# An interval matrix game and its extensions to fuzzy and stochastic games

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**Abstract**

In this paper, we consider an interval matrix game with interval valued payoffs, which is the generation of the traditional matrix game. The "saddle-points"of this interval matrix game are defined and characterized as equilibrium points of corresponding non-zero sum parametric games. Numerical examples are given to illustrate our idea. These results are extended to the fuzzy matrix games. Also, we formulate two person zero-sum stochastic interval games.

**Keywords**: Interval game, saddle point, interval payoffs, fuzzy payoffs, equilibrium point, parametric game.

## 1 Introduction and notations

In usual matrix game theory(cf. [25, 26]), all the elements of the payoff matrix are assumed to be exactly given. But in a real application, we often encounter the case where the information on the required data includes imprecision or ambiguity because of the uncertain environment. In order to deal with such a case, it is more reasonable to estimate the elements of the payoff matrix by intervals. As for interval approaches to linear programming problem and decision processes, refer, for example, to [7, 21] and [9] respectively.

In this note, we consider the interval matrix game which is an interval generation of the traditional matrix game. The saddle points of the interval matrix game are defined and characterized as equilibrium points of corresponding non-zero sum parametric games. Also, these results are extended to the fuzzy matrix games. Recently, Kurano et al [10] have developed the theory of MDPs in which the immediate rewards are described by use of fuzzy sets. So that, we consider the question whether these results can be extended to stochastic games with interval or fuzzy payoffs. We shall formulate two person zero-sum stochastic interval games in which one-step payoffs are estimated by intervals.

In the reminder of this section, we shall give some notation on interval arithmetics (cf. [16]) and some preliminaries related to the preference relation on intervals.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{C}$ the set of all bounded and closed intervals in $\mathbb{R}$. Note that $\mathbb{R} \subset \mathbb{C}$ by identifying $a \in \mathbb{R}$ with $\boldsymbol{a} = [a, a] \in \mathbb{C}$. We will give a partial order $\succcurlyeq, \succ$ on $\mathbb{C}$ by the following definition.

For $[a,b],[c,d]\in\mathbb{C}$, $[a,b]\succcurlyeq[c,d]$ if $a\geqq c$ and $b\geqq d$ , and $[a,b]\succ[c,d]$ if $[a,b]\succcurlyeq[c,d]$ and $[a,b]\neq[c,d]$. The Hausdorff metric(cf. [13]) on $\mathbb{C}$ is defined by $\delta$, i.e.,

$$\delta([a,b],[c,d]):=|a-c|\vee|b-d| \quad \text{for} \quad [a,b],[c,d]\in\mathbb{C},$$

where $x\vee y=\max\{x,y\}$. Obviously, the metric space $(\mathbb{C},\delta)$ is complete.

The following arithmetics are used in the sequel.

For $[a,b],[c,d]\in\mathbb{C}$ and $\lambda\in\mathbb{R}$ $(\lambda\geq 0)$,

$$[a,b]+[c,d]=[a+c,b+d], \tag{1.1}$$

$$\lambda[a,b]=[\lambda a,\lambda b]. \tag{1.2}$$

Then, we have the following.

**Lemma 1.1** *For any $[a,b],[a',b'],[c,d],[c',d']\in\mathbb{C}$ and $\lambda\in\mathbb{R}$ $(\lambda\geq 0)$.*

(i) $\delta(\lambda[a,b],\lambda[a',b'])=\lambda\delta([a,b],[a',b'])$. *(scalar)*

(ii) $\delta([a,b]+[a',b'],[c,d]+[c',d'])\leq\delta([a,b],[c,d])+\delta([a',b'],[c',d'])$. *(triangle)*

(iii) $\delta([a,b]+[a',b'],[a,b]+[c',d'])=\delta([a',b'],[c',d'])$. *(shift)*

Let $\mathbb{C}_+:=\{\boldsymbol{a}\in\mathbb{C}\mid\boldsymbol{a}=[a,b]\succcurlyeq[0,0]\}$ be the set of nonnegative intervals. Let $\mathbb{C}^m$ and $\mathbb{C}^{m\times n}$ be the set of all $m$-dimensional column vectors and $m\times n$ matrices, called interval vectors and interval matrices respectively, whose elements are in $\mathbb{C}$, i.e.,

$$\mathbb{C}^m:=\{\boldsymbol{a}=(\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_m)^t\mid\boldsymbol{a}_i\in\mathbb{C}\ (1\leqq i\leqq m)\},$$

$$\mathbb{C}^{m\times n}:=\{\boldsymbol{A}=(\boldsymbol{a}_{ij})\mid(\boldsymbol{a}_{ij})\in\mathbb{C}\ (1\leqq i\leqq m,1\leqq j\leqq n)\}.$$

We shall identify $m\times 1$ interval matrices with interval vectors and $1\times 1$ interval matrices with intervals, so that $\mathbb{C}=\mathbb{C}^{1\times 1}$ and $\mathbb{C}^m=\mathbb{C}^{m\times 1}$. Also, we denote by $\mathbb{C}^m_+$ and $\mathbb{C}^{m\times n}_+$ the subsets of componentwise non-negative elements in $\mathbb{C}^m$ and $\mathbb{C}^{m\times n}$. We equip $\mathbb{C}^{m\times n}$ with componentwise relations $\preccurlyeq$, $\prec$, $\succcurlyeq$, $\succ$. Similarly, we can define $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$ as the set of real $m$-dimensional column vectors and real $m\times n$ matrices. Note that $\mathbb{R}^{m\times n}\subset\mathbb{C}^{m\times n}$.

For any $\boldsymbol{A}=(\boldsymbol{a}_{ij})\in\mathbb{C}^{m\times n}$ with $\boldsymbol{a}_{ij}=[a^-_{ij},a^+_{ij}]$, $\boldsymbol{A}$ will be denoted by $\boldsymbol{A}=[A^-,A^+]$, where $A^-=(a^-_{ij})\in\mathbb{R}^{m\times n}$, $A^+=(a^+_{ij})\in\mathbb{R}^{m\times n}$ and $[A^-,A^+]=\{A\in\mathbb{R}^{m\times n}\mid A^-\preccurlyeq A\preccurlyeq A^+\}$.

