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Property (p.g.p.) of Fuzzy Measures and Convergence in Measure*

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Abstract:
The purpose of this paper is to further investigate the convergence in measure of sequence of measurable functions on fuzzy measure spaces. Several classical results on measurable functions are generalized to fuzzy measure spaces by using the property (p.g.p.), null-additivity and exhaustivity of fuzzy measures.

Keywords:
Fuzzy measures; property (p.g.p.); convergence in measure; fundamental in measure.

1. Introduction

An important problem in measure theory is to describe the convergence of sequence of measurable functions on measure spaces. We hope that most of properties in classical measure theory remain valid for fuzzy measures, but it is very difficult to generalize those properties [1, 3, 6, 7, 8].

This paper is a continuation of our previous efforts [6]. Our purpose is to further investigate the convergence in measure of measurable functions on a fuzzy measure space

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possessing the property (p.g.p.). The main theorems are stated in Section 3. We show that a sequence of measurable functions converges in measure if and only if it is fundamental in measure. In Section 4, we introduce a new concept which is called to be *almost uniformly bounded*, and obtain some interesting results on the convergence of sequence of measurable functions.

In the last section, we prove that fuzzy mean convergence and fuzzy mean fundamental are equivalent. Some relevant examples are also given in this paper.

2. Preliminary

Let X be a non-empty set and $\mathcal{F}$ be a σ-algebra of subsets of X, and let $N = \{1, 2, \ldots,\}$ and $\overline{R_+} = [0, +\infty]$. Unless stated otherwise, all subsets are supposed to belong to $\mathcal{F}$ and we make the following conventions: $\sup\{i \mid i \in \emptyset\} = 0$, $\infty - \infty = 0$ and $0 \cdot \infty = 0$. The following terminology will be used without any further reference.

A set function $\mu: \mathcal{F} \to \overline{R_+}$ is said to be *exhaustive* if $\mu(E_n) \to 0$ for any infinite disjoint sequence $\{E_n\}_n$; *order continuous* if $\mu(E_n) \to 0$ whenever $E_n \searrow \emptyset$; *null-additive* if $\mu(E \cup F) = \mu(E)$ for any E whenever $\mu(F) = 0$; and *autocontinuous from above* if $\mu(F_n) \to 0$ implies $\mu(E \cup F_n) \to \mu(E)$ for any E.

A *fuzzy measure* is a set function $\mu: \mathcal{F} \to [0, +\infty]$ with the properties:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $A \subset B \Rightarrow \mu(A) \le \mu(B)$;

(FM3) $A_1 \subset A_2 \subset \cdots \Rightarrow$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n);$$

(FM4) $A_1 \supset A_2 \supset \cdots$, and there exists $n_0 \ge 1$ with $\mu(A_{n_0}) < +\infty$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

Definition 1. Set function μ is said to have the property (p.g.p.) if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that $\mu(E) \vee \mu(F) < \delta$ implies $\mu(E \cup F) < \epsilon$.

The following three propositions concerning properties of fuzzy measures are given in [2, 3].

Proposition 1. *Let μ be a fuzzy measure. Then μ is exhaustive if and only if it is order continuous.*

Proposition 2. *Any finite fuzzy measure is exhaustive.*

Proposition 3. *If a fuzzy measure is autocontinuous from above, then it has the property* (p.g.p.).

Let **F** be the class of all finite measurable functions on $(X,\mathcal{F})$, and let f, g, f_n, $g_n \in \mathbf{F}(n \geq 1)$. We denote that $\{f_n\}_n$ *everywhere converges to* f by $f_n \to f$. We say that $\{f_n\}_n$ *almost everywhere converges to* f *on* X if there is subset $E \subset X$ with $\mu(E)=0$ such that $f_n \to f$ on $X \setminus E$, and denote it by $f_n \xrightarrow{a.e.} f$; $\{f_n\}_n$ *almost uniformly converges to* f *on* X if for any $\epsilon > 0$ there is subset $E_\epsilon \subset X$ with $\mu(E_\epsilon) < \epsilon$ such that f_n converges to f uniformly on $X \setminus E_\epsilon$, and denote it by $f_n \xrightarrow{a.u.} f$.

