{for } \alpha \in (0, 1], \tag{3.3}$$

where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{\tilde{\tau}_\alpha \leq n\} \in \mathcal{M}_n$ *for $n \in \mathbf{N}$;*

(b) $\tilde{\tau}_\alpha(\omega) \leq \tilde{\tau}_{\alpha'}(\omega)$ a.a. $\omega \in \Omega$ *if $\alpha \geq \alpha'$;*

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ a.a. $\omega \in \Omega$ *if $\alpha > 0$;*

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ a.a. $\omega \in \Omega$.

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha \in [0,1]}$ be maps $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ satisfying the above (a) – (d). Define a map $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0, 1]$ by*

$$\tilde{\tau}(n, \omega) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{\{\tilde{\tau}_\alpha > n\}}(\omega)\}, \quad n \in \mathbf{N} \text{ and } \omega \in \Omega. \tag{3.4}$$

Then $\tilde{\tau}$ is a fuzzy stopping time.

4. A fuzzy optimal stopping problem

In this section, we discuss an optimal fuzzy stopping problem for the sequence of fuzzy random variables $\{\tilde{X}_n\}_{n \in \mathbf{N}}$ in the previous section. Let $\omega \in \Omega$. We consider about α-cuts of the sequence stopped at a fuzzy stopping time $\tilde{\tau}(n, \omega)$. The α-cuts must be $\tilde{X}_{\tilde{\tau}_\alpha(\omega),\alpha}(\omega)$ where $\tilde{\tau}_\alpha(\omega)$ is given by (3.3) and is a 'classical' stopping time for each $\alpha \in (0, 1]$. Let $g : \mathcal{I} \mapsto \mathbf{R}$ be an additive map, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S). \tag{4.1}$$

A weighting function satisfies this kind of linearity (4.1) and is used for the scalar estimation of fuzzy numbers (see Fortemps and Roubens [2]). Then we define a random variable

$$G_{\tilde{\tau}}(\omega) := \int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega))\, \mathrm{d}\alpha, \quad \omega \in \Omega, \tag{4.2}$$

where we write $\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega) := \tilde{X}_{\tilde{\tau}_\alpha(\omega),\alpha}(\omega)$ for simplicity. Note that the scalar-estimated value $g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega))$ is a real number and the map $\alpha \mapsto g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega))$ is left-continuous on $(0, 1]$, so that the right-hand integral of (4.2) is well-defined. (4.2) would be a scalar-estimated value of the sequence of fuzzy random variables stopped at a fuzzy stopping time $\tilde{\tau}(n, \omega)$. Therefore the expectation of (4.2) is

$$E(G_{\tilde{\tau}}) := E\left(\int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\cdot))\, \mathrm{d}\alpha\right) = \int_0^1 E(g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha})\, \mathrm{d}\alpha \tag{4.3}$$

for fuzzy stopping times $\tilde{\tau}$.

Definition 4.1.

(i) Let $\alpha \in [0,1]$. A finite stopping time τ^* is called α-optimal if $E(g(\tilde{X}_{\tau^*,\alpha})) \geq E(g(\tilde{X}_{\tau,\alpha}))$ for all finite stopping times τ.

(ii) A fuzzy stopping time $\tilde{\tau}^*$ is called optimal if $E(G_{\tilde{\tau}^*}) \geq E(G_{\tilde{\tau}})$ for all fuzzy stopping times $\tilde{\tau}$.

To give optimal stopping times for the stopping problems in Definition 4.1, we define a sequence of subsets $\{\Lambda_n\}_{n=0}^{\infty}$ of Ω by

$$\Lambda_n := \{\omega \in \Omega \mid g(\tilde{X}_{n,\alpha}(\omega)) \geq E(g(\tilde{X}_{n+1,\alpha})|\mathcal{M}_n)(\omega)\}, \quad n \in \mathbf{N}.$$

In this paper, we deal with the monotone case in the following assumption (Chow et al. [1]).