For $\boldsymbol{A}=(\boldsymbol{a}_{ij}),\boldsymbol{B}=(\boldsymbol{b}_{ij})\in\mathbb{C}^{m\times n}$ and $\lambda\in R_+$,

$$\boldsymbol{A}+\boldsymbol{B}=\{A+B\mid A\in\boldsymbol{A}\ \text{ and }\ B\in\boldsymbol{B}\} \tag{1.1'}$$

$$\lambda\boldsymbol{A}=\{\lambda A\mid A\in\boldsymbol{A}\}, \tag{1.2'}$$

where for $C=(c_{ij})$ and $D=(d_{ij})\in\mathbb{R}^{m\times n}, C+D=(c_{ij}+d_{ij})$. Observing $\boldsymbol{A}+\boldsymbol{B}=[A^-+B^-,A^++B^+]\in\mathbb{C}^{m\times n}$.

For any given $\mathbb{D}\subset\mathbb{C}$, $\boldsymbol{c}$ is called *a minimal (maximal) point* of $\mathbb{D}$ if

$$\{\boldsymbol{d}\in\mathbb{D}\mid\boldsymbol{d}\prec\ (\succ)\ \ \boldsymbol{c}\}=\emptyset. \tag{1.3}$$

The set of all minimal (maximal) point of $\mathbb{D}$ will be defined by $\min\mathbb{D}(\max\mathbb{D})$ (cf. [20, 24]).

Since the partial order $\preccurlyeq$ on $\mathbb{C}$ is equivalent to the vector ordering on $\mathbb{R}^2$ with $\mathbb{R}^2_+$ as the corresponding order cone, the following fact follows easily (cf. [2, 20]).

**Lemma 1.2** *Let $\mathbb{D}$ be a compact and convex subset of $\mathbb{C}$. Then $[a,b]\in\min\mathbb{D}(\max\mathbb{D})$ if and only if there exists $\gamma\in(0,1)$ such that $\gamma a+(1-\gamma)b\leqq(\geqq)\gamma a'+(1-\gamma)b'$ for all $[a',b']\in\mathbb{D}$.*

In Section 2, an interval matrix game is specified and its saddle points are characterized as equilibrium points of the corresponding non-zero sum parametric game. A fuzzy matrix game is investigated in Section 3. In Section 4, a numerical example is given to illustrate our arguments. In order to formulate the interval stochastic game we need the concept of a expectation of interval-valued random variables.

Let $(\Omega, \mathscr{B}, P)$ be a probability space and $\boldsymbol{r} : \Omega \to \mathbb{C}$ a discrete random quantity with its range $\mathscr{R}(\boldsymbol{r}) = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_l\} \subset \mathbb{C}$. Then, we define the expectation of $\boldsymbol{r}$ by

$$E[\boldsymbol{r}] = \sum_{i=1}^{l} \boldsymbol{c}_i P(\boldsymbol{r} = \boldsymbol{c}_i). \tag{1.4}$$

Note that arithmetics in (1.4) is given in (1.1) and (1.2) and $E[\boldsymbol{r}] \in \mathbb{C}$. The definition of (1.4) is corresponding to the discrete case of the expectation of general fuzzy random variables (cf. [18]).

In Section 5 stochastic interval games are specified and their saddle points are defined, which are characterized as equilibrium points of corresponding nonzero-sum parametric stochastic games in Section 6. In Section 7, stochastic interval game are extended to the case of the multi-dimensional fuzzy payoffs.

## 2 Interval matrix games

The two person interval matrix game is defined by the $m \times n$ interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{C}^{m \times n}$, where player 1(maximizer) and player 2(minimizer) have $m$ pure strategies $\{i \mid i = 1, 2, \ldots, m\}$ and $n$ pure strategies $\{j \mid j = 1, 2, \ldots, n\}$ and if player 1 and 2 select $i(1 \leqq i \leqq m)$ and $j(1 \leqq j \leqq n)$ respectively, the payoff for player 1 is estimated by the interval $\boldsymbol{a}_{ij} \in \mathbb{C}$.

Let $X$ and $Y$ be the set of all mixed strategies for player 1 and 2 respectively, i.e.,

$$X = \{x = (x_1, x_2, \ldots, x_m)^t \in \mathbb{R}^m_+ \mid \sum_{j=1}^{m} x_i = 1\},$$

$$Y = \{y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} y_i = 1\}.$$

Then, for any selected pair of strategies $(x, y) \in X \times Y$ the expected payoff for player 1 is estimated by

$$f(x, y) := x^t \boldsymbol{A} y = \sum_{i,j} x_i y_j \boldsymbol{a}_{ij}. \tag{2.1}$$

By arithmetics in (1.1') and (1.2') the following holds obviously.

**Lemma 2.1.** *For any $x \in X$ and $y \in Y$, it holds that*

$$f(x, y) = [x^t A^- y, x^t A^+ y] \in \mathbb{C}. \tag{2.2}$$

**Definition 1.** (cf. [[14], [24]]) Let $(x^*, y^*) \in X \times Y$ and $\boldsymbol{A} \in \mathbb{C}^{m \times n}$. Then $(x^*, y^*)$ is said to be a *saddle point of the interval matrix game* $\boldsymbol{A}$ if the following holds:

$$f(x^*, y^*) \in \max f(X, y^*) \cap \min f(x^*, Y), \tag{2.3}$$

where for any $(x, y) \in X \times Y$, $f(X, y) = \{f(x', y) \mid x' \in X\}$ and $f(x, Y) = \{f(x, y') \mid y' \in Y\}$.

We note that $f(X, y)$ and $f(x, Y)$ are compact and convex subset of $\mathbb{C}$.

In order to characterize the saddle point of the interval matrix game $\boldsymbol{A}$, we introduce a parametric matrix game $A(\gamma)$. For each $\gamma \in (0, 1)$ and $\boldsymbol{A} = [A^-, A^+] \in \mathbb{C}^{m \times n}$, let $A(\gamma) = \gamma A^+ + (1 - \gamma)A^-$.

