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A Saddle Point of an Inventory Problem

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ABSTRACT

When the demand distribution is unknown, that is, most of the informations of it are insufficient for a definite probability distribution, a minimax procedure is sometimes used in an inventory problem. The minimax inventory problem is considered as a 2-person zero-sum game, in which one player (manager) decides his ordering level and other player (nature) chooses her distribution in the prescribed class of distributions. We shall show the necessary and sufficient conditions for the existence of saddle points in this problem, and give the explicit formula for a set of saddle points and a saddle value. A multi-period model is also discussed.

1. INTRODUCTION

Most of authors have made precise analyses in the general inventory problem which assume that the demand distribution is completely known. Especially, an optimal inventory policy is sensitive to the form of the ordering cost functions, therefore several types of policies have been derived and examined with their properties. Sufficient conditions under which they are optimal have also been derived.

When the demand distribution is unknown, that is, most of the informations of it are insufficient for a definite probability distribution,

Journal of Information & Optimization Sciences
Vol. 2 (1981), No. 2, pp. 181—191

a Bayesian approach or a minimax procedure is sometimes used. On the minimax inventory problem, one can refer to the work done in [1, 5—9]. A minimax policy is one that minimizes the maximum expected cost, where the maximum is taken over a prescribed class of distributions. For example, Scarf [9] assumed that only the mean and the variance of the distribution were known.

In this paper, we consider this minimax inventory problem as a 2-person zero-sum game, in which one player (manager) decides his ordering level and other player (nature) chooses her distribution in the prescribed class of distributions. By a game theoretic approach, we shall show the necessary and sufficient conditions for the existence of saddle points in this problem, and give the explicit formula for a set of saddle points and a saddle value.

2. DESCRIPTION OF THE MODEL

Let us define the one-period expected loss function by a inventory level y after ordering and a demand distribution F as follows :

$$L(y,\ F)=cy+\int_{[0,\ \infty)}\{h\,(y-t)^{+}+p\,(t-y)^{+}\}\,dF(t) \tag{1}$$

where $x^{+}=max\,(x,0)$. Here, c, h and p are the unit ordering cost, the unit holding cost and the unit penalty cost, respectively. And it is assumed that $p>c\geqslant 0$, $c+h>0$ and that the demand t is a non-negative random variable with the distribution F ($F(0-)=0$).

When F is known precisely, the value of y° which minimizes (1) is easily solved. Let $F^{-1}(z)=\{y\ ;\ F(y)=z,\ \ y\in[0,\infty)\}$, $0\leqslant z\leqslant 1$, and $q\equiv(p-c)/(p+h)$. Then

$$\begin{cases} y^{\circ}\in F^{-1}(q) & \text{if } F^{-1}(q)\neq\phi,\text{ and} \\ y^{\circ}=\text{the minimum } y \text{ which satisfies } F(y)>q & \text{if } F^{-1}(q)=\phi. \end{cases} \tag{2}$$

In any specific problem, most of the informations of the demand distribution F are insufficient for a definite probability distribution. So it is assumed that F is unknown but belongs to some class $\mathcal{J}$ of distributions, which is constituted by the partial information prescribed. Then the minimax problem

$$\inf_{y\geqslant 0}\ \sup_{F\in\mathcal{J}} L(y,F) \tag{3}$$

was considered for serveral classes of distributions ([1], [5]—[9]). In these literatures, a minimax value of L and a minimax policy of y in

(3) were derived. It is naturally extended to consider a saddle point of this problem.

$$\inf_{y\geqslant 0}\ \sup_{F\in\mathcal{J}} L(y,F)=\sup_{F\in\mathcal{J}}\ \inf_{y\geqslant 0} L(y,F)\ (=L(y^{*},F^{*})\equiv L^{*}) \tag{4}$$

As we show in the next section, all the classes $\mathcal{J}$ given by the above literatures are known to satisfy the conditions for the existence of saddle points. Hence, instead of deriving a minimax solution of (3) in which treatments of F are not came out, what we do in this paper is to derive a saddle value L^{*} and a set of sadddle points (y^{*},F^{*}) of (4) in the explicit form by solving the maximin problem. This is reduced to the maximization problem by (2)

$$\sup_{F\in\mathcal{J}} L(y^{\circ},F). \tag{5}$$

The general treatments of duality theorem are given in Ishii [4].