Assumption A (Monotone case). The following conditions hold almost surely:

$$\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=0}^{\infty} \Lambda_n = \Omega.$$

In order to characterize α-optimal stopping times, we let

$$\gamma_n^{\alpha} := \operatorname*{ess\,sup}_{\tau:\ \text{finite stopping times},\, \tau \geq n} E(g(\tilde{X}_{\tau,\alpha})|\mathcal{M}_n) \quad \text{for } n \in \mathbf{N}, \tag{4.4}$$

where we refer to Neveu [5, Proposition 6-1-1] and Chow et al. [1, Chapter 1-6] for the definition of the essential supremum. We define a map $\tilde{\sigma}_{\alpha}^* : \Omega \mapsto \mathbf{N}$ by

$$\tilde{\sigma}_{\alpha}^*(\omega) := \inf\left\{n \mid g(\tilde{X}_{n,\alpha}(\omega)) = \gamma_n^{\alpha}(\omega)\right\} \tag{4.5}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the next lemma is given by Chow et al. [1].

Lemma 4.1 ([1, Theorems 4.1 and 4.5]). *Suppose Assumption A holds. Then, for* $\alpha \in [0,1]$ *the following (i) and (ii) hold:*

(i) $\gamma_n^{\alpha}(\omega) = \max\{g(\tilde{X}_{n,\alpha}(\omega)), \gamma_{n+1}^{\alpha}(\omega)\}$ a.a. $\omega \in \Omega$ *for* $n \in \mathbf{N}$;

(ii) *almost all* $\omega \in \Omega$. *If* $P(\tilde{\sigma}_{\alpha}^* < \infty) = 1$, *then* $\tilde{\sigma}_{\alpha}^*$ *is* α*-optimal and* $E(\gamma_0^{\alpha}) = E(g(\tilde{X}_{\tilde{\sigma}_{\alpha}^*,\alpha}))$.

In order to construct an optimal fuzzy stopping time from the α-optimal stopping times $\{\tilde{\sigma}_{\alpha}^*\}_{\alpha\in[0,1]}$, we need the following regularity condition.

Assumption B (Regularity). For almost all $\omega \in \Omega$, the map $\alpha \mapsto \tilde{\sigma}_{\alpha}^*(\omega)$ is non-increasing and $\lim_{\alpha'\uparrow\alpha} \tilde{\sigma}_{\alpha'}^*(\omega) = \tilde{\sigma}_{\alpha}^*(\omega)$ if $\alpha > 0$.

Under Assumption B, we can define a map $\tilde{\sigma}^* : \mathbf{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^*(n,\omega) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{\{\tilde{\sigma}^*_\alpha > n\}}(\omega)\}, \quad n \in \mathbf{N} \text{ and } \omega \in \Omega. \tag{4.6}$$

Theorem 4.1. *Suppose Assumptions A and B hold. If $P(\tilde{\sigma}^*_0 < \infty) = 1$, then $\tilde{\sigma}^*$ is an optimal fuzzy stopping time.*

Proof. ¿From Assumption B and Lemma 3.2, $\tilde{\sigma}^*$ is a fuzzy stopping time. ¿From Lemma 4.1, for all fuzzy stopping times $\tilde{\tau}$ we obtain

$$\begin{aligned} E(G_{\tilde{\tau}}) &= E\left(\int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha})\,\mathrm{d}\alpha\right) \\ &\le \int_0^1 \sup_{\tau:\text{ finite stopping times}} E\left(g(\tilde{X}_{\tau,\alpha})\right)\mathrm{d}\alpha \\ &= \int_0^1 E(\gamma_0^\alpha)\mathrm{d}\alpha \\ &= \int_0^1 E(g(\tilde{X}_{\tilde{\sigma}^*_\alpha,\alpha}))\mathrm{d}\alpha \\ &= E(G_{\tilde{\sigma}^*}). \end{aligned}$$

Therefore $\tilde{\sigma}^*$ is optimal for the stopping problem in Definition 4.1(ii). $\square$

In the rest of this section, we discuss ε-optimality. This method is effective when the condition $P(\tilde{\sigma}^*_\alpha < \infty) = 1$ is not neccessarily satisfied (see Lemma 4.1(ii)).

Definition 4.2. Let $\varepsilon > 0$.