Definition 2. $\{f_n\}_n$ is said to be convergent in measure to f on fuzzy measure space $(X,\mathcal{F},\mu)$, if

$$\lim_{n\to+\infty} \mu(\{x : |f_n(x)-f(x)| \geq \epsilon\}) = 0$$

for any $\epsilon > 0$, denoted by $f_n \xrightarrow{\mu} f$; and be fundamental in measure if

$$\lim_{n,m\to+\infty} \mu(\{x : |f_n(x)-f_m(x)| \geq \epsilon\}) = 0$$

for any $\epsilon > 0$.

We know that the convergence in measure and fundamental in measure are equivalent if μ is a σ-additive measure. The relations among convergences of sequence of measurable function on fuzzy measure spaces are discussed [8]. Some generalizations are demonstrated in [6].

3. Convergence in measure

We generalize some results about convergence in measure from classical measure spaces to fuzzy measure spaces.

Theorem 1. *Let μ be a fuzzy measure with the property* (p.g.p.). *If $\{f_n\}_n$ is fundamental in measure, then there are $f \in \mathbf{F}$ and a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $f_{n_k} \xrightarrow{a.e.} f$, $f_{n_k} \xrightarrow{a.u.} f$ and $f_{n_k} \xrightarrow{\mu} f$.*

Proof. From the property (p.g.p.) of μ, there is $\delta_1 \in (0,1/2)$ such that

$$\mu(E) \vee \mu(F) < \delta_1 \Rightarrow \mu(E \cup F) < \frac{1}{2}.$$

For above δ_1, there exist $\delta_2 \in (0, \delta_1 \wedge 1/2^2)$ and $n_1 \geq 1$ satisfying

$$\mu(E) \vee \mu(F) < \delta_2 \Rightarrow \mu(E \cup F) < \delta_1$$

and

$$\mu\left(\left\{x : \left|f_n(x) - f_{n_1}(x)\right| \geq \frac{1}{2}\right\}\right) < \delta_1 \ (\forall\, n \geq n_1)$$

since $\{f_n\}_n$ is fundamental in measure. If we take $n_2 > n_1$ to satisfy

$$\mu\left(\left\{x : |f_n(x) - f_{n_2}(x)| \geq \frac{1}{2^2}\right\}\right) < \delta_2 \ (\forall\ n \geq n_2),$$

then

$$\mu\left(\left\{x : |f_n(x) - f_{n_2}(x)| \geq \frac{1}{2^2}\right\} \cup \left\{x : |f_{n_2}(x) - f_{n_1}(x)| \geq \frac{1}{2}\right\}\right) < \frac{1}{2}.$$

For above δ_2, there exists $\delta_3 \in (0, \delta_2 \wedge 1/2^3)$ such that

$$\mu(E) \vee \mu(F) < \delta_3 \Rightarrow \mu(E \cup F) < \delta_2.$$

If we take $n_3 > n_2$ to satisfy

$$\mu\left(\left\{x : |f_n(x) - f_{n_3}(x)| \geq \frac{1}{2^3}\right\}\right) < \delta_3 \ (\forall\ n \geq n_3),$$

then

$$\mu\left(\left\{x : |f_n(x) - f_{n_3}(x)| \geq \frac{1}{2^3}\right\} \cup \left\{x : |f_{n_3}(x) - f_{n_2}(x)| \geq \frac{1}{2^2}\right\}\right) < \delta_1$$

and

$$\mu\left(\left\{x : |f_{n_3}(x) - f_{n_2}(x)| \geq \frac{1}{2^2}\right\} \cup \left\{x : |f_{n_2}(x) - f_{n_1}(x)| \geq \frac{1}{2}\right\}\right) < \frac{1}{2}.$$

Repeating this procedure, we can obtain $n_{k+1} > n_k > n_{k-1} > \ldots > n_1$ and $\delta_k < \delta_{k-1} \wedge 1/2^k$ such that

$$\mu\left(\bigcup_{i=k}^{r+1} \left\{x : |f_{n_{i+1}}(x) - f_{n_i}(x)| \geq \frac{1}{2^i}\right\}\right) < \delta_{k-1} \ (k = 2, 3, \ldots, r+1)$$

for any $r \geq 1$. If we put $E_k = \{x : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 1/2^k\}$, $B_k = \bigcup_{i=k}^{+\infty} E_i$ and $E = \overline{\lim}_{k \to +\infty} E_k = \bigcap_{k=2}^{+\infty} B_k$, then $B_k \searrow E$, and

$$\mu(B_k) = \mu\left(\bigcup_{i=k}^{+\infty} E_i\right) \leq \delta_{k-1} \ (\forall\ k \geq 2).$$