3, CONDITIONS FOR SADDLE POINTS

For the class $\mathcal{J}$ of distributions, we assume that

$$(a)\qquad \sup_{F\in\mathcal{J}}\int_{[0,\ \infty)} t\,dF(t)<\infty.$$

This assumption (a) only makes $L(y,F)$ of (1.1) finite for any fixed y to avoid a trivial case, The function $L(y,F)$ is defined on $[0,\infty)\times\mathcal{J}$ with finite non-negative values. For any fixed $F\in\mathcal{J}$, it is seen that $L(y,F)$ is convex in y and $\lim_{y\to\infty} L(y,F)=\infty$. Then, we may restrict the domain of y is compact and convex. Furthermore, we assume that

(b) $\mathcal{J}$ is convex,

i.e., if $F_1, F_2, \in\mathcal{J}$, then $\lambda F_1+(1-\lambda)F_2\in\mathcal{J}$ for any real number λ $(0\leqslant\lambda\leqslant 1)$. The following lemmas are easily derived from the general minimax theorem in [3].

Lemma 1. *If assumptions* (a), (b) *are satisfied, then*

$$\min_{y}\ \sup_{F} L(y,F)=\sup_{F}\ \min_{y} L(y,F)\qquad (=L^{*}), \tag{6}$$

and there exists a saddle value L^{*}. *And additionally that*

(c) $\mathcal{J}$ *is compact with respect to the Lévy metric*

is satisfied, then the function $L(y, F)$ possesses at least one saddle point (y^*, F^*) on $[0, \infty)\times\mathcal{J}$ and

$$\min_y \max_F L(y, F) = \max_F \min_y L(y, F) = L(y^*, F^*) = L^* \tag{7}$$

Lemma 2. *If assumptions (a), (b), (c) are satisfied, then $(y^*, F^*) \in [0, \infty)\times\mathcal{J}$ is a saddle point of $L(y, F)$ if and only if*

(i) $L'(y^*, F^*; y-y^*)$

$$= \begin{cases} (y-y^*)\,[c-p+(h+p)\,F^*(y^*)\,] \geqslant 0 \text{ for any } y > y^*, \\ (y-y^*)\,[c-p+(h+p)\,F^*(y^*-)\,] \geqslant 0 \text{ for any } y < y^*, \end{cases}$$

where L' is a Gâteaux-differential. That is,

$$\begin{cases} y^* = 0 \text{ and } F^*(0) \geqslant (p-c)/(p+h) \equiv q, \\ y^* > 0 \text{ and } F^*(y^*) \geqslant q, \quad F^*(y^*-) \leqslant q, \end{cases}$$

(ii) $L(y^*, F^*) \geqslant L(y^*, F)$ for any $F \in \mathcal{J}$.

From Lemma 2, we can see that F^* has more than q of probability on the interval $[0, y^*]$ and has less than $1-q$ of probability on the interval (y^*, ∞).

4. THE CLASS $\mathcal{J}_\mu$

Now we consider the next class $\mathcal{J}_\mu$ of distributions in a class $\mathcal{J}$ which satisfies assumptions (a), (b), (c)

$$\mathcal{J}_\mu = \{F \in \mathcal{J} \mid \int dF=1, \int tdF=\mu\}. \tag{8}$$

The equation (1) is reduced to as follows by condition (i) of Lemma 2.

$$L^*=(p+h)\,[y^*\{F^*(y^*)-(p-c)/(p+h)\}-\int_{[0, y^*]} tdF^*]+p\mu$$

$$=p\mu-(p+h)\int_{[0, y^*\}} tdF^*,$$

where $\int_{[0, y^*\}} dF^*$ represents the integral from 0 to y^* until just q of probability with respect to the distribution F^*. Since a saddle value L^* is given by (5), then the following theorem holds,

Theorem 1.

$$L^*=p\mu-(p+h)\min_{F\in\mathcal{J}_\mu}\int_{[0, y^\circ\}} tdF, \tag{9}$$

where

$$\begin{cases} y^\circ=0 & \text{if } F(0) \geqslant q, \\ y^\circ \in F^{-1}(q) & \text{if } F(0) < q \text{ and } F^{-1}(q) \neq \phi, \\ y^\circ = \textit{the minimum } y \textit{ which satisfies } F(y) > q \\ \quad \textit{if } F(0) < q \textit{ and } F^{-1}(q) = \phi. \end{cases}$$

Now we can divide the theorem into two cases.