**Definition 2.** For any $\gamma, \gamma' \in [0, 1]$, the point $(x^*, y^*) \in X \times Y$ is said to be a $(\gamma, \gamma')$-*equilibrium point* for a non-zero sume game $(A(\gamma), A(\gamma'))$ if the following (i)–(ii) holds:

(i) $x^t A(\gamma) y^* \leqq x^{*t} A(\gamma) y^*$ for all $x \in X$,

(ii) $x^{*t} A(\gamma') y \geqq x^{*t} A(\gamma') y^*$ for all $y \in Y$.

We note that the $(\gamma, \gamma)$-equilibrium point $(x^*, y^*)$ for a non-zero sume game $(A(\gamma), A(\gamma))$ means that $(x^*, y^*)$ is a saddle point for the zero sume matrix game $A(\gamma)$, i.e.,

$$x^t A(\gamma) y^* \leqq x^{*t} A(\gamma) y^* \leqq x^{*t} A(\gamma) y \quad \text{for all} \quad x \in X \quad \text{and} \quad y \in Y. \tag{2.4}$$

Also, every non-zero sum finite game has an equilibrium point(cf. [15, 26]), so that for any $\gamma, \gamma' \in [0, 1]$, a $(\gamma, \gamma')$-equilibrium point exists. Applying Lemma 1.1 and 2.1, we have the following.

**Theorem 2.1.** *A point $(x^*, y^*) \in X \times Y$ is a saddle point for the interval matrix game $\boldsymbol{A}$ if and only if there exist $\gamma, \gamma' \in (0, 1)$ such that $(x^*, y^*)$ is a $(\gamma, \gamma')$-equilibrium point. for a non-zero sume game $(A(\gamma), A(\gamma'))$*

**Proof.** By Lemma 1.1 and 2.1, that $f(x^*, y^*) \in \min f(x^*, Y)$ means that there exists $\gamma' \in (0, 1)$ satisfying

$$\gamma' x^{*t} A^+ y^* + (1 - \gamma') x^{*t} A^- y^* \leqq \gamma' x^{*t} A^+ y + (1 - \gamma') x^{*t} A^- y \quad \text{for all} \quad y \in Y. \tag{2.5}$$

Obviously, (2.5) is rewritten as follows.

$$x^{*t} A(\gamma') y^* \leqq x^{*t} A(\gamma') y \quad \text{for all} \quad y \in Y. \tag{2.6}$$

which is corresponding with (ii) of Definition 2.

Similarly, $f(x^*, y^*) \in \max f(X, y^*)$ means that there exists $\gamma \in (0, 1)$ such that

$$x^{*t} A(\gamma) y^* \geqq x^t A(\gamma) y^* \quad \text{for all} \quad x \in X. \tag{2.7}$$

Thus, the proof is complete. □

The following results easily follow from Theorem 2.1.

**Corollary 2.1.** *If the point $(x^*, y^*) \in X \times Y$ is a saddle point for the matrix game $A(\gamma)$ ($\gamma \in (0, 1)$), $(x^*, y^*)$ is a saddle point for the interval matrix game $\boldsymbol{A}$.*

**Corollary 2.2.** *For any $\boldsymbol{A} = ([a_{ij}^-, a_{ij}^+]) \in \mathbb{C}^{m \times n}$ with $a_{ij}^+ - a_{ij}^- = c$ independent of $i$ and $j$ $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$, the saddle point $(x^*, y^*)$ of $\boldsymbol{A}$ is uniquely determined as a saddle point for the matrix game $A^- = (a_{ij}^-)$.*

**Proof.** We note that $A(\gamma)$ is rewritten as $A(\gamma) = A^- + \gamma(A^+ - A^-)$. So that if $A^+ - A^- = cE$, $A(\gamma)$ and $A(\gamma')$ is essentially equivalent for any $\gamma, \gamma' \in (0, 1)$, where all the elements of $E \in \mathbb{R}^{m \times n}$ are 1. Thus, the statement of Corollary 2.2 follows obviously. □

The following is useful in finding the saddle point of the interval matrix game $\boldsymbol{A}$.

**Corollary 2.3** [cf. [22]]. The point $(x^*, y^*) \in X \times Y$ is a saddle point for the interval matrix game $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ if and only if there exist $\gamma, \gamma' \in (0,1)$ such that $(x^*, y^*)$ is a part of a solution to

$$
(2.8) \qquad \begin{cases} A(\gamma')y + \nu = x^t A(\gamma')y 1_m \\ x^t A(\gamma) - \mu^t = x^t A(\gamma) y 1_n^t \\ \nu^t x = 0, \qquad \mu^t y = 0 \\ x^t 1_m = 1, \quad y^t 1_n = 1 \\ x, \nu \in \mathbb{R}^m_+, \quad y, \mu \in \mathbb{R}^n_+, \end{cases}
$$

where $1_m = (1, \ldots, 1)' \in \mathbb{R}^m_+$ and $1_n = (1, \ldots, 1)' \in \mathbb{R}^n_+$.

**Remark.** On the interval matrix game $\boldsymbol{A}$, if player 1(2) selects the stratrgy $i(j)$ player 1(2) receives(loses) the iterval valued payoff $\boldsymbol{a}_{ij} = [a^-_{ij}, a^+_{ij}] \in \mathbb{C}$, where the actual value of the payoff is not known precisely for both players like a value of a beautiful ancient urn. In general, $[a^-_{ij}, a^+_{ij}] + [-a^+_{ij}, -a^-_{ij}] = [a^-_{ij} - a^+_{ij}, a^+_{ij} - a^-_{ij}] \neq 0$ ($\ni 0$) then the interval matrix game $\boldsymbol{A}$ is not a zero sum game in the strict sense of the word. The player 1 and 2 may consider the interval game $\boldsymbol{A}$ as a non-zero sum game $(A(\gamma), A(\gamma'))$ for some parameters $\gamma$ and $\gamma'$, where the parametric game $A(\gamma)$ and $A(\gamma')$ for player 1 and 2 may be their subjective values for the intervlal game $\boldsymbol{A}$. Consider the extreme case $(\gamma, \gamma') = (0, 1)$, a $(0,1)$-equilibrium point $(x^*, y^*) \in X \times Y$ means that

(i) $x^t A^- y^* \leqq x^{*t} A^- y^*$ for all $A \in \boldsymbol{A}$ and $x \in X$,

(ii) $x^{*t} A^+ y \geqq x^{*t} A^+ y^*$ for all $A \in \boldsymbol{A}$ and $y \in Y$.