Hence, $\mu(E) = 0$. For any fixed $\sigma > 0$ and $\epsilon > 0$, there exist $k_\sigma > 2$ and $k_\epsilon > 2$ such that

$$\mu(B_{k_\epsilon}) < \epsilon \text{ and } \frac{1}{2^{k-1}} < \sigma \ (\forall\ k \geq k_\sigma).$$

If $x_0 \in X \setminus E$, then there is $k_1 \geq k_\sigma$ such that

$$x_0 \notin \bigcup_{i=k_1}^{+\infty} \left\{x : \left|f_{n_{i+1}}(x) - f_{n_i}(x)\right| \geq \frac{1}{2^i}\right\}$$

and, therefore,

$$\left|f_{n_{k+r}}(x_0) - f_{n_k}(x_0)\right| \leq \sum_{i=1}^{r} \left|f_{n_{k+i}}(x_0) - f_{n_{k+i-1}}(x_0)\right| \leq \frac{1}{2^{k-1}} < \sigma \ (\forall\, r \geq 1, k \geq k_1),$$

i.e., $\{f_{n_k}(x_0)\}_k$ is Cauchy sequence of real numbers for any $x_0 \in X \setminus E$. Put

$$f(x) = \begin{cases} \lim_{k\to+\infty} f_{n_k}(x) & \text{if } x \in X \setminus E \\ 0 & \text{if } x \in E. \end{cases}$$

Then, $f \in \mathbf{F}$ and $f_{n_k} \xrightarrow{a.e.} f$.

Furthermore, we note that

$$\bigcup_{r=1}^{+\infty} \{x : \left|f_{n_{k+r}}(x) - f_{n_k}(x)\right| \geq \sigma\} \subset B_k \ (\forall\, k \geq k_\sigma).$$

Thus, for any $k \geq k_\sigma$,

$$\left|f_{n_{k+r}}(x) - f_{n_k}(x)\right| < \sigma \ (\forall\, r \geq 1, x \in X \setminus B_k)$$

and, therefore,

$$\left|f(x) - f_{n_k}(x)\right| \leq \sigma \ (\forall\, x \in X \setminus B_k, k \geq k_\sigma \vee k_\epsilon).$$

Hence, f_{n_k} converges to f uniformly on $X \setminus B_{k_\epsilon}$, and

$$\mu(\{x : \left|f(x) - f_{n_k}(x)\right| \geq 2\sigma\}) \leq \mu(B_k) < \epsilon$$

whenever $k \geq k_\sigma \vee k_\epsilon$. This shows $f_{n_k} \xrightarrow{a.u.} f$ and $f_{n_k} \xrightarrow{\mu} f$.

Theorem 2. *Let* μ *have the property* (p.g.p.) *and* $\{f_n\}_n$ *be fundamental in measure. If there exist* $f \in \mathbf{F}$ *and a subsequence* $\{f_{n_k}\}_k$ *of* $\{f_n\}_n$ *such that* $f_{n_k} \xrightarrow{\mu} f$, *then* $f_n \xrightarrow{\mu} f$.

Proof. For any fixed $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) \vee \mu(F) < \delta \Rightarrow \mu(E \cup F) < \epsilon.$$

Since $\{f_n\}_n$ is fundamental in measure and $f_{n_k} \xrightarrow{\mu} f$, there are n_0, $k_0 \geq 1$ such that $n_{k_0} \geq n_0$,

$$\mu\left(\left\{x : |f_n(x) - f_m(x)| \geq \frac{\sigma}{2}\right\}\right) < \delta \ (\forall\, m, n \geq n_0)$$

and

$$\mu\left(\left\{x : \left|f_{n_{k_0}}(x) - f(x)\right| \geq \frac{\sigma}{2}\right\}\right) < \delta \ (\forall\, k \geq k_0).$$

Thus, we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \sigma\}) \leq \mu\left(\left\{x : \left|f_n(x) - f_{n_{k_0}}(x)\right| \geq \frac{\sigma}{2}\right\} \cup \left\{x : \left|f_{n_{k_0}}(x) - f(x)\right| \geq \frac{\sigma}{2}\right\}\right) < \epsilon$$

whenever $n \geq n_0$. This shows $f_n \xrightarrow{\mu} f$.