(i) Let $\alpha \in [0,1]$. A finite stopping time τ^* is called (ε,α)-optimal if $E(g(\tilde{X}_{\tau^*,\alpha})) \ge E(g(\tilde{X}_{\tau,\alpha})) - \varepsilon$ for all finite stopping times τ.

(ii) A fuzzy stopping time $\tilde{\tau}^*$ is called ε-optimal if $E(G_{\tilde{\tau}^*}) \ge E(G_{\tilde{\tau}}) - \varepsilon$ for all fuzzy stopping times $\tilde{\tau}$.

We define a map $\tilde{\sigma}^\varepsilon_\alpha : \Omega \mapsto \mathbf{N}$ by

$$\tilde{\sigma}^\varepsilon_\alpha(\omega) := \inf\left\{n \mid g(\tilde{X}_{n,\alpha}(\omega)) \ge \gamma_n^\alpha(\omega) - \varepsilon\right\} \tag{4.7}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the next lemma is given by Neveu [5, Chapter 6-1] or Chow et al. [1, Chapter 4-5].

Lemma 4.2 ([5, Proposition 6-1-3]). *Suppose Assumption A holds. Then, for $\varepsilon > 0$ and $\alpha \in [0,1]$ the following (i) and (ii) hold:*

(i) *$P(\tilde{\sigma}^\varepsilon_\alpha < \infty) = 1$;*

(ii) *$\tilde{\sigma}^\varepsilon_\alpha$ is (ε,α)-optimal and $E(g(\tilde{X}_{\tilde{\sigma}^\varepsilon_\alpha,\alpha})) \ge E(\gamma_0^\alpha) - \varepsilon$.*

In order to construct an optimal fuzzy stopping time from the (ε,α)-optimal stopping times $\{\tilde{\sigma}^\varepsilon_\alpha\}_{\alpha\in[0,1]}$, we need the following regularity condition.

Assumption B' (Regularity). For almost all $\omega\in\Omega$, the map $\alpha\mapsto\tilde{\sigma}^\varepsilon_\alpha(\omega)$ is non-increasing
and $\lim_{\alpha'\uparrow\alpha}\tilde{\sigma}^\varepsilon_{\alpha'}(\omega)=\tilde{\sigma}^\varepsilon_\alpha(\omega)$ if $\alpha>0$.

Under Assumption B', we can define a map $\tilde{\sigma}^\varepsilon:\mathbf{N}\times\Omega\mapsto[0,1]$ by

$$\tilde{\sigma}^\varepsilon(n,\omega):=\sup_{\alpha\in[0,1]}\min\{\alpha,1_{\{\tilde{\sigma}^\varepsilon_\alpha>n\}}(\omega)\},\quad n\in\mathbf{N}\text{ and }\omega\in\Omega. \tag{4.8}$$

Theorem 4.2. *Suppose Assumptions A and B' hold. $\tilde{\sigma}^\varepsilon$ is an ε-optimal fuzzy stopping time.*

Proof. ¿From Assumption B' and Lemma 3.2, $\tilde{\sigma}^\varepsilon$ is a fuzzy stopping time. ¿From Lemma 4.2, for all fuzzy stopping times $\tilde{\tau}$ we obtain

$$\begin{aligned}
E(G_{\tilde{\tau}}) &= E\left(\int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha})\,\mathrm{d}\alpha\right)\\
&\le \int_0^1 \sup_{\tau:\ \textbf{finite stopping times}} E\left(g(\tilde{X}_{\tau,\alpha})\right)\mathrm{d}\alpha\\
&= \int_0^1 E(\gamma_0^\alpha)\mathrm{d}\alpha\\
&\le \int_0^1 E(g(\tilde{X}_{\tilde{\sigma}^*_\alpha,\alpha}))\mathrm{d}\alpha+\varepsilon\\
&= E(G_{\tilde{\sigma}^*})+\varepsilon.
\end{aligned}$$

Therefore $\tilde{\sigma}^\varepsilon$ is ε-optimal for the stopping problem in Definition 4.2(ii). $\square$