This shows that $(x^*, y^*)$ guarantees the best in the worst case for both players. Thus, $(0,1)$-equilibrium point $(x^*, y^*)$ will be called a pessimistic-pessimistic pair. By the same discussion as the above, the $(1,0)$-equilibrium point $(x^*, y^*)$ will be called an optimistic-optimistic pair. Then the parameter $\gamma$ $(0 \leq \gamma \leq 1)$ is a grade of optimism for player 1 or a grade of pessimism for player 2.

# 3 Extensions to fuzzy games

In this section, the results in the preceding section will be extended to the multi-dimensional fuzzy payoff games.

We write a fuzzy set on $\mathbb{R}^p$ by its membership function $\widetilde{s} : \mathbb{R}^p \to [0,1]$ (see Novák [17] and Zadeh [27]). The $\alpha$-cut $(\alpha \in [0,1])$ of the fuzzy set $\widetilde{s}$ on $\mathbb{R}^p$ is defined as

$$\widetilde{s}_\alpha := \{x \in \mathbb{R}^p \mid \widetilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \widetilde{s}_0 := \mathrm{cl}\{x \in \mathbb{R}^p \mid \widetilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\widetilde{s}$ is called convex if

$$\widetilde{s}(\lambda x + (1-\lambda)y) \geq \widetilde{s}(x) \wedge \widetilde{s}(y) \quad x, y \in \mathbb{R}^p, \ \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\widetilde{s}$ is convex if and only if the $\alpha$-cut $\widetilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^p)$ be the set of all convex fuzzy sets whose membership functions $\widetilde{s} : \mathbb{R}^p \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x\in\mathbb{R}^p} \widetilde{s}(x) = 1$) and have a compact support. In the one-dimensional case $n = 1$, $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. Let $\mathbb{C}(\mathbb{R}^p)$ be the set of all compact convex subsets of $\mathbb{R}^p$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^p)$ are as follows: For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$ and $\lambda \geq 0$,

$$(\widetilde{s}+\widetilde{r})(x) := \sup_{\substack{x_1,x_2\in\mathbb{R}^p \\ x_1+x_2=x}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\}, \tag{3.1}$$

$$(\lambda\widetilde{s})(x) := \begin{cases} \widetilde{s}(x/\lambda) & \text{if } \lambda > 0 \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^p), \tag{3.2}$$

where $\mathbf{1}_{\{\cdot\}}(\cdot)$ is an indicator.

By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}^p$, the following holds immediately.

$$(\widetilde{s}+\widetilde{r})_\alpha = \widetilde{s}_\alpha + \widetilde{r}_\alpha \quad \text{and} \quad (\lambda\widetilde{s})_\alpha = \lambda\widetilde{s}_\alpha \quad (\alpha \in [0,1]). \tag{3.3}$$

Let $K$ be a non-empty cone of $\mathbb{R}^p$. Using this $K$, we can define a pseudo order relation $\preccurlyeq_K$ on $\mathbb{R}^p$ by $x \preccurlyeq_K y$ if and only if $y - x \in K$. We introduce a pseudo order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^p)$ (cf. [8]). Let $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$. The relation $\widetilde{s} \preccurlyeq_K \widetilde{r}$ means the following (i) and (ii):

(i) For any $x \in \mathbb{R}^p$, there exists $y \in \mathbb{R}^p$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \leq \widetilde{r}(y)$.

(ii) For any $y \in \mathbb{R}^p$, ther exists $x \in \mathbb{R}^p$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \geq \widetilde{r}(y)$.

For any $a \in \mathbb{R}^p$ and $\boldsymbol{d} \in \mathbb{C}(\mathbb{R}^p)$, the product of $a$ and $\boldsymbol{d}$ is defined as

$$a\boldsymbol{d} = \{a^t d \mid d \in \boldsymbol{d}\}. \tag{3.4}$$

We note that $a\boldsymbol{d} \in \mathbb{C}$.

**Lemma 3.1** [8]. *For any $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ if and only if $a\widetilde{s}_\alpha \leq a\widetilde{r}_\alpha$ for all $a \in K^+$ and $\alpha \in [0,1]$.*

Here, we consider the two person fuzzy matrix game defined by the $m \times n$ fuzzy matrix $\widetilde{\boldsymbol{A}} = (\widetilde{\boldsymbol{a}}_{ij}) \in \mathcal{F}(\mathbb{R}^p)^{m\times n}$. For any $x = (x_1, x_2, \ldots, x_m)^t \in X$ and $y = (y_1, y_2, \ldots, y_n)^t \in Y$, the expected payoff for player 1 is estimated (cf. [18]) by

$$f(x,y) := x^t\widetilde{\boldsymbol{A}}y = \sum x_i y_j \widetilde{\boldsymbol{a}}_{ij}. \tag{3.5}$$

We note that $f(x,y) \in \mathcal{F}(\mathbb{R}^p)$ and its $\alpha$-cut is given by

$$f(x,y)_\alpha = \sum x_i y_j \widetilde{\boldsymbol{a}}_{ij,\alpha} \in \mathbb{C}(\mathbb{R}^p), \tag{3.6}$$

where $\widetilde{\boldsymbol{a}}_{ij,\alpha}$ is the $\alpha$-cut of $\widetilde{\boldsymbol{a}}_{ij}$.

The saddle point of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ is defined similarly as that of the interval matrix game (see Definition 1 in Section 2).

