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ON A STOPPING PROBLEM INVOLVING REFUSAL AND FORCED STOPPING

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Abstract

Although the usual optimal stopping problem is described as a Markov decision process with two decisions, stop and continue, we shall consider a model which distinguishes the observer's strategy from the system's two decisions. The observer can select a strategy defined on an action space, and the decision of the system to stop or continue is determined by a prescribed conditional probability. For this model, it may happen that the strategy (a) to stop is refused, or (b) to continue is forcibly stopped. This is a slight modification of the one-dimensional stopping problem by involving refusal and forced stopping. The model is motivated by the uncertain secretary choice problem of Smith (1975) and the multivariate stopping problem of Kurano, Yasuda and Nakagami (1980), (1982).

OPTIMAL STOPPING; SECRETARY PROBLEM; REFUSAL AND FORCED STOPPING

1. Formulation

Suppose that the discrete-time parameter process $X_n\,(=X(n))$, $n=1,2,\cdots$ is observed, and one selects a strategy from an action space A at each period. This strategy determines stochastically the system's decision to stop or continue. If the decision is to stop, one gets a reward for interrupting the observation; if the decision is to continue, one observes the next value, and pays the cost. To be explicit, let $X(n)$, $n=1,2,\cdots$ be a stochastic process with a state space $E\subset R$ and let an action space A be a finite topological space. (1) To start the first observation, one must pay a cost $c\in R$. Then, observing $X(1)$, one selects a strategy $\sigma_1\in A$. (2) Observing $X(n)$ at the nth period and selecting a strategy $\sigma_n\in A$, one gets a net gain $X(n)-nc$ if the process decides to stop. If not, one incurs the cost c and observes $X(n+1)$.

The strategy σ_n at the nth period is an A-valued $\mathscr{B}(X_1,\cdots,X_n)$-measurable random variable with its distribution $\phi_n(a)=P(\sigma_n=a)$, $a\in A$, where $(\Omega,\mathscr{B},P)$

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is an underlying probability space and $\mathscr{B}(X_1,\cdots,X_n)\subset\mathscr{B}$. The strategy σ denotes the infinite sequence $(\sigma_1,\cdots,\sigma_n,\cdots)$, and Σ is the set of all strategies.

Let us denote the system's decision by the variables (S_n):

$$S_n=\begin{cases}1 & \text{if the decision of the process is stop at the } n\text{th period,}\\ 0 & \text{if continue.}\end{cases}\tag{1.1}$$

The decision (S_n) is determined only by the strategy σ_n at the nth period, with the conditional probability

$$\gamma_n(a)=P(S_n=1\mid\sigma_n=a),\qquad a\in A\tag{1.2}$$

where $\gamma_n(a)$ is a given amount.

Assumption 1.1. We assume that $\gamma_n(a)$ is independent of n, so that

$$\gamma(a)=\gamma_n(a)\qquad\text{for all } n.\tag{1.3}$$

For the space A, there exist

$$\alpha=\min\gamma(a)=\gamma(a_0),\quad \beta=\max\gamma(a)=\gamma(a_1),\qquad a_0,a_1\in A.\tag{1.4}$$

To avoid a trivial case, assume $\gamma(a)$, $a\in A$ is not constant so that

$$0\leqq\alpha<\beta\leqq 1.\tag{1.5}$$

According to the setup of our model in the finite N-horizon case, the stopping time is defined by

$$t_N(\sigma)=\begin{cases}\text{first }\{n\leqq N;\ S_n=1\}\\ N \qquad \text{if }\{\ \}\text{ is empty}\end{cases}\tag{1.6}$$

where $\sigma\in\Sigma$ is a strategy.

Our aim in the finite-horizon stopping problem is to maximize the expected gain

$$E[X(t_N(\sigma))-ct_N(\sigma)]$$

subject to the strategy $\sigma\in\Sigma$. The optimal value V_N is defined by

$$V_N=\sup E[X(t_N(\sigma))-ct_N(\sigma)].\tag{1.7}$$

The optimal strategy $^*\sigma$ is such that $E[X(t_N(^*\sigma))-ct_N(^*\sigma)]=V_N$.