5. A numerical example

An example is given to illustrate the results of the fuzzy stopping problem in the previous section. Let $\{Y_n\}_{n\in\mathbf{N}}$ be a uniform integrable sequence of independent, identically distributed real random variables. Let c and d be constants satisfying $0<d<3c$. Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence given by $a_n:=d\cdot n$ for $n\in\mathbf{N}$. Put

$$M_n:=\max\{Y_0,Y_1,Y_2,\cdots,Y_n\}-c\cdot n,\quad n\in\mathbf{N}.$$

Hence we take a sequence of fuzzy random variables $\{\tilde{X}_n\}_{n\in\mathbf{N}}$ as follows:

$$\tilde{X}_n(\omega)(x):=\begin{cases} L((M_n(\omega)-x)/a_n) & \text{if } x\le M_n(\omega)\\ L((x-M_n(\omega))/a_n) & \text{if } x\ge M_n(\omega)\end{cases}$$

for $n\in\mathbf{N}$, $\omega\in\Omega$ and $x\in\mathbf{R}$, where the shape function $L(x):=\max\{1-|x|,0\}$ for $x\in\mathbf{R}$. Then their α-cuts are

$$\tilde{X}_{n,\alpha}(\omega)=[M_n(\omega)-(1-\alpha)a_n,M_n(\omega)+(1-\alpha)a_n],\quad\omega\in\Omega$$

for $n \in \mathbf{N}$ and $\alpha \in [0,1]$. Let a function $g([x,y]) := (x+2y)/3$ for $x, y \in \mathbf{R}$ satisfying $x \le y$. Then g satisfies (4.1), and we can easily check

$$g(\tilde{X}_{n,\alpha}(\omega)) \quad = M_n(\omega) + \tfrac{1-\alpha}{3} a_n, \quad \omega \in \Omega$$

for $\alpha \in [0,1]$, and

$$G_n(\omega) = \int_0^1 g(\tilde{X}_{n,\alpha}(\omega))\,\mathrm{d}\alpha = M_n(\omega) + \frac{1}{6} a_n, \quad \omega \in \Omega.$$

Next we check Assumptions A and B. Let $\alpha, \alpha' \in [0,1]$ satisfy $\alpha' \le \alpha$ and let $\omega \in \Omega$. If $g(\tilde{X}_{n,\alpha'}(\omega)) = \gamma_n^{\alpha'}(\omega)$ for some n, then we have

$$\begin{aligned} g(\tilde{X}_{n,\alpha}(\omega)) &= M_n(\omega) + \tfrac{1-\alpha}{3} A_n(\omega) \\ &= M_n(\omega) + \tfrac{1-\alpha'}{3} A_n(\omega) + \tfrac{\alpha'-\alpha}{3} A_n(\omega) \\ &= g(\tilde{X}_{n,\alpha'}(\omega)) + \tfrac{\alpha'-\alpha}{3} A_n(\omega) \\ &= \gamma_n^{\alpha'}(\omega) + \tfrac{\alpha'-\alpha}{3} A_n(\omega) \\ &\ge E(g(\tilde{X}_{\tau,\alpha'}) \mid \mathcal{M}_n)(\omega) + \tfrac{\alpha'-\alpha}{3} E(A_\tau \mid \mathcal{M}_n)(\omega) \\ &= E\left(g(\tilde{X}_{\tau,\alpha'}) + \tfrac{\alpha'-\alpha}{3} A_\tau \mid \mathcal{M}_n\right)(\omega) \\ &= E(g(\tilde{X}_{\tau,\alpha}) \mid \mathcal{M}_n)(\omega) \quad \text{a.a. } \omega \in \Omega \end{aligned}$$

for all finite stopping times τ such that $\tau \ge n$. It follows $g(\tilde{X}_{n,\alpha}(\omega)) = \gamma_n^\alpha(\omega)$. Therefore we obtain $\tilde{\sigma}_\alpha^*(\omega) \le \tilde{\sigma}_{\alpha'}^*(\omega)$ for almost all $\omega \in \Omega$, and Assumption B holds. On the other hand, we have

$$\begin{aligned} g(\tilde{X}_{n,\alpha}(\omega)) &= M_n(\omega) + \tfrac{1-\alpha}{3} a_n \\ &= \max\{Y_0(\omega), Y_1(\omega), Y_2(\omega), \cdots, Y_n(\omega)\} - c \cdot n + \tfrac{1-\alpha}{3} d \cdot n \\ &= \max\{Y_0(\omega), Y_1(\omega), Y_2(\omega), \cdots, Y_n(\omega)\} - \left(c - \tfrac{1-\alpha}{3} d\right) n. \end{aligned}$$

