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ON A RANDOMIZED STRATEGY IN NEVEU'S STOPPING PROBLEM

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In Neveu's variant of the stopping problem, a randomized strategy is considered in order to relax a condition on values of two stochastic sequences. We shall describe the variant of the problem as a zero sum two person sequential game and show that a solution for a recursive equation of the game value exists. Neveu's condition reduces the equilibrium solution to a Markov time among the class of randomized strategies.

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optimal stopping problem * game variant * zero sum two person game * randomized strategy

1. Neveu's stopping problem

In the Chapter 6 of Neveu's (1975) book, a modification of the game conceived by Dynkin (1969) is presented as an optimization problem in martingale theory. The game variant of the stopping problem by Dynkin is as follows: Two players observe a stochastic sequence $X(n)$, $n = 1, 2, \dots$. If each of them chooses a strategy, λ, μ respectively both Markov times, the payoff is given by

$$R(\lambda, \mu) = \begin{cases} X(\lambda) & \text{on } \{\lambda \leq \mu\}, \\ X(\mu) & \text{on } \{\lambda > \mu\}. \end{cases} \tag{1.1}$$

The first player is to maximize the expected value of (1.1) and the other is to minimize. Dynkin proved the existence of the game value and optimal strategy with a restriction on the moves of the game. Under the same formulation, Kiefer (1971) obtained another existence condition. Neveu modified the payoff in the Dynkin's problem as follows: There are two preassigned stochastic sequences $X(n)$, $Y(n)$, and for each strategy of the two players λ and μ (which are Markov times and without Dynkin's constraint on moves), the payoff equals

$$R(\lambda, \mu) = \begin{cases} X(\lambda) & \text{on } \{\lambda \leq \mu\}, \\ Y(\mu) & \text{on } \{\lambda > \mu\}, \end{cases} \tag{1.2}$$

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with the condition

$$X(n) \leq Y(n) \quad \text{for each } n. \tag{1.3}$$

Under a regularity condition for the integrability of supremum or infimum of the sequences he proved the existence of the game (min-max) value and ε-optimal (equilibrium) strategy. Assuming the essential condition (1.3), several authors such as Krylov (1970) and Bismut (1977) considered the stopping game problem of random processes.

In this paper we assign three stochastic sequences to the payoff of two players: one sequence is used when Player 1 stops sooner than the other; the second is when Player 2 is faster and the third is when both stop simultaneously. Let $X(n)$, $Y(n)$ and $W(n)$ denote these sequences respectively. This problem is formulated as the multi-stage two person zero sum game (2×2 Matrix game) with the intention of extending a strategy from a Markov time (a pure strategy) to a randomized one. Although the extension is meaningless in a one person problem (Chow, Robbins, Siegmund (1971, in Chap 5.3)), a randomization should be considered in the situation. Neveu discusses the infinite horizon case but we firstly consider the finite horizon case, referring to Chow, Robbins, Siegmund. Then we give a condition for the existence of a game value in the infinite horizon case with a discount factor. This problem resembles Everett's Recursive game in Luce, Raiffa (1972) and the arbitration problem in Chatterjee (1981). A similar stochastic game, in which stopping can occur by mutual agreement, is discussed by Sakaguchi (1980).

2. Randomized strategy

Let $(\Omega, \mathscr{F}, P)$ be a probability space with an increasing sub σ-field $\mathscr{F}(n)$ (alternately $\mathscr{F}_n$) of $\mathscr{F}$. Suppose we are given the trio: $X(n)$, $Y(n)$ and $W(n)$, which are sequences of integrable random variables adapted to $\mathscr{F}(n)$ for each n. We consider the following game: There are two players and each of them chooses as his strategy a stopping time. If λ and μ are the stopping times of the first and the second player respectively, then the corresponding payoff is of the form

$$R(\lambda, \mu) = X(\lambda) I_{\{\lambda<\mu\}} + W(\lambda) I_{\{\lambda=\mu\}} + Y(\mu) I_{\{\lambda>\mu\}} \tag{2.1}$$

where I is an indicator function. That is, the process is stopped when either of the two players declares stop and the payoffs are given according to their declaration. However, without conditions such as (1.3), an equilibrium pair of Markov times does not exist. So we propose that the class of strategies should be extended from a pure one (a Markov time) to a randomized one. The adapted scheme in this paper is as follows:

Definition 2.1. A strategy for each player is a random sequence $p = (p_n) \in \mathscr{P}$ or $q = (q_n) \in \mathscr{Q}$ such that, for each n,

(i) p_n and q_n are adapted to $\mathscr{F}(n)$,

(ii) $0 \leqslant p_n, q_n \leqslant 1$ with probability 1.
If each random variable equals either 0 or 1, we call it a pure strategy.

Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be independent identically distributed random variables of uniform distribution on [0, 1] and independent of $\bigvee_{n=1}^{\infty} \mathscr{F}(n)$. Let $\mathscr{G}(n)$ be the σ-field generated by $\mathscr{F}(n)$, $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$. The probability space is, without loss of generality, rich enough to support the additional randomisation. A randomized stopping time $\lambda(p)$ for a strategy $p = (p_n) \in \mathscr{P}$ and $\mu(q)$ for a strategy $q = (q_n) \in \mathscr{Q}$ are defined, respectively, by

$$\lambda(p) = \inf\{n \geqslant 1: A_n \leqslant p_n\}, \qquad \mu(q) = \inf\{n \geqslant 1: B_n \leqslant q_n\}. \tag{2.2}$$

Definition 2.2. For a strategy $p = (p_n) \in \mathscr{P}$ and $q = (q_n) \in \mathscr{Q}$ of each player, define a payoff:

$$R(p, q) = \begin{cases} X(\lambda(p)) & \text{if } \{\lambda(p) < \mu(q)\}, \\ W(\lambda(p)) = W(\mu(q)) & \text{if } \{\lambda(p) = \mu(q)\}, \\ Y(\mu(q)) & \text{if } \{\lambda(p) > \mu(q)\} \end{cases} \tag{2.3}$$

provided $\lambda(p)$ and $\mu(q)$ are randomized ($\{\mathscr{G}(n)\}$-measurable) stopping times.

Clearly if each p_n is either zero or one, then $\lambda(p)$ is in fact an $\{\mathscr{F}(n)\}$-stopping time, and the strategy is pure and corresponds to a Markov time. In particular an $\{\mathscr{F}(n)\}$-stopping time λ corresponds to the strategy $p = (p_n)$ with $p_n = I_{\{\lambda = n\}}$ where I is an indicator function.

The aim of Player 1 (respectively Player 2) is to make the expectation of the payoff as large (as small) as possible. Firstly we consider a finite horizon case, which is restricted to N stages. Let

$$\mathscr{P}_n^N = \{p = (p_n) \in \mathscr{P};\, p_1 = \cdots = p_{n-1} = 0,\, p_N = 1\} \tag{2.4}$$

denotes all of the strategies between stage n and N for Player 1. Similarly $\mathscr{Q}_n^N$ denotes all of the strategies between n and N for Player 2.

Let

$$\bar{\gamma}_n^N = \operatorname*{essinf}_{\mathscr{Q}_n^N} \operatorname*{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}_n}[R(p, q)], \qquad \underline{\gamma}_n^N = \operatorname*{esssup}_{\mathscr{P}_n^N} \operatorname*{essinf}_{\mathscr{Q}_n^N} E^{\mathscr{F}_n}[R(p, q)]. \tag{2.5}$$

A pair of strategies which coincides with the minimax (infsup in (2.5)) and maximin (supinf in (2.5)) strategy is called an equilibrium strategy. Note that, by (2.2),

$$\tilde{\gamma}_n^N = \operatorname*{essinf}_{M_n^N} \operatorname*{esssup}_{\Lambda_n^N} E^{\mathscr{F}_n}[R(\lambda, \mu)] \quad \text{and} \quad \underset{\sim}{\gamma}_n^N = \operatorname*{esssup}_{\Lambda_n^N} \operatorname*{essinf}_{M_n^N} E^{\mathscr{F}_n}[R(\lambda, \mu)]$$

where Λ_n^N, M_n^N are a class of Markov times such that $n \leqslant \lambda$, $\mu \leqslant N$, satisfy

$$\tilde{\gamma}_n^N > \bar{\gamma}_n^N > \underline{\gamma}_n^N > \underset{\sim}{\gamma}_n^N. \tag{2.6}$$