The difference from the usual stopping problem is that a conditional probability $\gamma(a)$ has been introduced into the connection between the observer's strategy and system's decision. Roughly, the observer's strategy, which determines the system's decision, is interrupted by this. Two extremal probabilities are significant: $1-\beta=1-\max\gamma(a)$, that is, the probability of refusal to stop the process

and $\alpha=\min\gamma(a)$, that is, the probability of forced stopping. If $\alpha=0$ and $\beta=1$ (no interruption), then the problem reduces to the usual one. The model is motivated by the uncertain secretary choice problem of Smith (1975) with $\beta=p$ $(0<p\leqq 1)$ and $\alpha=0$, and also the multivariate stopping problem of Kurano, Yasuda and Nakagami (1980), (1982) including both refusal and forced stopping. These secretary choice problems are discussed in Section 4.

2. Optimal strategy

Assumption 2.1. (i) Let X, $X(n)$, $n=1,2,\cdots$ denote independent identically distributed (i.i.d.) random variables with $E|X|<\infty$. Denote their distribution function by F and let $\mu=\int_{-\infty}^{\infty}x dF(x)=E(X)$. (ii) Assume that $\mu<\sup\{x;F(x)<1\}$.

The first assumption is not essential to our argument and we shall treat the non-identically distributed case in the example of Section 4.

Using the notation

$$T_{\alpha,\beta}(x)=E(X-x)^+\beta-E(X-x)^-\alpha\tag{2.1}$$

where $(a)^+=\max(a,0)$ and $(a)^-=(-a)^+$, define the sequence (μ_n) as follows:

$$\begin{aligned}&\mu_1=E(X),\\ &\mu_n=\mu_{n-1}-c+T_{\alpha,\beta}(\mu_{n-1}-c),\qquad n=2,3,\cdots.\end{aligned}\tag{2.2}$$

In the special case, $\beta=1$ and $\alpha=0$, (2.1) implies $T_{0,1}(x)=E(X-x)^+=\int_x^{\infty}(y-x)dF(y)$. This appears frequently in the ordinary stopping problem. Clearly $T_{\alpha,\beta}(x)=(\beta-\alpha)T_{0,1}(x)+\alpha(\mu-x)$. Also $T_{\alpha,\beta}(x)$ is a continuous, convex function of x and has two asymptotes. If $\alpha=0$, then $T_{0,\beta}(x)\geqq 0$ but generally it varies over $(-\infty,\infty)$. Therefore we note that the sequence (μ_n) is not monotone increasing in the case of $\alpha\neq 0$.

Theorem 2.1. The optimal strategy $^*\sigma=(^*\sigma_1,\cdots,{}^*\sigma_N,\cdots)\in\Sigma$ is given by

$$^*\sigma_n(\omega)=\begin{cases}a_1 & \text{if } X_n(\omega)\geqq\mu_{N-n}-c\\ a_0 & \text{if } X_n(\omega)<\mu_{N-n}-c\end{cases}\tag{2.3}$$

for $n=1,2,\cdots$, and the optimal value is

$$V_N=\mu_N-c.\tag{2.4}$$

Proof. In the case of $N=1$, the optimal value is clearly

$$V_1=E(X_1)-c=\mu_1-c$$

because the reward is X_1 and the cost incurred for the observation is c. In the usual dynamic programming procedure, we assume inductively that (2.4) holds and consider the parameter N as a time-period left in the sequential decision process. When $N-n$ time-periods are left, one must select $\sigma_n = a$ from $a \in A$. If $S_n = 1$ occurs, then one gets X_n, otherwise $V_{N-n} = \mu_{N-n} - c$ since it reduces to the $(N-n)$th period problem. One selects a strategy σ_n at the nth period so as to maximize

$$E[X_n P(S_n = 1 \mid X_n, \sigma_n) + V_{N-n} P(S_n = 0 \mid X_n, \sigma_n)].$$

