MULTI-VARIATE STOPPING PROBLEMS
WITH A MONOTONE RULE

Masami Yasuda
Chiba University

Junichi Nakagami
Chiba University

Masami Kurano
Chiba University

(Received January 30, 1981; Final July 9, 1982)

Abstract  A monotone rule is introduced to sum up individual declarations in a multi-variate stopping problem. The rule is defined by a monotone logical function and is equivalent to the winning class of Kadane. This paper generalizes the previous works on a majority rule. The existence of an equilibrium stopping strategy and the associated gain are discussed for the finite and infinite horizon cases.

1. Formulation

Let \( \mathbf{X}_n \), \( n \geq 1 \), be \( p \)-dimensional random vectors on a probability space \( (\Omega, \mathcal{B}, P) \). The process \( \{X_n\} \) can be interpreted as a sequence of payoffs to a group of \( p \) players. Each of \( p \) players observes sequentially the values of \( \{X_n\} \). Its distribution is assumed to be known to all of the \( p \) players. A player must make a declaration to either "stop" or "continue" on the basis of the observed value at each stage. A group decision whether to stop the process or not is determined by summing up from the individual declarations.

If the group decision is to stop at stage \( n \), then the player \( \ell \)'s net gain is

\[
\sigma^\ell_n = \mathbf{X}_n^\ell - n \sigma^\ell
\]

where \( \sigma^\ell \) is a constant observation cost. According to the individual declarations, let us define a random variable \( \tilde{d}_n^\ell, n \geq 1, \ell = 1, \ldots, p \) by

\[
\tilde{d}_n^\ell = \begin{cases} 1 & \text{if player } \ell \text{ declares to stop (continue).} \\
0 & \text{if player } \ell \text{ continues (stop).} 
\end{cases}
\]

We assume, for each \( n \) and \( \ell \),

\[
\tilde{d}_n^\ell \in \mathcal{B}(\mathbf{X}_n)
\]
MULTI-VARIATE STOPPING PROBLEMS
WITH A MONOTONE RULE

Masami Yasuda  Junichi Nakagami
Chiba University  Chiba University
Masami Kurano
Chiba University

(Received January 30, 1981; Final July 9, 1982)

Abstract A monotone rule is introduced to sum up individual declarations in a multi-variate stopping problem. The rule is defined by a monotone logical function and is equivalent to the winning class of Kadane. This paper generalizes the previous works on a majority rule. The existence of an equilibrium stopping strategy and the associated gain are discussed for the finite and infinite horizon cases.

1. Formulation

Let \( X_n, n \geq 1 \), be \( p \)-dimensional random vectors on a probability space \((\Omega, \mathcal{F}, P)\). The process \( \{X_n\}_{n=1}^{\infty} \) can be interpreted as a sequence of payoffs to a group of \( p \) players. Each of \( p \) players observes sequentially the values of \( X_n \). Its distribution is assumed to be known to all of the \( p \) players. A player must make a declaration to either "stop" or "continue" on the basis of the observed value at each stage. A group decision whether to stop the process or not is determined by summing up from the individual declarations.

If the group decision is to stop at stage \( n \), then the player \( \xi \)'s net gain is

\[
\pi_n^\xi = X_n^\xi - n \sigma^\xi
\]

where \( \sigma^\xi \) is a constant observation cost. According to the individual declarations, let us define a random variable \( d_n^{\xi}, \xi = 1, \ldots, p \) by

\[
d_n^{\xi} = 1 \quad (0) \quad \text{if player } \xi \text{ declares to stop (continue).}
\]

We assume, for each \( n \) and \( \xi \),

\[
d_n^\xi \in \mathcal{B}(X_n)
\]

where \( \mathcal{B}(X_n) \) denotes the \( \sigma \)-algebra generated by \( X_n \).

Definition 1.1. An individual (stopping) strategy is a sequence of random variables

\[
d_n^\xi = (d_1^\xi, d_2^\xi, \ldots, d_n^\xi, \ldots)
\]

satisfying (1.3). \( \mathcal{A}_n^\xi \) denotes the set of all individual strategies for player \( \xi \). A \( p \)-dimensional and \((0,1)\)-valued random vector

\[
d_n = (d_1_n, d_2_n, \ldots, d_n^p)
\]

denotes the declarations of \( p \) players at stage \( n \). A (stopping) strategy is the sequence

\[
d = (d_1, d_2, \ldots, d_n, \ldots)
\]

and \( \mathcal{A} \) denotes the whole set of the strategies.