Theorem 3. *Let* μ *have the property* (p.g.p.). *Then,* $f_n \xrightarrow{\mu} f$ *if and only if* $\{f_n\}_n$ *is fundamental in measure.*

Proof. The "only if" part is trivial by

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \sigma\}) \leq \mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\sigma}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| \geq \frac{\sigma}{2}\right\}\right)$$

and the property (p.g.p.) of μ. If $\{f_n\}_n$ is fundamental in measure, then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and $f \in \mathbf{F}$ such that

$$f_{n_k} \xrightarrow{\mu} f \ (\text{as } k \to +\infty)$$

by Theorem 1. Thus, $f_n \xrightarrow{\mu} f$ by Theorem 2.

Remark 1. The property (p.g.p.) of Theorem 3 is inevitable as shown in the following example.

Example 1. Let $X = [0, +\infty)$, $\mathcal{F}$ be the Borel σ-algebra of X, and m be the Lebesgue measure. Put

$$\mu(E) = \begin{cases} 0 & \text{if } 0 \notin E \\ m(E) & \text{if } 0 \in E. \end{cases}$$

Then, fuzzy measure μ has not the property (p.g.p.) [4]. Put $f(x) \equiv 1$ and

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (0, n] \\ 0 & \text{otherwise.} \end{cases}$$

For any fixed $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f_{n+p}| \geq \epsilon\}) \leq \mu((n, n+p]) = 0, \ (\forall\, n, p \geq 1)$$

and

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(\{0\} \cup (n, +\infty)) = +\infty, \ (\forall\, n \geq 1).$$

Therefore, $f_n \xrightarrow{a.e.} f$ and $\{f_n\}_n$ is fundamental in measure on $(X, \mathcal{F}, \mu)$, but $f_n \xrightarrow{a.u.} f$ and $f_n \xrightarrow{\mu} f$ are not true.

Remark 2. Any σ-additive measure is a fuzzy measure with the property (p.g.p.), but the converse is not true.

Example 2. Let $x = [0,1]$, $\mathcal{F}$ be the Borel σ-algebra of X, and m be the Lebesgue measure, and let

$$\mu(E) = \tan\left(\frac{\pi \cdot m(E)}{2}\right) \ (\forall\, E \in \mathcal{F}).$$

Then, μ has the property (p.g.p.), but it is not σ-additive.

Theorem 4. *Let μ be a fuzzy measure with the property* (p.g.p.), *and let f, $f_n \in \mathbf{F}$. Then $f_n \xrightarrow{\mu} f$ if and only if, for any subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$, there exists a subsequence $\{f_{n_{k_i}}\}_i$ of $\{f_{n_k}\}_k$ such that $f_{n_{k_i}} \xrightarrow{a.u.} f$.*

Proof. The "only if" part is trivial by Theorems 1 and 3. Suppose that "if" part is not true. Then there are ϵ_0, $\delta_0 > 0$ and a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that

$$\mu(\{x \,|\, |f_{n_k}(x) - f(x)| \geq \epsilon_0\}) \geq \delta_0$$

for any $k \geq 1$. For sequence $\{f_{n_k}\}_k$, there exists a subsequence $\{f_{n_{k_i}}\}_i$ of $\{f_{n_k}\}_k$ such that $f_{n_{k_i}} \xrightarrow{a.u.} f$ by the supposition of the theorem. It follows that there exists $B_{\delta_0} \in \mathcal{F}$, and an integer i_0 such that $\mu(B_{\delta_0}) < \delta_0$ and

$$|f_{n_{k_i}}(x) - f(x)| < \epsilon_0 \ (\forall\, i \geq i_0, x \in X \setminus B_{\delta_0}).$$

Hence, we have

$$\mu(\{x \,|\, |f_{n_{k_i}}(x) - f(x)| \geq \epsilon_0\}) \leq \mu(B_{\delta_0}) < \delta_0$$

whenever $i \geq i_0$. This is a contradiction with the fact that

$$\mu(\{x \,|\, |f_{n_{k_i}}(x) - f(x)| \geq \epsilon_0\}) \geq \delta_0$$

for any $i \geq 1$. The theorem is now proved.

From Theorems 3 and 4, we can immediately obtain the following corollaries.