Since $P(S_n = 1 \mid X_n, \sigma_n) = P(S_n = 1 \mid \sigma_n)$ and $\gamma(a) = P(S_n = 1 \mid \sigma_n = a)$, one is to maximize

$$E\left[\sum_{a \in A} (X_n - V_{N-n})\gamma(a)\phi_n(a)\right] + V_{N-n}$$

for $0 \leqq \phi_n(a) \leqq 1$ over all the densities. Hence if $X_n - V_{N-n} \geqq 0$,

$$\phi_n(a) = 1 \quad \text{if } a = a_1, \quad \text{and} \quad \phi_n(a) = 0 \quad \text{otherwise}$$

and if $X_n - V_{N-n} < 0$,

$$\phi_n(a) = 1 \quad \text{if } a = a_0, \quad \text{and} \quad \phi_n(a) = 0 \quad \text{otherwise.}$$

That is, the pure strategy (2.3) is optimal. Its maximum equals

$$E[(X_n - V_{N-n})^+\beta - (X_n - V_{N-n})^-\alpha] + V_{N-n} = T_{\alpha,\beta}(\mu_{N-n} - c) + \mu_{N-n} - c$$
$$= \mu_{N-n+1}.$$

The total optimal value, with a cost c per observation, is

$$V_{N-n+1} = \mu_{N-n+1} - c.$$

This proves the theorem by letting $n = 1$.

3. Infinite-horizon problem

Define a stopping time $t(\sigma)$ by

(3.1) $$t(\sigma) = \begin{cases} \inf\{n \geqq 1;\ S_n = 1\}, \\ \infty \quad \text{if } \{\ \} \text{ is empty} \end{cases}$$

for the strategy $\sigma \in \Sigma$. Let $X(t(\sigma)) = X(n)$ on $t(\sigma) = n$, $X(t(\sigma)) = \limsup X(n)$ on $t(\sigma) = \infty$. The optimal value V^* is defined by

(3.2) $$V^* = \sup E[X(t(\sigma)) - ct(\sigma)].$$

Assumption 3.1. We assume (i) $\alpha > 0$ and c is any real number, or (ii) $\alpha = 0$ and $c > 0$.

Lemma 3.1. Under Assumptions 1.1, 2.1 and 3.1, the limit of the sequence (μ_n) of (2.2) exists:

(3.3) $$\lim \mu_n = v^* + c$$

where v^* is the unique solution of the equation

(3.4) $$T_{\alpha,\beta}(v) = c.$$

Proof. Let $v_n = \mu_n - c$. The iteration (2.2) implies $v_n = v_{n-1} + T_{\alpha,\beta}(v_{n-1}) - c$. It is clear that the function $v + T_{\alpha,\beta}(v)$ of v is continuous, convex and monotone increasing. Also $g(v)$, the asymptote of $T_{\alpha,\beta}(v)$ as $v \to \infty$, is $g(v) = \alpha\mu + (1-\alpha)v$. Therefore (3.4) has a unique finite solution for $\alpha > 0$ and for any c. Under the conditions $\alpha = 0$ and $c > 0$, it holds similarly.

The property (3.3) is called stable by Ross (1970); we can therefore say the forced stopping problem is stable.

A necessary and sufficient condition that the solution v^* of (3.4) satisfies $v^* \geqq \mu$ is that $E(X-\mu)^+ \geqq c/(\beta - \alpha)$. If $c = 0$, the result is trivial and the following inequality holds:

(3.5) $$\mu \leqq v^* \leqq \sup\{x;\ F(x) < 1\}.$$

Examples of the solution v^* in (3.4) with $c = 0$ are as follows.

(i) Normal distribution $N(0,1)$: $0 \leqq v^* \leqq \infty$,

$$\Psi(v) = \alpha v/(\beta - \alpha)$$

where $\Psi(v) = \phi(v) - v\Phi(v)$, $\Phi(v) = \int_v^\infty \phi(x)dx$ and $\phi(x)$ is a density function.