Now we shall define a stopping rule by which the group decision is determined from the declarations of players at each stage. A \( p \)-variate and \((0,1)\)-valued logical function

\[
\pi = \pi(x^1, \ldots, x^p) : (0,1)^p \rightarrow \{0,1\}
\]

is said to be monotone (cf. Fishburn [2]) if

\[
\pi(x^1, \ldots, x^p) \leq \pi(y^1, \ldots, y^p)
\]

whenever \( x^\xi \leq y^\xi \) for each \( \xi \).

Definition 1.2. A monotone rule is a non-constant logical function \( \pi \), which is

(i) monotone with

(ii) \( \pi(1,1,\ldots,1) = 1 \).

In this paper a rule does not mean "when to stop the process" but means "how to sum up" the whole players' declarations. The property (ii) is called unanimity in Fishburn [2]. Its dual property \( \pi(0,0,\ldots,0) = 0 \) is not needed to be assumed here. A constant function makes the problem trivial because the decision is always to stop from (i).

The monotone rule has a wide variety in choice systems of our real life. Some examples for the monotone rule are given as follows.

Example 1.1. (i) (Equal majority rule) In a group of \( p \) players, if no less than \( p \quad (\leq p) \) members declare to stop, then the group decision is to stop the process. That is,

\[
\pi(d_n^1, \ldots, d_n^p) = 1 \quad (0) \quad \text{if } \sum_{\xi=1}^{p} d_n^\xi = (\leq) p.
\]
For instance, a simple majority rule for three players, i.e., \((p, r) = (3, 2)\), is
\[
\pi(d_1^0, d_2^0, d_3^0) = d_1^0 + d_2^0 + d_3^0 - d_1^0 \cdot d_2^0 \cdot d_3^0
\]
where \(+\) is a logical sum and \(-\) is a logical product. The stopping problem of the majority rule is discussed in Kurano, Yasuda and Nakagami [5].

(ii) (Unequal majority rule) A straightforward extension of (1.9) is
\[
\pi(d_1^0, \ldots, d_n^0) = 1 \iff \sum_{i=1}^{p} w_i d_i^0 > (n - r)
\]
where \(w_i > 0\), \(i = 1, \ldots, p\), are given weighting constants. See Table 3.1 in Section 3 for several rules with \(p = 3\).

(iii) (Hierarchical rule) A hierarchical system or Murakami's representative system (cf. Fishburn [2]) is regarded as a composed rule. Since the composition of two monotone logical functions is monotone and satisfies the property (ii) of Def.1.2, the hierarchical rule is also a monotone rule.

**Definition 1.3.** For a strategy \(d = (d_1^0, \ldots, d_n^0) \in \mathcal{E}\) with \(d_n^0 = (d_1^0, \ldots, d_p^0, 0, \ldots, 0)\) and a monotone rule \(\pi\), a stopping time \(\tau(d)\) is defined by
\[
\tau(d) = \begin{cases} 
\min \{ n \geq 1 \mid \pi(d_1^0, \ldots, d_n^0) = 1 \} \quad &\text{if no such } n \text{ exists.}
\end{cases}
\]

For any stopping time \(\tau(d)\), let
\[
\mathcal{I}^d(\tau(d)) = \begin{cases} 
\tau(d) \quad &\text{if } \tau(d) = n.
\limsup_{n \to \infty} \frac{\tau(d)}{n} \quad &\text{if } \tau(d) = \infty.
\end{cases}
\]

When the group decision is to stop at the time \(\tau(d)\), player \(i\) gets
\[
\mathcal{I}^d(\tau(d))
\]
as a net gain.

**Definition 1.4.** Let \(\pi\) be a monotone rule. We call \(d = (d_1^0, \ldots, d_p^0)\) an equilibrium strategy with respect to \(\pi\) if, for each \(i\) and any \(d \in \mathcal{E}\),
\[
\pi(d) = \pi(d_i^0, \ldots, d_p^0)
\]
and our objective is to find an equilibrium strategy \(d \in \mathcal{E}\) for a given monotone rule \(\pi\). The notion of the equilibrium owes to the non-cooperative game theory by Nash [6].