Corollary 1. *Let a fuzzy measure have the property* (p.g.p.). *Then,* $\{f_n\}_n$ *is fundamental in measure if and only if there is* $f \in \mathbf{F}$ *and, for any subsequence* $\{f_{n_k}\}_k$ *of* $\{f_n\}_n$, *there exists a subsequence* $\{f_{n_{k_i}}\}_i$ *of* $\{f_{n_k}\}_k$ *such that* $f_{n_{k_i}} \xrightarrow{a.u.} f$.

Corollary 2. *Let* μ *be an order continuous fuzzy measure with the property* (p.g.p.). *Then,* $\{f_n\}_n$ *is fundamental in measure if and only if there is* $f \in \mathbf{F}$ *and, for any subsequence* $\{f_{n_k}\}_k$ *of* $\{f_n\}_n$, *there exists a subsequence* $\{f_{n_{k_i}}\}_i$ *of* $\{f_{n_k}\}_k$ *such that* $f_{n_{k_i}} \xrightarrow{a.e.} f$.

Remark 3. (1) The property (p.g.p.) in Corollary 1 is inevitable as shown in Example 1.

(2) The order continuity in Corollary 2 is inevitable, as shown in Example 3. It is possible for a fuzzy measure, which has neither null-addtivity nor the property (p.g.p.), that there exist f and a sequence $\{f_n\}_n$ of measurable functions which is fundamental in measure and $f_n \xrightarrow{a.e.} f$ (see Example 1).

Example 3. Let $X = [0, +\infty)$, $\mathcal{F}$ be the Borel σ-algebra of X, and let μ be the Lebesgue measure m. Then, μ is null-additive and has the property (p.g.p.). Since $E_n = [n, +\infty) \searrow \emptyset$ as $n \to +\infty$ and $\mu([n, +\infty)) = +\infty$, we know that μ is not order continuous. Put

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, n] \\ 0 & \text{if } x \in (n, +\infty). \end{cases}$$

Then, $f_n \to 1$, but

$$\mu(\{x : |f_n(x) - f_{n+p}| \geq \epsilon\}) = ((n, n+p]) = p, \ (\forall\ n, p \geq 1)$$

for any fixed $\epsilon \in (0,1)$. i.e., $\{f_n\}_n$ is not fundamental in measure on the fuzzy measure space $(X, \mathcal{F}, \mu)$.

Let E, $E_n \in \mathcal{F}$, and let

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases}$$

and

$$\chi_{E_n}(x) = \begin{cases} 0 & \text{if } x \notin E_n \\ 1 & \text{if } x \in E_n. \end{cases}$$

Then, χ_{E_n}, $(n \geq 1)$, are measurable functions on measure space $(X, \mathcal{F})$ and

$$\mu(\{x : |\chi_{E_n}(x) - \chi_{E_m}(x)| \geq \epsilon\}) = \mu((E_n \setminus E_m) \cup (E_m \setminus E_n)) = \mu(E_n \,\Delta\, E_m) \ (\forall\, m, n \geq 1)$$

for any $\epsilon \in (0,1)$. Thus, the following lemmas are easy to prove.

Lemma 1. *Let μ be a fuzzy measure and $\{E_n\}_n \subset \mathcal{F}$. Then, $\{\chi_{E_n}\}_n$ is fundamental in measure if and only if* $\lim_{m,n\to+\infty} \mu(E_n \,\Delta\, E_m) = 0$.

Lemma 2. *Let μ be a fuzzy measure and $E,\ E_n \in \mathcal{F}$. Then, $\chi_{E_n} \xrightarrow{\mu} \chi_E$ if and only if* $\lim_{n\to+\infty} \mu(E_n \,\Delta\, E) = 0$.

Theorem 5. *Let μ have the property* (p.g.p.) *and $\{E_n\}_n \subset \mathcal{F}$. Then, $\{\chi_{E_n}\}_n$ is fundamental in measure if and only if there is $E \in \mathcal{F}$ such that* $\lim_{n\to+\infty} \mu(E_n \,\Delta\, E) = 0$.