(ii) Exponential distribution with a density function $\lambda \exp(-\lambda x)$, $\lambda > 0$:

$$1/\lambda \leqq v^* \leqq \infty, \qquad (\exp(-\lambda v))/(1 - \lambda v) = -\alpha/(\beta - \alpha).$$

(iii) Uniform distribution on a unit interval (0,1); $0.5 \leqq v^* \leqq 1$,

$$v^* = 1/(1 + \sqrt{(\alpha/\beta)}).$$

Lemma 3.2. The functional equation of $V(x)$, $x \in R$:

(3.6) $$V(x) = \max(\gamma(a)x + (1-\gamma(a))\{E(V(X)) - c\})$$

where $\gamma(a)$, $a \in A$ is in (1.4), has a unique solution in a functional space $\{V(x),\ x \in R;\ E(V(X)) < \infty\}$ under Assumption 3.1. It is given by

(3.7) $$V(x) = (x - v^*)^+\beta - (x - v^*)^-\alpha + v^*$$

where v^* is determined by Lemma 3.1.

Proof. We can show by straightforward calculation that (3.7) satisfies $E(V(X)) < \infty$ and (3.6). The uniqueness can be proved from the fundamental property of 'max' mapping in (3.6), as in Bellman (1957).

Theorem 3.3. In the infinite-horizon case under Assumptions 1.1, 2.1 and 3.1, the strategy ${}^*\sigma = ({}^*\sigma_1, \cdots, {}^*\sigma_n, \cdots) \in \Sigma$ with

$$(3.8) \qquad {}^*\sigma_n(\omega) = \begin{cases} a_1 & \text{if } X_n(\omega) \geqq v^*, \\ a_0 & \qquad\quad < v^*, \end{cases}$$

$n = 1, 2, \cdots$ is optimal and the optimal value V^* is given by

$$(3.9) \qquad V^* = v^*.$$

Proof. Let $V(x)$ denote the optimal value when the first $X_1 = x$ is observed. By the optimality principle, $V(x)$ satisfies the optimality equation (3.6). It follows that, with the incurred cost c, the optimal value equals $V^* = E[V(X)] - c$. Hence (3.9) is immediately obtained from (3.7) and $E[V(X)] = v^* + c$.

Theorem 3.4. In the case of $c = 0$, a sufficient condition that $P(t({}^*\sigma) < \infty) = 1$ is that $\alpha > 0$.

Proof. Since $X(k)$, $k = 1, 2, \cdots$, are i.i.d.,

$$P(t({}^*\sigma) = n) = P(X(k) < v^*,\ k = 1, \cdots, n-1,\ X(n) \geqq v^*)$$
$$= (1 - F(v^* -))(F(v^* -))^{n-1}.$$

Now $c = 0$ implies that $E(X - v^*)^+ / E(X - v^*)^- = \alpha/\beta$. If $\alpha > 0$, then $E(X - v^*)^+ > 0$ yields $v^* < \sup\{x; F(x) < 1\}$ and so $F(v^*) < 1$. From these, the conclusion is immediate.

4. Application to a secretary choice problem

Let the observation cost $c = 0$ and let $X(n)$, $n = 1, \cdots, N$ be independent random variables such that

$$(4.1) \qquad X(n) = \begin{cases} n/N & \text{with probability } n^{-1} \\ 0 & \text{with probability } 1 - n^{-1}. \end{cases}$$

The stopping problem for this process (4.1) is called the secretary choice problem by Chow, Robbins and Siegmund (1971). We shall not assume (1.3) and use α_n, β_n instead of α, β in (1.4). It is seen, indexing the time parameter, that results similar to Theorem 2.1 hold. Define

$$(4.2) \qquad T^{(n)}(x) = E(X_{N-n} - x)^+ \beta_n - E(X_{N-n} - x)^- \alpha_n$$

for $n = 0, \cdots, N-1$ in place of (2.1). From (4.1),

$$T^{(n)}(x) = \begin{cases} \beta_n/N - \beta_n x & \text{if } x \leqq 0, \\ \beta_n/N - (\alpha_n + (\beta_n - \alpha_n)/N_n)x & \text{if } 0 \leqq x \leqq N_n/N, \\ \alpha_n/N - \alpha_n x & \text{if } N_n/N \leqq x, \end{cases}$$

where $N_n = N - n$. Since $c = 0$, $V_n = \mu_n$ holds by (2.4), and so consider the following sequence similar to (2.2):

$$(4.3) \qquad \begin{aligned} V_1 &= E[X_N] = 1/N, \\ V_n &= V_{n-1} + T^{(n-1)}(V_{n-1}), \qquad n = 2, 3, \cdots. \end{aligned}$$

This is different from the usual problem; if $\alpha_n \neq 0$, we note that the sequence V_n is not generally monotone increasing.