In order to denote a stopping event of the process for a given rule, we need a set-valued function on \(\mathcal{E}(X_n)\). For \(d = (d_1, d_2, \ldots)\), we shall call
\[
\mathcal{R}^d_n = \{ \omega \in \Omega \mid d_n^0(\omega) = 1 \} \in \mathcal{E}(X_n)
\]
an individual stopping event for player \(i\) at stage \(n\). If \(\mathcal{R}^d_n\) occurs, i.e., \(\omega \in \mathcal{R}^d_n\), then player \(i\) declares to stop. So
\[
\mathcal{R}^d_n = I_{\mathcal{R}^d_n}
\]
where \(I_{\mathcal{R}^d_n}\) is the indicator of a set \(\mathcal{R}^d_n\) on \(\Omega\). Hence there exists a set-valued function \(\mathcal{R}^d_n\) on \(\mathcal{E}(X_n)\) corresponding to a logical function \(\pi\) on \(\{0, 1\}^p\), such that
\[
\pi(d_1^0, \ldots, d_p^0) = \pi(I_{\mathcal{R}^d_n}, \ldots, I_{\mathcal{R}^d_n}) = I_{\{d_1^0, \ldots, d_p^0\}}
\]

Clearly two functions \(\pi\) and \(\pi\) are related to each other. For example, \(\pi(d_1^0, d_2^0, d_3^0) = d_1^0 + d_2^0 - d_3^0\) corresponds to
\[
\Pi(d_1^0, d_2^0, d_3^0) = I_{\{d_1^0, d_2^0, d_3^0\}}
\]

The stopping event of the process at stage \(n\) is denoted by
\[
\mathcal{D}_n = \{ \omega \in \Omega \mid \mathcal{R}^d_n = 1 \} = \{d_1^0, \ldots, d_p^0\}
\]

Note that, if \(\pi\) is monotone, \(d \subseteq d\) for each \(i\) implies
\[
\mathcal{I}^d_i \subseteq \mathcal{I}^d_i
\]

from (1.18).

For a given (monotone) rule \(\pi\), a corresponding set-valued function \(\pi\) is determined only by the union and the intersection of sets.

Next, a one-stage stopping model is considered to clarify a strategy of our problem. Each player observes a random variable \(X = (X_1, \ldots, X_p)\) with \(E|X|^2|\leq \infty\), and player \(i\) receives a net gain \(X_i - \delta_i\) if the group decision is to stop, or \(\mathcal{I}^d_i - \delta_i\) if not, where \(\mathcal{I}^d_i\) is a given constant. For a monotone rule \(\pi\), the stopping event of the process becomes \(\pi(d_1^0, \ldots, d_p^0)\) for \(d \in \mathcal{E}\), \(i = 1, \ldots, p\). Then the expected net gain for player \(i\) is expressed by
\[
\mathbb{E}(X_i - \delta_i)|\pi(d_1^0, \ldots, d_p^0) + \mathbb{E}(\pi(d_1^0, \ldots, d_p^0))(\mathcal{I}^d_i - \delta_i)
\]

Since a logical function can be written generally as
\[
\pi(x_1, \ldots, x_p) = x_1^f \cdot \pi(x_1^f, \ldots, x_p^f) + x_2^f \cdot \pi(x_1^f, \ldots, x_p^f),
\]
it holds
\[
\pi(d_1^0, \ldots, d_p^0) = (d_1^o \cap \Pi(d_1^0, \ldots, d_p^0)) \cup (d_1^o \cap \Pi(d_1^0, \ldots, d_p^0))
in terms of the events. A substitution of this for the last expression of (1.20) yields

\[ \int D^2 [X^2 - Y^2]I(P_1, \ldots, P_r) dP + \int D^2 [X^2 - Y^2] dP + Y^2 - X^2. \]

By (1.19), it is clear that \( I(P_1, \ldots, P_r) \geq 0 \). Therefore we can derive the next proposition.

**Proposition 1.1.** When \( P_1, \ldots, P_{r-1}, P_{r+1}, \ldots, P_r \) are fixed, the player \( i \)'s maximum expected net gain subject to \( P_i \in \Omega(X) \) is attained by

\[ Y_i = \max(x, 0), \]

and it equals

\[ \int D^2 [X^2 - Y^2]I(P_1, \ldots, P_{r-1}, P_{r+1}, \ldots, P_r) dP + \int D^2 [X^2 - Y^2] I(P_1, \ldots, P_r) dP + Y^2 - X^2, \]

where \( x = \max(x, 0) \) and \( x = \max(-x, 0). \) Especially, when \( \Omega(X_1, \ldots, X_r) = \Omega^2, \ldots, X_r) \), player \( i \)'s expected net gain (1.22) and (1.24) is constant not depending on \( P_i \).