Following the usual conventions, for example, Luce, Raiffa (1957), let us denote a value of the two person zero sum game with the payoff matrix A by val$[A]$. That is, for a 2×2 matrix $A=(a_{ij})$ with each $\{\mathscr{F}(n)\}$-measurable element,

$$\begin{aligned}\mathrm{val}[A] &= \operatorname*{esssup}_{p_n} \operatorname*{essinf}_{q_n}\{p_nq_na_{11}+p_n(1-q_n)a_{12}+(1-p_n)q_na_{21} \\ &\quad +(1-p_n)(1-q_n)a_{22}\} \\ &= \operatorname*{essinf}_{q_n} \operatorname*{esssup}_{p_n}\{p_nq_na_{11}+p_n(1-q_n)a_{12}+(1-p_n)q_na_{21} \\ &\quad +(1-p_n)(1-q_n)a_{22}\}.\end{aligned}$$

Recursively define the sequence $\gamma_n^N, \gamma_{N-1}^N, \ldots, \gamma_1^N$ by setting

$$\begin{cases}\gamma_n^N = W(N), \\ \gamma_n^N = \mathrm{val}\begin{bmatrix} W(n) & X(n) \\ Y(n) & E^{\mathscr{F}n}[\gamma_{n+1}^N]\end{bmatrix}, \quad n=N-1, N-2, \ldots, 1.\end{cases} \tag{2.7}$$

Theorem 2.1. *$\gamma_n^N = \bar{\gamma}_n^N = \underline{\gamma}_n^N$, $n=1,2,\ldots N$ holds.*

Proof. Firstly we show as in the proof of Proposition VI-6-9 in Neveu (1975) that the operations $\operatorname{essinf}_{\mathscr{Q}_n^N} \operatorname{esssup}_{\mathscr{P}_n^N}$ and the integral with respect to $E^{\mathscr{F}n-1}$ are exchangeable. If $p\in\mathscr{P}_1^N$ and $q\in\mathscr{Q}_1^N$, the random variable $R(p,q)$ is integrable as it is dominated in the absolute value by $\max_{1\le i\le N}\{X(i)+Y(i)+W(i)\}$. Clearly, for every n and q, the family $E^{\mathscr{F}n}[R(p,q)]$, $p\in\mathscr{P}_n^N$ is an increasing directed set. That is, for p^1 and p^2, there exist p such that $E^{\mathscr{F}n}[R(p,q)]=\max\{E^{\mathscr{F}n}[R(p^1,q)], E^{\mathscr{F}n}[R(p^2,q)]\}$. Hence $\operatorname{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}n}[R(p,q)]$ is also integrable. Because

$$R(p,q)\le\{X(\lambda(p))+\max_{n\le i\le N} W(i)\} \quad \text{for all } p\in\mathscr{P}_n^N,\ q\in\mathscr{Q}_n^N,$$

we have

$$E^{\mathscr{F}_{n-1}}[\operatorname*{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}_n}[R(p,q)]] = \operatorname*{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}_{n-1}}[R(p,q)] \tag{2.8}$$

by Proposition VI-1-1 of Neveu (1975). For every n the family of $\operatorname{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}_n}[R(p,q)]$ as q varies over $\mathscr{Q}_n^N$ is a decreasing directed set and so its essential supremum $\bar{\gamma}_n^N$ is again integrable. Therefore we have obtained the formula:

$$E^{\mathscr{F}_{n-1}}[\bar{\gamma}_n^N] = \operatorname*{essinf}_{\mathscr{Q}_n^N} E^{\mathscr{F}_{n-1}}[\operatorname*{esssup}_{\mathscr{P}_n^N} E^{\mathscr{F}_n}[R(p,q)]]. \tag{2.9}$$