For any $a \in \mathbb{R}^p$, noting $a\widetilde{\boldsymbol{a}}_{ij,\alpha} \in \mathbb{C}$, we denote $a\widetilde{\boldsymbol{a}}_{ij,\alpha}$ by $[\widetilde{a}^-_{ij,\alpha}(a), \widetilde{a}^+_{ij,\alpha}(a)]$ and set $A^-_\alpha(a) := (\widetilde{a}^-_{ij,\alpha}(a)) \in \mathbb{R}^{m\times n}$ and $A^+_\alpha(a) := (\widetilde{a}^+_{ij,\alpha}(a)) \in \mathbb{R}^{m\times n}$. Here, for $\alpha \in [0,1], \gamma \in (0,1)$ and $a \in \mathbb{R}^p$, we put

$$A_{\alpha,a}(\gamma) = \gamma A^+_\alpha(a) + (1-\gamma)A^-_\alpha(a). \tag{3.7}$$

Then, the saddle points of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ will be characterized in the following theorem, whose proof is done by applying Lemma 3.1 and the ideas used in Section 2.

**Theorem 3.1.** *A point $(x^*, y^*) \in X \times Y$ is a saddle point of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ if and only if there exist two functions $\gamma, \gamma' : [0,1] \times K^+ \to (0,1)$ such that*

$$\begin{aligned} x^t A_{\alpha,a}(\gamma(\alpha,a))y^* &\leqq x^{*t} A_{\alpha,a}(\gamma(\alpha,a))y^* \\ x^{*t} A_{\alpha,a}(\gamma'(\alpha,a))y &\geqq x^{*t} A_{\alpha,a}(\gamma'(\alpha,a))y^* \end{aligned} \tag{3.8}$$

*for all $\alpha \in [0,1]$ and $a \in K^+$.*

## 4 Numerical Example

Here, we give numerical examples.

**Example 1.** Let $\boldsymbol{A} = \begin{pmatrix} [2,4], & [-2,0] \\ [0,2], & [1,3] \end{pmatrix} \in \mathbb{C}^{2\times 2}$.

Noting that $A^- = \begin{pmatrix} 2, & -2 \\ 0, & 1 \end{pmatrix}$ and $A^+ = \begin{pmatrix} 4, & 0 \\ 2, & 3 \end{pmatrix}$ and $A^+ - A^- = \begin{pmatrix} 2, & 2 \\ 2, & 2 \end{pmatrix}$. Thus, by Corollary 2.2, a saddle point $(x^*, y^*)$ of $\boldsymbol{A}$ is unique and given by a sadle point for $A^-$. After a simple calculation, we find that $x^* = \left(\frac{1}{5}, \frac{4}{5}\right)$, $y^* = \left(\frac{3}{5}, \frac{2}{5}\right)$ and $f(x^*, y^*) = \left[\frac{2}{5}, \frac{12}{5}\right]$.

**Example 2.** Let $\boldsymbol{A} = \begin{pmatrix} [3,4], & \left[-\frac{3}{2}, \frac{1}{2}\right] \\ \left[\frac{1}{2}, \frac{3}{2}\right], & [1,2] \end{pmatrix}$ with $A^- = \begin{pmatrix} 3, & -\frac{3}{2} \\ \frac{1}{2}, & 1 \end{pmatrix}$ and $A^+ = \begin{pmatrix} 4, & \frac{1}{2} \\ \frac{3}{2}, & 2 \end{pmatrix}$.

Noting $A(\gamma) = \begin{pmatrix} \gamma+3, & 2\gamma - \frac{3}{2} \\ \gamma + \frac{1}{2}, & \gamma+1 \end{pmatrix}$, for each $\gamma \in (0,1)$, we solve the parametric equation (2.8) and find that the $(\gamma, \gamma')$-equilibrium point $(x^*, y^*)$ is given by

$$x^* = \left(\frac{1}{10-2\gamma} \frac{9-2\gamma}{10-2\gamma}\right), y^* = \left(\frac{5-2\gamma'}{1-2\gamma'}, \frac{5}{1-2\gamma'}\right) \text{ with}$$

$$f(x^*, y^*) = \left[\frac{2\gamma\gamma' - 15\gamma' - 15\gamma + 75}{(10-2\gamma)(10-2\gamma')}, \frac{6\gamma\gamma' - 35\gamma - 35\gamma' + 75}{(10-2\gamma)(10-2\gamma')}\right].$$

By Theorem 2.1, the set of all saddle points is specified by the set of all $(\gamma, \gamma')$-equilibrium points. Some saddle points and their values are given in Table 1.

| | $x^*$ | $y^*$ | $f(x^*, y^*)$ |
|---|---|---|---|
| $\gamma = \frac{1}{3}\ \gamma' = \frac{1}{3}$ | $(\frac{3}{28}, \frac{25}{28})$ | $(\frac{13}{28}, \frac{15}{28})$ | $[\frac{587}{784}, \frac{177}{98}]$ |
| $\gamma = \frac{2}{3}\ \gamma' = \frac{1}{3}$ | $(\frac{3}{26}, \frac{23}{26})$ | $(\frac{13}{28}, \frac{15}{28})$ | $[\frac{68}{91}, \frac{1317}{728}]$ |
| $\gamma = \frac{1}{3}\ \gamma' = \frac{2}{3}$ | $(\frac{3}{28}, \frac{25}{28})$ | $(\frac{11}{26}, \frac{15}{26})$ | $[\frac{68}{91}, \frac{1317}{728}]$ |
| $\gamma = \frac{2}{3}\ \gamma' = \frac{2}{3}$ | $(\frac{3}{26}, \frac{23}{26})$ | $(\frac{11}{26}, \frac{15}{26})$ | $[\frac{503}{676}, \frac{306}{169}]$ |

Table 1. Saddle points and their values.

## 5 Interval stochastic game

In this section, we formulate two person zero-sum stochastic games with interval payoffs, called *interval stochastic games*, and define the saddle points under a criterion of discounted interval gains.

A two person interval stochastic game is determined by five objects:

$$\{S, A, B, \boldsymbol{r}, q\}.$$

Where $S = \{1, 2, \ldots, N\}$ denotes the state space, $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$ denote the set of actions available to player 1 (maximizer) and player 2 (minimizer) respectively. An interval-valued map $\boldsymbol{r} : S \times A \times B \to \mathbb{C}$ denotes interval estimate of one-step payoff function and $q = \{q_{ss'}(i, j) \mid s, s' \in S, i \in A, j \in B\}$ is a transition law, i.e., $q_{ss'}(i, j) \geq 0$ and $\sum_{s' \in S} q_{ss'}(i, j) = 1$ for $s, s' \in S, i \in A, j \in B$.