By Prop.1.1, we have solved a one-stage problem where the seeking equilibrium strategy is given as (1.23) and we have shown that the player \( i \)'s individual strategy depends only on the \( i \)-th component \( X^i \) of the \( P \)-dimensional vector \( X \). Because the larger he perceives his own value, the larger will be his net gain, he is eager to declare to stop when his observed value is high. This situation holds under a monotonicity of the rule, but does not hold under other rules including a negation. It is known that the monotone logical function does not include a negation and vice versa. Another essential point is the "non-cooperative" character in a reward, so other players' net gains do not affect his gain. Therefore, he observes his own value closely.

In the end of this section we shall now refer to the winning class of Kadane [4]. He proved the conjecture of Nakagami [8], that is, the reversibility in the juror problem by many of the players. To prove the reversibility affirmative, Kadane used a notion of the winning class as a choice rule.

**Definition 1.6.** Let \( P \) denote a number of players. A family \( \mathcal{W} \) of subsets of integers \( \{1, 2, \ldots, P\} \) is called a winning class if

\[ \{1, 2, \ldots, P\} \in \mathcal{W} \]

and

\[ \mathcal{W} \in \mathcal{W}, \mathcal{W} \subset \mathcal{W} \text{ implies } \mathcal{W} \in \mathcal{W}. \]

Assume that \( P \) players, e.g., player \( 1, \ldots, P \) declare to stop. Then the process must be stopped if the set \( 1, \ldots, P \) is an element of \( \mathcal{W}, \) or continued otherwise.

For a non-empty subset \( \mathcal{W}(1, \ldots, P) \) of \( \{1, 2, \ldots, P\} \) there corresponds a vertex \( z \) of the \( P \)-dimensional unit cube whose \( 1 \)-, \( 2 \)-, \ldots, and \( P \)-th components are equal to \( 1 \) and the remaining components \( 0. \) Concerning to the two correspondences between \( \mathcal{W} \), \( \mathcal{W} \subset \mathcal{W}, \) and \( x_1, x_2, \ldots, x_P \) respectively, a necessary and sufficient condition for \( x_1 \leq x_2 \) (component-wise) is that \( x_1 \leq x_2 \) (component-wise). Let \( V \) be a set of vertices corresponding to a winning class \( \mathcal{W} \). Define a logical function \( \pi \) by

\[ \pi(x_1, \ldots, x_P) = 1 \text{ if } (x_1, \ldots, x_P) \in V, \]

\[ = 0 \text{ otherwise.} \]

Accordingly the following proposition clearly holds.

**Proposition 1.2.** The stopping rule defined by a winning class of players, Def.1.6, is equivalent to the one by a monotone logical function, Def.1.2.

2. Finite Horizon Case

Consider a finite horizon case restricted by a prescribed number \( H < \infty \). Our objective is to find an equilibrium strategy for a given monotone rule and determine the associated expected net gain under the situation formulated in the previous section.

**Assumption 2.1.** (a) For any \( d \in \{d_1, \ldots, d_n\} \in \mathcal{D}, d_k = 1 \) for \( k = 1, \ldots, n \) with probability 1. (b) Random vectors \( X_1, \ldots, X_n \) are independent and \( E[x_{i,n}^2] = \) for each \( i \) and \( t \). (c) \( \pi \) is a monotone rule and \( \Pi \) is the corresponding rule of events.

Let us consider a sequence of vectors \( V_n = (V_1, \ldots, V_n) \) defined by

\[ V_n^i = v_{i,n}^i - \gamma_i \pi(X_{i,n}) + \pi(I_{i,n}) \]

and

\[ E[(x_{i,n}^2 - v_{i,n}^2)^2] = E[(x_{i,n}^2 - v_{i,n}^2)^2], \]

where \( \gamma_i \) is a positive constant.
\[ v_n^f = E[X_n^f] - \sigma_f, \]

where \( v_n^f = (v_n^1, \ldots, v_n^{i-1}, v_n^{i+1}, \ldots, v_n^p) \in R^{p-1}, \quad i = 2, \ldots, p. \)