Assumption 4.1. Let α_n, β_n satisfy the conditions:
(i) $0 \leqq \alpha_n < \beta_n \leqq 1$,
(ii) $\beta_n \geqq \beta_{n+1}$,
(iii) $\alpha_n - \alpha_{n+1} + \alpha_n \alpha_{n+1} \leqq 0$
for each n.

Lemma 4.1. Under Assumption 4.1, if $n/N \geqq V_{N-n}$ for some n, then it holds also for later n.

Proof. If V_n is concave in n, the lemma is immediately proved since the boundary V_N at $n = 0$ is strictly positive, that is, the initial position is above the straight line n/N. To prove this, it is enough to show that

$$(4.4) \qquad T^{(n)}(V_n) - T^{(n-1)}(V_{n-1}) \leqq 0.$$

First, we show that $T^{(n)}(x) \leqq T^{(n-1)}(x)$, $0 \leqq x < \infty$; this follows because $T^{(n)}(x)$ is a convex function of x and is composed of three line segments. Hence it is sufficient to consider the inequality at $x = N_{n+1}/N$ and $x = N_n/N$. The result is immediate at these points for the increasing α_n and decreasing β_n following from Assumptions 4.1 (i), (iii).

To prove (4.4), we restrict β_n to be a constant in n, without loss of generality. Because, for a general β_n, the gradient of $T^{(n)}(x)$ on $0 \leqq x \leqq N_n/N$ decreases, the above arguments hold independently of β_n on $x \geqq N_n/N$. Consider a function of x:

$$S^{(n)}(x) = T^{(n+1)}(x + y) - T^{(n)}(x)$$

where $y = T^{(n)}(x)$. On $0 \leqq x \leqq N_n/N$, if $y = T^{(n)}(x) \leqq 0$, $S^{(n)}(x) \leqq 0$ follows by considering

$$S^{(n)}(N_n/N) = \alpha_{n+1}/N - \alpha_{n+1}(N_n/N + y) - (-\alpha_n N_{n+1}/N)$$
$$= (\alpha_n - \alpha_{n+1} + \alpha_n \alpha_{n+1}) N_{n+1}/N \leqq 0.$$

If $y \geqq 0$, clearly $T^{(n+1)}(x+y) \leqq T^{(n+1)}(x) \leqq T^{(n)}(x)$ holds by the monotone decreasing property of $T^{(n)}(x)$ in n and x. For $x > N_n/N$, we easily see that $y = T^{(n)}(x) < 0$ and

$$S^{(n)}(x) = \alpha_{n+1}/N - \alpha_{n+1}(x+y) - y = (\alpha_n - \alpha_{n+1} + \alpha_n\alpha_{n+1})(x - 1/N) \leqq 0$$

by Assumption 4.1(iii). We have thus obtained $S^{(n)}(x) \leqq 0$ on $0 \leqq x$ and so completed the proof of the lemma.

The optimal policy $^*\sigma$ is, by (2.3) in Theorem 2.1, such that $^*\sigma_n = a_1$ if $X_n \geqq V_{N-n}$ occurs or $n/N \geqq V_{N-n}$; that is, we declare 'stop' if the relative best applicant has appeared. Define

$$n^* = \inf\{n\,;\, n/N \geqq V_{N-n}\}. \tag{4.5}$$

By Assumption 4.1 and Lemma 4.1, the optimal strategy of the problem considered is the OLA policy (refer to Ross (1970)). The result is summarized as follows.

Theorem 4.1. The optimal strategy of the secretary choice problem is $^*\sigma_n = a_0$ for $n = 1, \cdots, n^*-1$ and $^*\sigma_n = a_1$ for $n = n^*, \cdots, N$. That is, observe applicants until n^*-1 and then declare 'stop' if one who has appeared is relatively the best.