Proof. The "if" part is trivial by Lemma 2 and Theorem 3. Now, we prove the "only if" part. If $\{\chi_{E_n}\}_n$ is fundamental in measure, then there exist $f \in \mathbf{F}$ and a subsequence $\{\chi_{E_{n_k}}\}_k$ of $\{\chi_{E_n}\}_n$ by Corollary 1 such that

$$\chi_{E_{n_k}} \xrightarrow{a.u.} f \ (\text{as } k \to +\infty)$$

and, therefore,

$$\chi_{E_{n_k}} \xrightarrow{a.e.} f \ (\text{as } k \to +\infty).$$

Put $E = \overline{\lim}_{k\to+\infty} E_{n_k}$. Then, $f = \chi_E \ (a.e.)$. Hence,

$$\chi_{E_{n_k}} \xrightarrow{a.u.} \chi_E \ (\text{as } k \to +\infty)$$

and

$$\chi_{E_n} \xrightarrow{\mu} \chi_E \ (\text{as } n \to +\infty)$$

by the property (p.g.p.) of μ and Theorem 2. We can immediately obtain $\lim_{n\to+\infty} \mu(E_n \,\Delta\, E) = 0$ by Lemma 2. The theorem is now proved.

Corollary 3. *Let μ have the property* (p.g.p.) *and let $\{E_n\}_n \subset \mathcal{F}$. Then,* $\lim_{m,n\to+\infty} \mu(E_n \,\Delta\, E_m) = 0$ *if and only if there is $E \in \mathcal{F}$ such that* $\lim_{n\to+\infty} \mu(E_n \,\Delta\, E) = 0$.

Proof. Trivial by Lemma 1 and Theorem 5.

4. Almost uniform bound

Let μ be order continuous and $f \in \mathbf{F}$. It is easy to prove that for any $\epsilon > 0$ there exist $M > 0$ and set $E_\epsilon \in \mathcal{F}$ with $\mu(E_\epsilon) < \epsilon$ such that $|f(x)| \leq M$ for any $x \in X \setminus E_\epsilon$). By using the property (p.g.p.) of μ, we have the following lemma.

Lemma 3. *Let an order continuous fuzzy measure, μ, have the property* (p.g.p.) *and $f_i \in \mathbf{F}$ $(i = 1, 2, \ldots, n_0)$. Then, for any $\epsilon > 0$ there exist $M_\epsilon > 0$ and set $A_\epsilon \in \mathcal{F}$ with $\mu(A_\epsilon) \leq \epsilon$ such that*

$$|f_i(x)| \leq M_\epsilon \ (\forall\, x \in X \setminus A_\epsilon,\ i = 1, 2, \ldots, n_0).$$

For the above mentioned fact, we introduce a new concept, which is called to be *almost uniformly bounded*, and obtain some results on convergence of sequence of measurable functions on fuzzy measure spaces.

Definition 3. Let $\{f_n\}_n \subset \mathcal{F}$. We say that $\{f_n : n \geq 1\}$ is almost uniformly bounded, if, for any fixed $\epsilon > 0$, there exist $M_\epsilon > 0$ and set $A_\epsilon \in \mathcal{F}$ such that $\mu(A_\epsilon) \leq \epsilon$ and $|f_i(x)| \leq M_\epsilon$ for any $\forall\, x \in X \setminus A_\epsilon$ and $n \geq 1$.

Lemma 4. *Let μ have the property* (p.g.p.), *and let $\{f_n\}_n$ and $\{g_n\}_n$ be almost uniformly bounded. Then, for any real numbers α and β, $\{\alpha \cdot f_n + \beta \cdot g_n : n \geq 1\}$ and $\{f_n \cdot g_n : n \geq 1\}$ are almost uniformly bounded.*

Proof. We only prove that $\{f_n \cdot g_n : n \geq 1\}$ is almost uniformly bounded. In fact, for any fixed $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \vee \mu(F) < \delta \Rightarrow \mu(E \cup F) < \epsilon.$$

Since $\{f_n\}_n$ and $\{g_n\}_n$ are almost uniformly bounded, there exist M_f, $M_g > 0$ and E_f, $E_g \in \mathcal{F}$ such that

$$\mu(E_f) \vee \mu(E_g) < \delta,$$
$$|f_n(x)| \leq M_f \ (\forall\, x \in X \setminus E_f,\ n \geq 1),$$

and

$$|g_n(x)| \leq M_g \ (\forall\, x \in X \setminus E_g,\ n \geq 1).$$

Put $M_\epsilon = M_f \cdot M_g$ and $E_\epsilon = E_f \cup E_g$. Then, $\mu(E_\epsilon) < \epsilon$ and

$$|f_n(x) \cdot g_n(x)| \leq M_\epsilon \ (\forall\, x \in X \setminus E_\epsilon,\ n \geq 1).$$

The lemma is now proved.