(2.3) \( E[X_n^f](v_n^f|X_{-n}^f) = \Pi(v_n^{i-1}^{-1} \cdot \cdot \cdot v_n^{i-1}, v_n^{i+1} \cdot \cdot \cdot v_n^{p}) \cdot (v_n^{i-1} \cdot \cdot \cdot v_n^{i-1}, v_n^{i+1} \cdot \cdot \cdot v_n^{p}) \cdot \cdot \cdot (v_n^{i} \cdot \cdot \cdot v_n^{i}) \cdot \cdot \cdot (v_n^{p} \cdot \cdot \cdot v_n^{p}). \)

(2.4) \( E[X_n^f](v_n^f|X_{-n}^f) = \Pi(v_n^{i-1}^{-1} \cdot \cdot \cdot v_n^{i-1}, v_n^{i+1} \cdot \cdot \cdot v_n^{p}) \cdot (v_n^{i-1} \cdot \cdot \cdot v_n^{i-1}, v_n^{i+1} \cdot \cdot \cdot v_n^{p}) \cdot \cdot \cdot (v_n^{i+1} \cdot \cdot \cdot v_n^{p}) \cdot \cdot \cdot (v_n^{p} \cdot \cdot \cdot v_n^{p}). \)

and \( v_n^f = (v_n^1, \ldots, v_n^p) \in \mathcal{B}(X_{-n}^f), \quad i = 2, \ldots, p. \)

From Assum.2.1 (a) and (c), \( p(\tau_n^i|\omega) = 1 \) holds for all \( \omega \in \mathcal{B} \) even if the corresponding observation cost is negative.

Theorem 2.1. By the sequence \( \tau_n^i = (v_n^1, \ldots, v_n^p), \quad n > 0 \) in (2.1) and (2.2), let us define a strategy \( \tau_n^i \in \mathcal{B} \) as follows: For \( n=1, \ldots, N-1, \)

(2.5) \( \tau_n^0(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathcal{B}_n, \quad \text{i.e., } X_n(\omega) \geq v_n^f - \frac{\sigma_f}{2}, \\ 0 & \text{otherwise} \end{cases} \)

and

(2.6) \( \tau_n^0(\omega) = 1, \quad \text{a.e., } \omega \in \Omega. \)

Then, under Assum.2.1, \( \tau_n^i \) is an equilibrium strategy and

(2.7) \( E[Y(\tau_n^i(z))|Z_n] = v_n^f \)

holds. That is, \( v_n^f \) is an equilibrium expected net gain for player \( \tau_n. \)

Proof: Define

\[ \tau_n = \tau_n^i(\tau_n^i) \text{ is first } \Pi \text{ such that } \Pi(n-1) > 1 \]

for \( n=1, \ldots, N. \) Clearly \( n \leq \tau_n^i \leq \tau_n \) and \( n-1 \leq \tau_n^i \leq \tau_n. \) We shall show that

(2.8) \( E[Y(t_n^i(z))|Z_n] = v_n^f (\tau_n^i), \quad i = 1, \ldots, p. \)

by backward induction on \( n. \)

From \( \tau_n^0 = \tau_n^0(\tau_n^0) \), it is trivial for \( n \leq N. \) Assume that it is true for \( n+1. \) From the definition of \( \tau_n^0 = \Pi(v_n^{i-1}, \ldots, v_n^p) \in \mathcal{B}(X_n^f), \)

\[ \tau_n = \begin{cases} \tau_n^0 & \text{on } \mathcal{B}_n, \\ v_n^f(\tau_n^0) & \text{on } \mathcal{B}_n^c. \end{cases} \]

Hence

\( E[Y(t_n^i(z))|Z_n] = E[Y(t_n^0(z)|Z_n) + E[Y(t_n^0(z)|Z_n)] \]

where \( E[Y(z)] = \int r_i f(z) dp. \) Since \( X_{n+1}, X_{n+2}, \ldots \) are independent of \( X_n, \)

\( E[Y(t_n^0(z)|Z_n)] = E[Y[t_n^0(z)|Z_n] + E[Y(t_n^0(z)|Z_n)] \]

Therefore we have the recursion:

\( E[Y(t_n^i(z))] = E[Y(t_n^0(z)|Z_n) + P(d_n)E[Y(t_n^0(z)|d_n)]. \)

By induction, it is equal to

\( E[Y(t_n^0(z)|Z_n) + P(d_n)E[Y(t_n^0(z)|d_n)] = E[Y(t_n^0(z)|Z_n) + v_n^f - \sigma_f]. \)

The first term of the right hand side in the above is rewritten as

\( E[X_n - \sigma_f]^2 n - \sigma_f] + (v_n^f - \sigma_f) - (n-1) \sigma_f. \)

The above expression is equal to

\( E[X_n - \sigma_f]^2 n - \sigma_f] + (v_n^f - \sigma_f) - (n-1) \sigma_f. \)

So, from (2.1),

\( v_n^f = E[X_n - \sigma_f]^2 n - \sigma_f] + (v_n^f - \sigma_f). \)

This implies (2.8) and we have just proved the latter part of the theorem by letting \( n = 1 \) in (2.8).