Since $c - \frac{1-\alpha}{3} d \ge c - \frac{1}{3} d > 0$, this is the monotone case (Assumption A) from Chow et al. [1, Chapter 3-5 (3.22)]. Then the finite α-optimal stopping times in the previous section are given by

$$\tilde{\sigma}_\alpha^*(\omega) = \inf\left\{n \mid g(\tilde{X}_{n,\alpha}(\omega)) = \gamma_n^\alpha(\omega)\right\} = \inf\{n \mid Y_n(\omega) \ge \beta^\alpha\}, \quad \omega \in \Omega,$$

where β^α is a constant and the unique solution of the equation

$$E\left((Y_1 - \beta)^+\right) = c - \frac{1-\alpha}{3} d$$

for $\alpha \in [0,1]$. Therefore the optimal fuzzy stopping time is

$$\tilde{\sigma}^*(n,\omega) = \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\sigma}_\alpha^* > n\}}(\omega)\} = \sup\{\alpha \in [0,1] \mid Y_n(\omega) < \beta^\alpha\},$$

for $n \in \mathbf{N}$ and $\omega \in \Omega$.

Finally we calculate the optimal expected value $E(G_{\tilde{\sigma}^*})$. ¿From Chow et al. [1, Chapter 3-5 (3.22)], we have

$$E(g(\tilde{X}_{\tilde{\sigma}^*_\alpha,\alpha})) = \beta^\alpha$$

for $\alpha \in [0,1]$. Therefore we obtain

$$E(G_{\tilde{\sigma}^*}) = \int_0^1 E(g(\tilde{X}_{\tilde{\sigma}^*_\alpha,\alpha}))\,\mathrm{d}\alpha = \int_0^1 \beta^\alpha\,\mathrm{d}\alpha.$$

For example, if $Y_0, Y_1, Y_2, \cdots$ are independent and uniformly distributed on [0,1], then

$$\beta^\alpha = \begin{cases} \frac{1}{2} - c + \frac{1-\alpha}{3}d & \text{if } \frac{1}{2} - c + \frac{1-\alpha}{3}d < 0 \\ 1 - \sqrt{2\left(c - \frac{1-\alpha}{3}d\right)} & \text{if } \frac{1}{2} - c + \frac{1-\alpha}{3}d \geq 0. \end{cases}$$

Let $c = 1/2$ and $d = 1$. Then we can easily check that the optimal expected values for the stopping problems (i) and (ii) of Definition 4.1 are

$$E(G_{\tau^*}) = \beta^\alpha = 1 - \sqrt{1 - \frac{2(1-\alpha)}{3}} \quad (\alpha \in [0,1])$$

and

$$E(G_{\tilde{\sigma}^*}) = \int_0^1 \beta^\alpha\,\mathrm{d}\alpha = (1/2)^{3/2}.$$

References

[1] Y.S.Chow, H.Robbins and D.Siegmund, *The theory of optimal stopping: Great expectations* (Houghton Mifflin Company, New York, 1971).

[2] P.Fortemps and M.Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[3] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[4] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, An approach to stopping problems of a dynamic fuzzy system, preprint.

[5] J.Neveu, *Discrete-Parameter Martingales* (North-Holland, New York, 1975).

[6] V.Novák, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[7] M.L.Puri and D.A.Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114** (1986) 409-422.

[8] M.L.Puri and D.A.Ralescu, Convergence theorem for fuzzy martingales, *J. Math. Anal. Appl.* **160** (1991) 107-122.

[9] M.Stojaković, Fuzzy conditional expectation, *Fuzzy Sets and Systems* **52** (1992) 53-60.

[10] G.Wang and Y.Zhang, The theory of fuzzy stochastic processes, *Fuzzy Sets and Systems* **51** (1992) 161-178.

[11] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.