Corollary 1. *If there is a distribution F with $F(0) \geqslant q$ in the class $\mathcal{J}_\mu$, then*

$$L^*=p\mu, \qquad y^*=0. \tag{10}$$

Corollary 2. *If all the distributions F in the class $\mathcal{J}_\mu$ are satisfied with $F(0) < q$, then there exists a unique y_F ($y_F > 0$) such that $\int_{[0, y^\circ\}} tdF=qy_F$.*

And $$L^*=p\mu-(p-c)\min_{F\in\mathcal{J}_\mu} y_F. \tag{11}$$

Hence, this case is reduced to the problem which determines the minimum value of y_F by other conditions of the class $\mathcal{J}_\mu$.

5. EXAMPLES

The examples mentioned in this section can be easily checked to satisfy assumptions (a), (b), (c). Some of the examples were treated as a minimax problem (3) by the literatures, in which case L^* and y^* have been derived on the guarantee of the existence of saddle points, but F^* does not come out. We, therefore, derive an L^* and a set of (y^*, F^*) from Corollary 1, or Corollary 2 and other conditions of the class $\mathcal{J}_\mu$.

Example 1. The class $\mathcal{J}(\mu)$ of distributions of which only the mean μ is assumed to be known.

$$\mathcal{J}(\mu)=\left\{\int dF=1, \int tdF=\mu\right\} \quad (\mu > 0).$$

The following results are immediately seen from Corollary 1.

$$\begin{cases} F^*(t)=q+\bar{q}G(t) & (t \geqslant 0), \\ y^*=0, \quad L^*=p\mu, \end{cases} \tag{12}$$

where $\bar{q}=(1-q)$ and $G(t)$ is an arbitrary distribution with $G(0-)=0$ such that $\int tdG=\mu/\bar{q}$.

Example 2. ([5], [6], [8], [9]). The class $\mathcal{J}(\mu, \sigma)$ of distributions of which only the mean μ and variance σ^2 are assumed to be known.

$$\mathcal{J}(\mu, \sigma)=\left\{ \int dF=1, \int tdF=\mu, \int t^2dF=\mu^2+\sigma^2 \right\} \quad (\mu, \sigma > 0).$$

It is noted that the distribution G which has a minimum variance in (12) is one that degenerates at a point $t=\mu/\bar{q}$. And F^* in (12) satisfies that $\sigma^2/\mu^2 \geqslant q/\bar{q}$, *i.e.*, $\mu^2/(\mu^2+\sigma^2) \leqslant \bar{q}$. Then, this example is divided into two cases according to Corollary 1, 2.

Case 1: $\mu^2/(\mu^2+\sigma^2) \leqq \bar{q}$.

$$\begin{cases} F^*(t)=q+\bar{q}G(t) & (t \geqslant 0), \\ y^*=0, \; L^*=p\mu, \end{cases} \tag{13}$$

where, $G(t)$ is an arbitary distribution with $G(0-)=0$ such that $\int tdG=\mu/\bar{q}$, $\int t^2dG=\mu^2+\sigma^2/\bar{q}$.

Case 2: $\mu^2/(\mu^2+\sigma^2) > \bar{q}$.

From the definition of y_F in Corollary 2, the following Schwartz inequalities hold.

$$\left(\int_{[0, y^\circ\}} tdF\right)^2 \leqslant \left(\int_{[0, y^\circ\}} dF\right)\left(\int_{[0, y^\circ\}} t^2dF\right),$$

$$\left(\int_{\{y^\circ, \infty)} tdF\right)^2 \leqslant \left(\int_{\{y^\circ, \infty)} dF\right)\left(\int_{\{y^\circ, \infty)} t^2dF\right).$$

After some simple calculations,

$$\mu-\sigma\sqrt{q/\bar{q}} \leqslant y_F \leqslant \mu+\sigma\sqrt{q/\bar{q}}.$$

Then,

$$\min_{F \varepsilon \mathcal{J}(\mu, \sigma)} y_F = \mu-\sigma\sqrt{q/\bar{q}} \quad (>0).$$