Theorem 6. *Let μ be order continuous and have the property* (p.g.p.). *If $f_n \xrightarrow{a.u.} f$, then $\{f_n : n \geq 1\}$ is almost uniformly bounded.*

Proof. By using the property (p.g.p.) of μ, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) \vee \mu(F) < \delta \Rightarrow \mu(E \cup F) < \epsilon .$$

Since $f_n \xrightarrow{a.u.} f$, there are set E_δ and integer n_0 such that $\mu(E_\delta) < \delta$ and

$$|f_n(x)| \le |f(x)| + 1, (\forall\, x \in X \setminus E_\delta,\ n \ge n_0).$$

Furthermore, by Lemma 3, we may find $M_\delta > 0$ and a set F_δ such that $\mu(F_\delta) < \delta$ and

$$|f_i(x)| \le M_\delta\ (\forall\, x \in X \setminus F_\delta,\ 0 \le i \le n_0)$$

where $f_0 = f$. Put $M_\epsilon = M_\delta + 1$ and $E_\epsilon = E_\delta \cup F_\delta$. Then $\mu(E_\epsilon) < \epsilon$ and

$$|f_n(x)| \le M_\epsilon\ (\forall\, x \in X \setminus E_\epsilon,\ n \ge 1).$$

The theorem is now proved.

By using Egoroff's theorem given in [6], and Theorems 1, 4 and 6, we can immediately obtain the following results.

Theorem 7. *Let μ be order continuous and have the property* (p.g.p.). *If $f_n \xrightarrow{a.e.} f$, then $\{f_n : n \ge 1\}$ is almost uniformly bounded.*

Theorem 8. *Let μ be order continuous and have the property* (p.g.p.). *If $\{f_n\}_n$ is fundamental in measure, then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $\{f_{n_k} : k \ge 1\}$ is almost uniformly bounded.*

5. Fuzzy mean convergence

Let $\mathbf{F}_+$ be the class of all non-negative finite measurable functions on $(X, \mathcal{F})$, $f \in \mathbf{F}_+$, and let μ be a fuzzy measure on $(X, \mathcal{F})$. The *fuzzy integral* of f with respect to μ is defined by

$$(S)\int_X f d\mu = \sup_{0 \le \alpha < +\infty} [\alpha \wedge \mu(\{x : f(x) \ge \alpha\})].$$

Definition 4. Let $f,\ f_n \in \mathbf{F}_+$, $n = 1, 2, \ldots$. We say that $\{f_n\}_n$ fuzzy mean converges (f-mean converges, for short) to f if

$$\lim_{n \to +\infty} \rho(f_n, f) = \lim_{n \to +\infty} (S)\int_X |f_n - f| d\mu = 0;$$

and $\{f_n\}_n$ be fuzzy mean fundamental, if

$$\lim_{m,n\to+\infty} \rho(f_n, f_m) = \lim_{m,n\to+\infty} (S)\int_X |f_n - f_m| d\mu = 0.$$

Let $\epsilon > 0$ and $f,\ g \in \mathbf{F}_+$. By using the definition and the elementary properties of fuzzy integral, we have

$$(S)\int_X |f-g| d\mu \geq \epsilon \wedge \mu(\{x : |f(x)-g(x)| \geq \epsilon\})$$

and

$$\mu\left(\left\{x : |f(x)-g(x)| \geq \frac{\epsilon}{2}\right\}\right) < \epsilon \Rightarrow (S)\int_X |f-g| d\mu < \epsilon.$$

Thus the following lemmas are easily proved.

Lemma 5. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f,\ f_n \in \mathbf{F}_+$, $n = 1, 2, \ldots$. Then $f_n \xrightarrow{\mu} f$ if and only if $\{f_n\}_n$ fuzzy mean converges to f* [2].

Lemma 6. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $\{f_n\}_n \subset \mathbf{F}_+$. Then, $\{f_n\}_n$ is fuzzy mean fundamental iff $\{f_n\}_n$ is fundamental in measure.*

Theorem 9. *Let μ be a fuzzy measure with the property* (p.g.p.). *Then, $\{f_n\}_n$ is fuzzy mean fundamental if and only if there exists $f \in \mathbf{F}_+$ such that $\{f_n\}_n$ fuzzy mean converges to f.*

Proof. Trivial by using Theorem 3 and Lemmas 5 and 6.

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