We now show that the sequence $\bar{\gamma}_n^N$, $n=1,2,\ldots,N$ satisfies (2.7) by the method of backward induction: For $n=N$ this is trivial by (2.3). Assume the equality of (2.7) holds for $n+1$. Note that, by (2.3) and (2.5),

$$P^{\mathscr{F}n}(A_n\le p_n, B_n\le q_n)=p_nq_n, \qquad E^{\mathscr{F}n}[R(p,q)\colon A_n\le p_n, B_n\le q_n]=W(n)$$

and similarly for other cases. Since

$$\operatorname*{essinf}_{q\in\mathcal{Q}_n^N}\operatorname*{esssup}_{p\in\mathcal{P}_n^N} E^{\mathcal{F}_n}[R(p,q)] = \operatorname*{essinf}_{q_n}\operatorname*{esssup}_{p_n}\operatorname*{essinf}_{q\in\mathcal{Q}_{n+1}^N}\operatorname*{esssup}_{p\in\mathcal{P}_{n+1}^N} E^{\mathcal{F}_n}[R(p,q)]$$

holds, it follows that

$$\bar{\gamma}_n^N = \operatorname*{essinf}_{q_n}\operatorname*{esssup}_{p_n}\{p_n q_n W(n) + p_n(1-q_n)X(n) + (1-p_n)q_n Y(n)$$

$$+(1-p_n)(1-q_n)\operatorname*{essinf}_{\mathcal{Q}_{n+1}^N} E^{\mathcal{F}_n}[\operatorname*{esssup}_{\mathcal{P}_{n+1}^N} E^{\mathcal{F}_{n+1}}[R(p,q)]]\}.$$

Therefore, using (2.8) and (2.9), $\bar{\gamma}_n^N$ satisfies (2.7). Similarly it is proved that $\underline{\gamma}_n^N$ satisfies the same equality. Hence the result is proved.

Corollary 2.2. (i) $\min\{W(n), X(n)\} \leq \gamma_n^N \leq \max\{W(n), Y(n)\}$ *and*
(ii) $\min\{Y(n), E^{\mathcal{F}_n}[\gamma_{n+1}^N]\} \leq \gamma_n^N \leq \max\{X(n), E^{\mathcal{F}_n}[\gamma_{n+1}^N]\}$ *for each n.*

Corollary 2.3. *If*

$$X(n) \leq W(n) \leq Y(n) \tag{2.10}$$

for each n with probability 1, *then*

$$\gamma_n^N = \begin{cases} Y(n) & \text{if } Y(n) \leq E^{\mathcal{F}_n}[\gamma_{n+1}^N], \\ E^{\mathcal{F}_n}[\gamma_{n+1}^N] & \text{if } X(n) \leq E^{\mathcal{F}_n}[\gamma_{n+1}^N] \leq Y(n), \\ X(n) & \text{if } E^{\mathcal{F}_n}[\gamma_{n+1}^N] \leq X(n). \end{cases} \tag{2.11}$$

That is,

$$\gamma_n^N - E^{\mathcal{F}_n}[\gamma_{n+1}^N] = (X(n) - E^{\mathcal{F}_n}[\gamma_{n+1}^N])^+ - (Y(n) - E^{\mathcal{F}_n}[\gamma_{n+1}^N])^- \tag{2.12}$$

where $(a)^+ = \max(a, 0)$ *and* $(a)^- = (-a)^+$. *Therefore a pair of the pure strategy* p^* *and* q^* *such that* $p_n^* = 1$ *if* $\gamma_n^N = X(n)$, $q_n^* = 1$ *if* $\gamma_n^N = Y(n)$ *and* $p_n^* = q_n^* = 0$ *otherwise for each n, gives an equilibrium one.*

The condition (2.10) reduces the equilibrium strategy to a pure one. The equalities (2.11) in the corollary provide a finite horizon case of Neveu's result. It is seen the ransom sequence $W(n)$, $n = 1, 2, \ldots$ is irrelevant for the recursive equation (2.12) because the declaration of both stopping never occurs. Similarly to (2.12), Dynkin's recursive equation is written as

$$\gamma_n^N - E^{\mathcal{F}_n}[\gamma_{n+1}^N] = I_{\{\phi_n>0\}}(X(n) - E^{\mathcal{F}_n}[\gamma_{n+1}^N])^+ - I_{\{\phi_n<0\}}(X(n) - E^{\mathcal{F}_n}[\gamma_{n+1}^N])^- \tag{2.13}$$

where the sequence $\{\phi_n\}$ denotes the moves of the game. That is, Player 1 (resp. Player 2) can select stop when $\phi_n > 0$ ($\phi_n < 0$).