In the rest of the section, we study the limiting procedure by allowing N to tend to ∞. Two special cases of the coefficients α_n and β_n are considered.

Case 1. Refusal and unforced stopping. Let

$$\beta_n = p \quad \text{and} \quad \alpha_n = 0 \tag{4.6}$$

where p is a constant $(0 < p \leqq 1)$. Since $\alpha_n = 0$, unforced stopping occurs, and this is the uncertain employment case considered by Smith (1975). By (4.3) and (4.5), we have

$$\begin{aligned} n^* = \inf \Big\{ n\,;\, p\Big[\frac{1}{n} + \Big(\frac{n+\bar p}{n}\Big)\frac{1}{n+1} + \cdots \\ + \Big(\frac{n+\bar p}{n}\Big)\Big(\frac{n+1+\bar p}{n+1}\Big)\cdots\Big(\frac{N-3+\bar p}{N-3}\Big)\frac{1}{N-2}\Big] \\ + \Big(\frac{n+\bar p}{n}\Big)\cdots\Big(\frac{N-2+\bar p}{N-2}\Big)\frac{1}{N-1} \leqq 1 \Big\} \end{aligned} \tag{4.7}$$

where $\bar p = 1 - p$. If $p = 1$, (4.7) becomes

$$n^* = \inf\{n\,;\, 1/n + 1/(n+1) + \cdots + 1/(N-1) \leqq 1\}$$

as is well known.

If $p < 1$ and $v_1 = p/N$, (4.3) and (4.5) imply

$$n^* = \inf\{n\,;\, p(1+\bar p/n)(1+\bar p/(n-1)) \cdots (1+\bar p/(N-1)) \leqq 1\}. \tag{4.8}$$

This result is obtained by Smith (1975). The limit is

$$\lim n^*/N = p^{1/(1-p)}. \tag{4.9}$$

This value holds for both the cases (4.7) and (4.8). This is seen in the next generalized situation.

Case 2. Refusal and forced stopping. Let

$$\beta_n = p \quad \text{and} \quad \alpha_n = q/(N-n) \tag{4.10}$$

where p and q are constants with $0 \leqq q < p \leqq 1$.

The situation in this secretary choice problem is that there are two observers: one is a young man who wants to choose a secretary and the other is his grandmother who also observes applicants. Each rank appears independently: we also assume that there is no relation between the two rankings. The problem is to find the best one with respect to the young man's rank. As a stopping rule, he could choose a candidate if he thinks she is best, in accordance with the possibility of refusal p. Aside from this case, there occurs forced stopping. That is, although he thinks that a candidate is not the best, he is forcibly stopped and must accept her when his grandmother thinks her the best one. The factor q denotes the strength of this effect.

Clearly this reduces to Case 1 if $q = 0$ and (4.10) satisfies Assumption 4.1. Now we proceed to calculate $\lim n^*/N$ as before where n^* is given in (4.5). By (4.3), if $N_n/N < V_n$,

$$V_{n+1} = V_n + \Big(\frac{1}{N} - \frac{V_n}{N_n}\Big)p - \frac{N_{n+1}}{N_n}\alpha_n V_n = p/N + \eta_n V_n$$

where $N_n = N - n$, $\eta_n = \bar\alpha_n + (\alpha_n - p)/N_n$ and $\bar\alpha_n = 1 - \alpha_n$. Hence we have, from the iteration (4.3) and the property of the optimal strategy, that

$$V_{n+1} = p(1 + \eta_n + \eta_n\eta_{n-1} + \cdots + \eta_n\eta_{n-1}\cdots\eta_1)/N + (1-p)\eta_n\eta_{n-1}\cdots\eta_1/N \tag{4.11}$$

$$\eta_n = 1 - (p+q)/N_n + q/N_n^2 = \delta_n N_{n+1}/N_n \tag{4.12}$$

and

$$\delta_n = \delta(n) = 1 + (\bar p - q)/N_{n+1} + q/(N_n N_{n+1}) = 1 + \bar p/N_{n+1} - q/N_n$$

and $\bar p = 1 - p$. Substituting (4.12) in (4.11), we obtain

$$V_{n+1} = \frac{N_{n+1}}{N}\Big\{p\Big(\frac{1}{N_{n+1}} + \frac{\delta_n}{N_n} + \frac{\delta_n\delta_{n-1}}{N_{n-1}} + \cdots + \frac{\delta_n\delta_{n-1}\cdots\delta_1}{N_1}\Big) + \bar p\,\frac{\delta_n\delta_{n-1}\cdots\delta_1}{N_1}\Big\}.$$