And,

$$L^*=c\mu+\sigma\sqrt{(c+h)(p-c)}. \tag{14}$$

Since the equalities hold in the Schwartz inequalities, F^* is restricted to the following two point distribution.

$$F^* \text{ has mass } q \text{ at } \underline{y} \equiv \mu-\sigma\sqrt{q/\bar{q}}$$

$$\text{and} \qquad \text{mass } \bar{q} \text{ at } \bar{y} \equiv \mu+\sigma\sqrt{q/\bar{q}}. \tag{15}$$

Then $y^* \varepsilon [\underline{y}, \bar{y})$ because of $L(y, F^*)=L^*$ for any $y \varepsilon [\underline{y}, \bar{y})$, and each $y \varepsilon [\underline{y}, \bar{y})$ is a candidate for y^* if $L(y, F^*) \geqslant L(y, F)$ for some F in $\mathcal{J}(\mu, \sigma)$ by condition (*ii*). Now, consider the two point distribution F which is shifted to the left (or right) from F^*, such that F has mass q' at $\mu-\sigma\sqrt{q'/\bar{q}'}$ and mass $\bar{q}'$ at $\mu+\sigma\sqrt{\bar{q}'/q'}$ where $q' < q$ (or $q' > q$). Simple calculations with concerning $L(y, F^*) \geqslant L(y, F)$ for $y \equiv \mu+\sigma x \varepsilon [\underline{y}, \bar{y}]$ yield that when q' tends to q from the left (or right),

$$x \leqslant (\text{or} \geqslant) \frac{d}{dq'}\left(-\sqrt{q'\bar{q}'}\right)\Big|_{q'=q}. \tag{16}$$

By Lemma 1, there exists at least one saddle point. Therefore, from (16),

$$y^*=\mu+\sigma\frac{(p-c)-(c+h)}{2\sqrt{(p-c)(c+h)}}. \tag{17}$$

Example 3. ([1]) The class (μ, δ) of distributions of which only the mean μ and the deviation δ are assumed to be known.

$$\mathcal{J}(\mu, \delta)=\left\{\int dF=1, \int tdF=\mu, \int |t-\mu| dF=\delta\right\}$$
$$(1 > \delta/2\mu \text{ and } \mu, \delta > 0).$$

This example is divided into following two cases according to Corollary 1, 2.

Case 1: $1-\delta/2\mu \leqslant \bar{q}$.

In this case F^* is in (12). Then

$$\begin{cases} F^*(t)=q+\bar{q}G(t) & (t \geqslant 0), \\ y^*=0, \quad L^*=p\mu, \end{cases} \tag{18}$$

where $G(t)$ is an arbitrary distribution with $G(0-)=0$ such that $\int tdG=\mu/\bar{q}$, $\int |t-\mu| dG=\mu+(\delta-\mu)/\bar{q}$.

Case 2 : $1-\delta/2\mu \geqslant \bar{q}$.

From the definition of y_F in Corollary 2, the following inequality holds.

$$\int_{[0,\, y^\circ\}} (\mu-t)dF=q(\mu-y_F) \leqslant \int_{[0,\, \mu]} (\mu-t)dF=\delta/2,$$

$$y_F \geqslant \mu-\delta/2q\ (>0).$$

And $L^*=c\mu+(p+h)\,\delta/2$. The equality holds if and only if $\int_{[0,\, y^\circ\}} (\mu-t)dF^*=\int_{[0,\, \mu]} (\mu-t)\, dF^*$. Therefore,

$$\begin{cases} F^*(t)=qG_1(t)+\bar{q}G_2(t), \\ y^*=\mu, \quad L^*=c\mu+(p+h)\delta/2, \end{cases} \tag{19}$$

where $G_1(t)$ is an arbitary distribution on $[0, \mu]$ such that $\int_{[0,\, \mu]} tdG_1=\mu-\delta/2q$ and $G_2(t)$ is an arbitary distribution on (μ, ∞) such that $\int_{(\mu,\, \infty)} tdG_2=\mu+\delta/2\bar{q}$.

Example 4. The class $\mathcal{J}(\mu, M)$ of distributions of which only the mean μ and the domain $[0, M]$ are assumed to be known. In [7], only the domain is assumed.

$$\mathcal{J}(\mu, M)=\left\{ \int_{[0,\, M]} dF=1, \quad \int tdF=\mu \right\} \quad (M>\mu>0):$$

This example is divived into following two cases according to Corollary 1, 2.