3. Infinite horizon problem with a discount factor

For a strategy $p \in \mathcal{P}_n = \mathcal{P}_n^\infty$ and $q \in \mathcal{Q}_n = \mathcal{Q}_n^\infty$, the terminal time $\tau(p, q)$ in the game equals $\min(\lambda(p), \mu(q))$. However, if the terminal time is not finite, then the payoff cannot be made. Here we set the payoff to be 0 when the game is not terminated. Also we incorporate a discount factor, a constant β with $0<\beta<1$, to assure the convergence of the payoff. Let

$$\bar{\gamma}_n = \operatorname*{essinf}_{\mathcal{Q}_n} \operatorname*{esssup}_{\mathcal{P}_n} E^{\mathcal{F}_n}[\beta^{\tau(p,q)-n} R(p, q)],$$

$$\underline{\gamma}_n = \operatorname*{esssup}_{\mathcal{P}_n} \operatorname*{essinf}_{\mathcal{Q}_n} E^{\mathcal{F}_n}[\beta^{\tau(p,q)-n} R(p, q)]. \tag{3.1}$$

Consider a recursive equation:

$$\gamma_n = \operatorname{val}\begin{bmatrix} W(n) & X(n) \\ Y(n) & \beta E^{\mathcal{F}_n}[\gamma_{n+1}] \end{bmatrix}, \quad n = 1, 2, \ldots \tag{3.2}$$

The next theorem corresponds to Proposition VI-6-9 of Neveu (1975) but we need not assume that $X(n) \leq W(n) = Y(n)$.

Theorem 3.1. *Assume that*

$$E[\sup_n |W(n)|] < \infty, E[\sup_n Y(n)^-] < \infty \quad \textit{and} \quad E[\sup_n X(n)^+] < \infty. \tag{3.3}$$

Then $\bar{\gamma}_n$ and $\underline{\gamma}_n$ coincide for all n and the sequence satisfies the above recursive equation.

Proof. First we note that the assumption (3.3) implies

$$\operatorname*{esssup}_{\mathcal{P}_{n+1}} E^{\mathcal{F}_n}[\beta^{\tau(p,q)-n} R(p, q)] = \beta E^{\mathcal{F}_n}[\operatorname*{esssup}_{\mathcal{P}_{n+1}} E^{\mathcal{F}_{n+1}}[\beta^{\tau(p,q)-n-1} R(p, q)]]$$

because a similar discussion as for (2.8) holds. Since $E^{\mathcal{F}_{n+1}}[\beta^{\tau(p,q)-n-1} R(p, q)]$ is independent of q_n, we obtain that

$$\bar{\gamma}_n = \operatorname*{essinf}_{q_n} \operatorname*{esssup}_{p_n} \{p_n q_n W(n) + p_n(1-q_n) X(n) + (1-p_n) q_n Y(n) + \beta(1-p_n)(1-q_n) E^{\mathcal{F}_n}[\bar{\gamma}_{n+1}]\},$$

similarly to (2.9). Hence the sequence $\bar{\gamma}_n$ satisfies the equation (3.2). Symmetrically it is proved that $\underline{\gamma}_n$ also satisfies (3.2).