By (4.5), we must find {first n such that $n/N \geqq V_{N-n}$} so

$$\inf\left\{n; p\left(\frac{1}{n}+\delta(N_{n+1})\frac{1}{n+1}+\cdots+\delta(N_{n+1})\delta(N_{n+2})\cdots\delta(1)\frac{1}{N-1}\right)\right.$$
$$\left.+\bar{p}\,\frac{\delta(N_{n+1})\cdots\delta(1)}{N-1}\leqq 1\right\}. \tag{4.13}$$

In the limiting procedure, it is enough to consider the relation between n and N:

$$1/p = 1/n + \left(1-\frac{\bar{p}-q}{n}+\frac{q}{n(n+1)}\right)\Big/(n+1)+\cdots$$
$$+\left(1+\frac{\bar{p}-q}{n}+\frac{q}{n(n+1)}\right)\cdots\left(1+\frac{\bar{p}-q}{N-2}+\frac{q}{(N-2)(N-1)}\right)\Big/(N-1)$$
$$+\frac{\bar{p}}{p}\,\frac{\delta(N_{n+1})\cdots\delta(1)}{N-1}. \tag{4.14}$$

From the principal terms of δ_n, we can write

$$\delta(N_{n+1}) = 1+(\bar{p}-q)/n+o(1/n)$$

where $o(1/n)$ denotes terms of order smaller than $1/n$. Hence (4.14) implies

$$1/p = 1/n+\left(1+\frac{\bar{p}-q}{n}\right)\Big/(n+1)+\cdots+\left(1+\frac{\bar{p}-q}{n}\right)\cdots\left(1+\frac{\bar{p}-q}{N-2}\right)\Big/(N-1)$$
$$+\frac{\bar{p}}{p}\,\frac{\delta(N_{n+1})\cdots\delta(1)}{N-1}+o(1)$$

where $o(1)$ is a term of negligible order as $n\to\infty$. Rearranging the sum in (4.14), we have

$$(1-q)/p = \left(1+\frac{\bar{p}-q}{n}\right)\cdots\left(1+\frac{\bar{p}-q}{N-1}\right)+\frac{\bar{p}(\bar{p}-q)}{p}\,\frac{\delta(N_{n+1})\cdots\delta(1)}{N-1}+o(1)$$

provided $p+q\neq 1$. The last two terms of the above equality are negligible. Using the approximation $1+x\cong\exp(x)$,

$$\log((1-q)/p) = (\bar{p}-q)\sum_{k=n}^{N-1}k^{-1}+o(1).$$

Therefore we have obtained the result that

$$\lim n^*/N = (p/(1-q))^{1/(1-p-q)} \qquad \text{for } p+q\neq 1. \tag{4.15}$$

If $p+q=1$, by (4.14), we have

$$1/p = 1/n+\cdots+1/(N-1)+o(1)$$

which implies

$$\lim n^*/N = \exp(-1/p). \tag{4.16}$$

In (4.15), since

$$(p/(1-q))^{1/(1-p-q)} = \exp\left(-\frac{\log(p)-\log(1-q)}{p-(1-q)}\right)$$

when $1-q\to p$, we have $\exp(-1/p)$. So there is no gap between (4.15) and (4.16). Letting $q=0$ in (4.15), this reduces to $p^{1/(1-p)}$ as in Smith's (1975) refusal and unforced stopping case, while letting $p=1$ in (4.15), it reduces to $(1-q)^{1/q}$ as in the forced stopping case. From this, we see that p and $1-q$ in $\beta_n=p$ and $\alpha_n=q/N_n$ have a dual property.

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