A game is played as follows: At each time of epoch, two players observe the current state $s \in S$ of the system and players 1 and 2 independently choose actions $i \in A$ and $j \in B$ respectively. Then two events happen; (i) player 1 receives an immediate payoff estimated by the interval $\boldsymbol{r}(s, i, j) \in \mathbb{C}$ and (ii) the system moves to a new state $s' \in S$ selected according to the distribution $q_{s\cdot}(i, j)$. This process is then repeated from the new state $s' \in S$.

The sample space is the product space $\Omega = (S \times A \times B)^\infty$ such that the projection $X_t, \Delta_t^A$ and $\Delta_t^B$ on the $t$-th factor $S, A$ and $B$ describe the state and the actions chooses respectively by players 1 and 2 at the $t$-th time of the process $(t = 1, 2, \ldots)$. Let $\mathcal{P}(A)$ and $\mathcal{P}(B)$ be the sets of all probability distributions on $A$ and $B$ respectively, i.e.,

$$\mathcal{P}(A) = \{x = (x_1, x_2, \ldots, x_m) \mid x_i \geq 0, \sum_{i=1}^{m} = 1\}$$

and

$$\mathcal{P}(B) = \{y = (y_1, y_2, \ldots, y_n) \mid y_j \geq 0, \sum_{j=1}^{n} = 1\}.$$

A (stationary) strategy $\pi$ and $\sigma$ for player 1 and 2 are sets of probability distributions $\{\pi(\cdot|s) \mid s \in S\} \subset \mathcal{P}(A)$ and $\{\sigma(\cdot|s) \mid s \in S\} \subset \mathcal{P}(B)$ respectively. The sets of all stationary strategies for player 1 and 2 will be denoted by $\Pi$ and $\Sigma$. We assume that for each pair $(\pi, \sigma) \in \Pi \times \Sigma$ with $s, s' \in S, i \in A, j \in B$ and $t \geq 1$,

$$\text{Prob}\{X_{t+1} = s' \mid X_1, \Delta_1^A, \Delta_1^B, \cdots, X_t = s, \Delta_t^A = i, \Delta_t^B = j\} = q_{ss'}(i, j),$$

$$\text{Prob}\{\Delta_t^A = i \mid X_1, \Delta_1^A, \Delta_1^B, \cdots, X_t = s\} = \pi(i|s)$$

and

$$\text{Prob}\{\Delta_t^B = j \mid X_1, \Delta_1^A, \Delta_1^B, \cdots, X_t = s\} = \sigma(j|s).$$

Then, the initial state $s \in S$ and the pair of strategies $(\pi, \sigma) \in \Pi \times \Sigma$ determine the probability measure $P_{\pi,\sigma}^s$ on $\Omega$ by the usual way.

Here, we consider the total expected payoff for player 1 in which the future payoff is discounted with a factor $\beta$ $(0 < \beta < 1)$. For any pair $(\pi, \sigma) \in \Pi \times \Sigma$ and any starting state $s \in S$, let

$$\mathscr{I}_T(s, \pi, \sigma) = \sum_{t=1}^{T} \beta^{t-1} E_{\pi,\sigma}^s[\boldsymbol{r}(X_t, \Delta_t^A, \Delta_t^B)], \tag{5.1}$$

where $E_{\pi,\sigma}^s$ is the expectation with respect to $P_{\pi,\sigma}^s$. Obviously, $\mathscr{I}_T(s, \pi, \sigma) \in \mathbb{C}$.

**Lemma 5.1** *For any pair $(\pi, \sigma) \in \Pi \times \Sigma$ and any starting state $s \in S$, $\{\mathscr{I}_T(s, \pi, \sigma)\}_{T=1}^\infty$ is a Cauchy sequence with respect to the Hausdorff metric $\delta \in \mathbb{C}$.*

**Proof.** For any $T > H$, it holds from Lemma 1.1 (iii) that

$$\begin{aligned}
&\delta(\mathscr{I}_T(s,\pi,\sigma), \mathscr{I}_H(s,\pi,\sigma)) \\
&= \delta(0, \sum_{t=H+1}^{T} \beta^{t-1} E^s_{\pi,\sigma}[\boldsymbol{r}(X_t, \Delta_t^A, \Delta_t^B)]) \\
&= \beta^H \delta(0, \sum_{t=H+1}^{T} \beta^{t-H-1} E^s_{\pi,\sigma}[\boldsymbol{r}(X_t, \Delta_t^A, \Delta_t^B)]) \\
&\le \beta^H \max_{s\in S, i\in A, j\in B} \delta(0, \boldsymbol{r}(s,i,j))/(1-\beta).
\end{aligned}$$

This completes the proof. $\square$

From Lemma 5.1, the infinite horizon total expected payoff for player 1 can be defined by

$$\mathscr{I}(s,\pi,\sigma) = \lim_{T\to\infty} \mathscr{I}_T(s,\pi,\sigma). \tag{5.2}$$

Since $\mathscr{I}(s,\pi,\sigma) \in \mathbb{C}$, it will be written as

$$\mathscr{I}(s,\pi,\sigma) = [\mathscr{I}^-(s,\pi,\sigma), \mathscr{I}^+(s,\pi,\sigma)]. \tag{5.3}$$

For any pair $(\pi,\sigma) \in \Pi \times \Sigma$ and $s \in S$, let

$$\mathscr{I}(s,\Pi,\sigma) = \{\mathscr{I}(s,\pi,\sigma) \mid \pi \in \Pi\} \quad \text{and} \quad \mathscr{I}(s,\pi,\Sigma) = \{\mathscr{I}(s,\pi,\sigma) \mid \sigma \in \Sigma\}.$$

The following is easily shown by applying the idea of Borkar's discounted occupation measure (cf. Theorem 1.2 [3]).