Let γ_n be a solution of (3.2) and let $p^* = (p_n^*)$, $q^* = (q_n^*)$ be an associated strategy. That is,

$$\gamma_n = p_n^* q_n^* W(n) + p_n^*(1-q_n^*) X(n) + (1-p_n^*) q_n^* Y(n) + \beta(1-p_n^*)(1-q_n^*) E^{\mathcal{F}_n}[\gamma_{n+1}] \tag{3.4}$$

for each n and $\tau^* = \tau(p^*, q^*)$. By (3.1) and (3.2), it follows that

$$E^{\mathcal{F}_n}[\beta^{\tau^*-n}R(p^*, q^*)] - \gamma_n$$
$$= \beta^{m-n+1}E^{\mathcal{F}_n}\left[\left\{\prod_{k=n}^{m}(1-p_k^*)(1-q_k^*)\right\}\right.$$
$$\left.\times (E^{\mathcal{F}_{m+1}}[\beta^{\tau^*-m-1}R(p^*, q^*)] - \gamma_{m+1})\right]$$

for any $m \geq n$. Since $\beta < 1$, by letting $m \to \infty$, it follows that

$$E^{\mathcal{F}_n}[\beta^{\tau^*-n}R(p^*, q^*)] = \gamma_n. \tag{3.5}$$

Consider a strategy $q^{(m)} = (q_1^{(m)}, q_2^{(m)}, \ldots)$ defined by, for each k,

$$q_k^{(m)} = q_k^*, \quad k > m, \qquad q_k^{(m)} = q_k, \quad k \leq m$$

with the pre-specified $q^* = (q_1^*, q_2^*, \ldots)$ and arbitrary strategy $q = (q_1, q_2, \ldots)$. Since

$$E^{\mathcal{F}_m}[\beta^{\tau(p^*,q^{(m)})-m}R(p^*, q^{(m)})] = E^{\mathcal{F}_m}[\beta^{\tau^*-m}R(p^*, q^{(m)})] = E^{\mathcal{F}_m}[\gamma_{m+1}]$$

is obtained by (3.5), it is immediately seen from (3.4) and the definition of (3.2) that $\gamma_m \leq E^{\mathcal{F}_m}[\beta^{\tau(p^*,q^{(m)})-m}R(p^*, q^{(m)})]$. Iteratively, $\gamma_n \leq E^{\mathcal{F}_n}[\beta^{\tau(p^*,q^{(m)})-n}R(p^*, q^{(m)})]$ holds for each $m \geq n$. Letting $m \to \infty$, we have $\gamma_n \leq E^{\mathcal{F}_n}[\beta^{\tau(p^*,q)-n}R(p^*, q)]$. As the strategy q is arbitrary,

$$\gamma_n \leq \operatorname*{essinf}_{\mathcal{Q}_n} E^{\mathcal{F}_n}[\beta^{\tau(p^*,q)-n}R(p^*, q)]$$
$$\leq \operatorname*{esssup}_{\mathcal{P}_n} \operatorname*{essinf}_{\mathcal{Q}_n} E^{\mathcal{F}_n}[\beta^{\tau(p,q)-n}R(p, q)] = \underline{\gamma}_n.$$

The other inequality $\gamma_n \geq \bar{\gamma}_n$ is proved symmetrically. Hence these show that $\bar{\gamma}_n = \underline{\gamma}_n$ and simultaneously that the solution γ_n in (3.2) constitute the unique sequence.

Corollary 3.2. *If $X(n) \leq W(n) \leq Y(n)$ for each n, than an ε-equilibrium pure strategy exists for arbitrary $\varepsilon > 0$.*

Note that the random variable $W(n)$ is irrelevant, as is seen in (2.12), because the declaration of both stopping never occurs in the case under consideration. To let $\varepsilon = 0$, we must show the terminal time is finite with probability 1, so conditions such as $\lim_n X(n) = -\infty$, $\lim_n Y(n) = \infty$ are needed (refer to Theorem 4.5 in Chow, Robbins, Siegmund). In this paper we give a condition in the next theorem, which is the due to the property of the matrix game.

Theorem 3.3. *In addition to the condition of Theorem* 3.1, *assume that*

$$S_1 \subset S_2 \subset \cdots, \qquad \bigcup_{n=1}^{\infty} S_n = \Omega \tag{3.6}$$

where $S_n = \{Y(n) \leq W(n) \leq X(n)\}$ for each n. Then the terminal time of the game is finite with probability 1 *and an equilibrium strategy exists.*

Proof. Because either player declares stop when the event S_n occurs, the result is immediate.

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