Case 1 : $\mu/M \leqslant \bar{q}$.
In this case F^* is in (12). Then

$$\begin{cases} F^*(t)=q + \bar{q}G(t) \quad (t \geqslant 0), \\ y^*=0, \quad L^*=p\mu, \end{cases} \tag{20}$$

where $G(t)$ is an arbitary distribution on $[0, M]$ such that

$$\int_{[0,\, M]} tdG=\mu/\bar{q}.$$

Case 2 : $\mu/M > \bar{q}$.

From the definition of y_F in Corollary 2, the following inequalities hold.

$$\int_{[0,\, y^\circ\}} tdF=qy_F, \quad \int_{\{y^\circ,\, M]} tdF \leqslant \bar{q}M.$$

$$y_F \geqslant (\mu-\bar{q}M)/q.$$

The equality holds if and only if F^* has mass $\bar{q}$ at a point M. Therefore

$$\begin{cases} F^*(t)=qG(t)+\bar{q}I_{[M,\, \infty)}(t), \\ y^*=M, \ L^*=cM+h(M-\mu), \end{cases} \tag{21}$$

where $I_{[\ \ )}(t)$ is the indicator function and $G(t)$ is an arbitar distribution on $[0, M]$ such that $\int_{[0,\, M]} tdG=(\mu-\bar{q}M)/q$.

6. MULTI-PERIOD MODEL

In this section, the demands in successive periods are assumed to form a sequence of random variables whose disributions are contained in $\mathcal{J}_\mu$ and can change from period to period. The multi-period model was treated as a minimax problem in [6-8].

As usual, we define that $f_n(x)$ is the discounted saddle valued cost over n periods as a function of the level x of inventory before ordering. Then the following theorem obviously holds by induction in applying Lemma 1.

Theorem 2, *$f_n(x)$ satisfies the functional equation*

$$f_n(x)=\underset{\substack{y \geqslant x \\ F \in \mathcal{J}_\mu}}{\text{value}} \Big[c(y-x)+\int_{[0,\, \infty)} \{h(y-t)^+ + p(t-y)^+ + \alpha f_{n-1}(y-t)\}\, dF(t)\Big] \quad \ldots(22)$$

where $0 \leqslant \alpha \leqslant 1$ and $f_0(x)=-cx$.

We have assumed $f_0(x)=-cx$ followed by Veinott [10] because a myopic policy for the multi-period inventory problem is optimal in the known demand distribution.

Now, we calculate $f_n(x)$ in this mypoic case, Let us define

$$\tilde{L}(y, F)=(1-\alpha)\, cy+\int_{[0,\, \infty)} \{h(y-t)^+ + p(t-y)^+\}\, dF(t) \tag{23}$$

similarly as (1). Then, (22) is reformed as for $n=1$

$$f_1(x) = \operatorname*{value}_{y \geq x,\, F \in \mathcal{F}_\mu} [\tilde{L}(y, F)] - cx + \alpha c\mu.$$

If $\mathcal{F}_\mu$ is the one of four examples in Section 4, a saddle value $\tilde{L}^*$ and $\tilde{y}^*$ of a set of saddle points $(\tilde{y}^*, \tilde{F}^*)$ of (23) are unique and explicitly solved.

Clearly

$$f_1(x) = \tilde{L}^* + \alpha c\mu - cx \qquad \text{for } x \leqslant \tilde{y}^*.$$

So, it holds that by induction for all $n \geqslant 1$

$$f_n(x) = (1 + \alpha + \dots + \alpha^{n-1}) \ (\tilde{L}^* + \alpha c\mu) - cx \quad \text{for } x \leqslant \tilde{y}^*. \qquad \dots(24)$$

Hence, the following theorem is established.

THEOREM 3. *For each class $\mathcal{F}_\mu$ in Section 4, if the level x of inventory before ordering at period n is less then $\tilde{y}^*$, a set of saddle points for the n-period problem (22) is $(\tilde{y}^*, \tilde{F}^*)$. That is, the policy is stationary and myopic. And the discounted saddle valued cost $f_n(x)$ over n periods is given by (24).*

ACKNOWLEDGEMENT

The authors would like to express thanks to Dr. M. Kurano of Chiba University for informing them the reference [1].

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Received December, 1980