**Lemma 5.2** *For any pair $(\pi,\sigma) \in \Pi \times \Sigma$ and $s \in S$, $\mathscr{I}(s,\Pi,\sigma)$ and $\mathscr{I}(s,\pi,\Sigma)$ are compact and convex subsets of $\mathbb{C}$.*

**Definition 1'** (cf. [20, 24]) Let $(\pi^*,\sigma^*) \in \Pi \times \Sigma$ and $s \in S$. Then, the pair $(\pi^*,\sigma^*)$ is said to be a saddle point at $s \in S$ for the interval stochastic game if the following holds.

$$\mathscr{I}(s,\pi^*,\sigma^*) \in \max \mathscr{I}(s,\Pi,\sigma^*) \cap \min \mathscr{I}(s,\pi^*,\Sigma).$$

## 6 Characterization of saddle points

In order to characterize the saddle point we introduce a parametric stochastic game.

For any *gamma* $\in [0,1]$, we put

$$r^\gamma(s,i,j) = \gamma \boldsymbol{r}^+(s,i,j) + (1-\gamma)\boldsymbol{r}^-(s,i,j) \in \mathbb{R} \quad (s \in S, i \in A, j \in B), \tag{6.1}$$

where $\boldsymbol{r}^-$ and $\boldsymbol{r}^+$ are extreme points of the interval $\boldsymbol{r}$ and $\boldsymbol{r} = [\boldsymbol{r}^-(s,i,j), \boldsymbol{r}^+(s,i,j)]$.

For any pair $(\pi,\sigma) \in \Pi \times \Sigma$ and $s \in S$, let

$$I^\gamma(s,\pi,\sigma) = \lim_{T\to\infty} I_T^\gamma(s,\pi,\sigma), \tag{6.2}$$

where

$$I_T^\gamma(s,\pi,\sigma) = \sum_{t=1}^{T} \beta^{t-1} E^s_{\pi,\sigma}[r^\gamma(X_t, \Delta_t^A, \Delta_t^B)] \quad (T \ge 1).$$

**Definition 2'** Let $\gamma, \gamma' \in [0,1]$ and $s \in S$. Then, the pair $(\pi^*,\sigma^*) \in \Pi \times \Sigma$ is said to be a $(\gamma,\gamma')$-equilibrium point at state $s \in S$ if the following (i) and (ii) hold:

(i) $I_T^{\gamma}(s,\pi,\sigma^*) \le I_T^{\gamma}(s,\pi^*,\sigma^*)$ for all $\pi \in \Pi$.

(ii) $I_T^{\gamma'}(s,\pi^*,\sigma) \ge I_T^{\gamma'}(s,\pi^*,\sigma^*)$ for all $\sigma \in \Sigma$.

Every finite noncooperative stochastic game has an equilibrium point (cf. [1]), so that for any $\gamma,\gamma' \in [0,1]$, a $(\gamma,\gamma')$-equilibrium point exists. The following lemma follows obviously from (2.2), (2.3) and (6.1).

**Lemma 6.1** *For any pair $(\pi,\sigma) \in \Pi \times \Sigma$ and $s \in S$,*

$$I^{\gamma}(s,\pi,\sigma) = \gamma \mathscr{I}^{+}(s,\pi,\sigma) + (1-\gamma)\mathscr{I}^{-}(s,\pi,\sigma).$$

**Theorem 6.1** *A pair $(\pi^*,\sigma^*) \in \Pi \times \Sigma$ is a saddle point at $s \in S$ if and only if there exist $\gamma,\gamma' \in (0,1)$ such that $(\pi^*,\sigma^*)$ is a $(\gamma,\gamma')$-equilibrium point at state $s \in S$.*

**Proof.** By Lemmas 1.2 and 6.1, that $\mathscr{I}(s,\pi^*,\sigma^*) \in \min \mathscr{I}(s,\pi^*,\Sigma)$ means that there exists $\gamma' \in (0,1)$ satisfying

$$\begin{aligned}(6.3)\qquad & \gamma'\mathscr{I}^{+}(s,\pi^*,\sigma^*) + (1-\gamma')\mathscr{I}^{-}(s,\pi^*,\sigma^*) \\ & \qquad \le \gamma'\mathscr{I}^{+}(s,\pi^*,\sigma) + (1-\gamma')\mathscr{I}^{-}(s,\pi^*,\sigma) \quad \text{for all } \sigma \in \Sigma.\end{aligned}$$

By Lemma 6.1, (6.3) is rewritten to (ii) of Definition 2'. Similarly, $\mathscr{I}(s,\pi^*,\sigma^*) \in \max \mathscr{I}(s,\Pi,\sigma^*)$ means that there exists $\gamma \in (0,1)$ for which (i) of Definition 2' holds. Thus, the proof is complete. □

The following results easily follow from Theorem 6.1.

**Corollary 6.1** *If a pair $(\pi^*,\sigma^*) \in \Pi \times \Sigma$ is a saddle point for a zero-sum game $\{I^{\gamma}(s,\pi,\sigma) \mid \pi \in \Pi, \sigma \in \Sigma\}, \gamma \in (0,1)$, the pair $(\pi^*,\sigma^*)$ is a saddle point at state $s \in S$ for the interval stochastic game.*

**Proof.** The saddle point $(\pi^*,\sigma^*)$ satisfies that $I^{\gamma}(s,\pi^*,\sigma) \ge I^{\gamma}(s,\pi^*,\sigma^*) \ge I^{\gamma}(s,\pi,\sigma^*)$, which implies that $(\pi^*,\sigma^*)$ is a $(\gamma,\gamma)$-equilibrium point. Thus, the statement of Corollary 6.1 follows from Theorem 6.1. □

**Corollary 6.2** *If $\boldsymbol{r}^{+}(s,i,j) - \boldsymbol{r}^{-}(s,i,j)$ (= const.) is independent of $s \in S, i \in A, j \in B$ the saddle point $(\pi^*,\sigma^*)$ for the interval stochastic game is uniquely determined as a saddle point for a zero-sum game $\{I^{0}(s,\pi,\sigma) \mid \pi \in \Pi, \sigma \in \Sigma\}$.*

**Proof.** We note from (6.1) that $r^{\gamma}(s,i,j) = \boldsymbol{r}^{-}(s,i,j) + \gamma(\boldsymbol{r}^{+}(s,i,j) - \boldsymbol{r}^{-}(s,i,j))$. So that if $\boldsymbol{r}^{+}(s,i,j) - \boldsymbol{r}^{-}(s,i,j)$ (= const.) is independent of $s \in S, i \in A, j \in B$, $I^{\gamma}(s,\pi,\sigma)$ and $I^{\gamma'}(s,\pi,\sigma)$ are essentially equivalent for any $\gamma,\gamma' \in (0,1)$. The proof is completed by observing Theorem 6.1. □

The following is useful in finding the saddle points for interval stochastic games.

**Corollary 6.3** (cf. [22]) *The pair $(\pi^*,\sigma^*) \in \Pi \times \Sigma$ is a saddle point at $s \in S$ if and only if there exist $\gamma,\gamma' \in (0,1)$ such that $\pi^*(\cdot|s) = (x_{s1}, x_{s2}, \ldots, x_{sm}) \in \mathcal{P}(A)$ and $\sigma^*(\cdot|s) =$*

$(y_{s1}, y_{s2}, \ldots, y_{sn}) \in \mathcal{P}(B)$ $(s \in S)$ *is a part of a solution to*

$$
(6.4) \qquad \begin{cases} v_s = \nu_{sj} + \sum_{i\in A}(r^{\gamma}(s,i,j) + \beta \sum_{s'\in S} q_{ss'}(i,j)v_{s'})x_{si} \quad (s \in S) \\ v'_s = \mu_{si} + \sum_{j\in B}(r^{\gamma'}(s,i,j) + \beta \sum_{s'\in S} q_{ss'}(i,j)v'_{s'})y_{sj} \\ \sum_{s\in S}\sum_{i\in A} \nu_{si}x_{si} = 0, \quad \sum_{s\in S}\sum_{j\in B} \mu_{sj}y_{sj} = 0 \\ \sum_{i\in A} x_{si} = 1, \quad \sum_{j\in B} y_{sj} = 1, \quad (s \in S) \\ x_{si} \geq 0, \quad y_{sj} \geq 0 \quad (s \in S, i \in A, j \in B). \end{cases}
$$

# 7 Extensions to fuzzy payoff cases

In this section, we consider the stochastic game similar to that specified in Section 5 except that for each $s \in S, i \in A$ and $j \in B$ the multi-dimensional fuzzy payoff $\widetilde{r}(s,i,j) \in \mathcal{F}(\mathbb{R}^p)$ is assigned.

Then, for a pair $(\pi, \sigma) \in \Pi \times \Sigma$ and $s \in S$, we let

$$
(7.1) \qquad \widetilde{\mathscr{I}}(s,\pi,\sigma) = \sum_{t=1}^{\infty} \beta^{t-1} E^s_{\pi,\sigma}[\widetilde{r}(X_t, \Delta^A_t, \Delta^B_t)],
$$

where the expectation of a fuzzy random variable is defined similarly as (1.3) by use of (3.1) and (3.2), and the convergence in (7.1) is taken with respect to the usual Hausdorff metric on $\mathcal{F}(\mathbb{R}^p)$ (cf. [4]).

We note that $\widetilde{\mathscr{I}}(s,\pi,\sigma) \in \mathcal{F}(\mathbb{R}^p)$ and its $\alpha$-cut $\widetilde{\mathscr{I}}(s,\pi,\sigma)_\alpha$ is given by

$$
(7.2) \qquad \widetilde{\mathscr{I}}(s,\pi,\sigma)_\alpha = \sum_{t=1}^{\infty} \beta^{t-1} E^s_{\pi,\sigma}[\widetilde{r}(X_t, \Delta^A_t, \Delta^B_t)_\alpha],
$$

where $\widetilde{r}(s,i,j)_\alpha$ is an $\alpha$-cut of $\widetilde{r}(s,i,j) \in \mathcal{F}(\mathbb{R}^p)$.

The saddle point of the stochastic game with fuzzy payoff is defined similarly as that of the interval stochastic game (see Definition 1' in Section 5).

For any $a \in \mathbb{R}^p$, since the product $a\widetilde{r}(s,i,j)_\alpha \in \mathbb{C}$, it will be written as

$$
a\widetilde{r}(s,i,j)_\alpha = [(a\widetilde{r}(s,i,j)_\alpha)^-, (a\widetilde{r}(s,i,j)_\alpha)^+].
$$

For $\alpha \in [0,1], d \in [0,1]$ and $a \in \mathbb{R}^p$, we put

$$
(7.3) \qquad r(s,i,j|\alpha,\gamma,a) = \gamma(a\widetilde{r}(s,i,j)_\alpha)^+ + (1-\gamma)(a\widetilde{r}(s,i,j)_\alpha)^-.
$$

For each pair $(\pi, \sigma) \in \Pi \times \Sigma$ and $s \in S$, we define

$$
(4.8) \qquad I(s,\pi,\sigma|\alpha,\gamma,a) = \sum_{t=1}^{\infty} \beta^{t-1} E^s_{\pi,\sigma}[r(X_t, \Delta^A_t, \Delta^B_t|\alpha,\gamma,a)].
$$

Then, the saddle point for the stochastic game with fuzzy payoff can be characterized in the following, whose proof is done by applying Lemma 3.1 and the idea used in Section 5.

**Theorem 7.1** *A pair* $(\pi^*, \sigma^*) \in \Pi \times \Sigma$ *is a saddle point for the stochastic game with fuzzy payoffs if and only if there exist two function* $\gamma, \gamma' : [0,1] \times K \to (0,1)$ *such that*

$$
(4.9) \qquad I(s,\pi,\sigma^*|\alpha,\gamma(\alpha,a),a) \leq I(s,\pi^*,\sigma^*|\alpha,\gamma(\alpha,a),a),
$$

$$
I(s,\pi^*,\sigma|\alpha,\gamma'(\alpha,a),a) \geq I(s,\pi^*,\sigma^*|\alpha,\gamma'(\alpha,a),a)
$$

*for all* $\pi \in \Pi, \sigma \in \Sigma, \alpha \in [0,1]$ *and* $a \in K^+$.

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