Next we must show that, for fixed \( i, \)

(2.9) \( E[Y(t_n^0(z)|Z_n)] \leq E[Y(t_n^i(z)|Z_n)] \)

where \( t_n^0(z) = (d_1, \ldots, d_n, d_n) \) and \( t_n^i(z) = (d_1, \ldots, d_n) \) is any individual strategy for player \( i. \) Define \( n_d, n = 0, 1, \ldots, N \) by

\[ n_{d} = \begin{cases} \quad d & \text{if } n = 0, \\ d & \text{if } n \neq 0. \end{cases} \]

using \( n_d^0 \) and \( n_d^i. \) This \( n_n^d \) is consistent with \( n_n^d \) after first \( N \) periods.

Also define a strategy \( n_n^d(z) \) by

\( n_n^d(z) = (d_1^0, \ldots, d_n^0). \)

Clearly \( n_n^d(z) = (n_n^0(z) \quad 0) \quad 0(z) = \tau_n^0. \)

We show

(2.10) \( E[Y(t_n^0(z)|Z_n)] \leq E[Y(t_n^i(z)|Z_n)] \)

for \( n = 0, 1, \ldots, N \) because (2.9) can be proved immediately from (2.10). By the strategy \( t_n^0(z) \), it is enough to consider a stopping time \( t_n \) instead of \( t. \)

It is seen that

\( E[Y(t_n^0(z)|Z_n)] = E[Y(t_n^0(z)|Z_n) + P(d_n)E[Y(t_n^0(z)|d_n)]], \)

where \( P_n(z) \) is a stopping event with respect to \( \tau_n^0(z), \tau_n^1(z), \ldots, \tau_n^p(z). \) Since \( t_n^0(z) = \tau_n^0(z) \) on \( d_n \) and \( E[Y(t_n^0(z)|d_n)] = v_n^f - \sigma_f, \) it becomes

\( E[Y(t_n^0(z)|Z_n) + P(d_n)(v_n^f - \sigma_f) = (n-1) \sigma_f. \)
From Thm.2.1, the equilibrium strategy \( \mathbf{v}^* \) of \( \mu \) is determined by the sequence of \( \mathbf{v}_n^* \), \( n \geq 1\), in (2.11), where \( \sigma^2 = 0 \) and \( \mathbf{v}_1^* = 1/N \). Since the rule \( \mathbf{v} \) of (2.13) is symmetric for players 2 and 3, \( \mathbf{v}_n^* = \mathbf{v}_n^* \) from Cor.2.3. Define

\[
\mathbf{v}_n^* = \inf \{ \mathbf{v}_1, \mathbf{v}_n^* : \mathbf{v}_n^* \leq \mathbf{v}, n \geq 1 \}, \quad \forall \mathbf{v}, n \geq 1.
\]

The strategy for player \( 1 \) is that he observes until the \( n-1 \)th stage and then declares to accept if the relatively best one appears. For player 2 and 3, the strategy is similar. Numerical results are as follows.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x^1 )</th>
<th>( x^2 )</th>
<th>( v_n^1 )</th>
<th>( v_n^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3642</td>
<td>0.1685</td>
<td>0.3642</td>
<td>0.1685</td>
</tr>
<tr>
<td>30</td>
<td>0.3649</td>
<td>0.0981</td>
<td>0.3649</td>
<td>0.0981</td>
</tr>
<tr>
<td>100</td>
<td>0.3673</td>
<td>0.0322</td>
<td>0.3673</td>
<td>0.0322</td>
</tr>
<tr>
<td>300</td>
<td>0.3677</td>
<td>0.0135</td>
<td>0.3677</td>
<td>0.0135</td>
</tr>
<tr>
<td>1000</td>
<td>0.3678</td>
<td>0.0050</td>
<td>0.3678</td>
<td>0.0050</td>
</tr>
<tr>
<td>10000</td>
<td>0.3679</td>
<td>0.0007</td>
<td>0.3679</td>
<td>0.0007</td>
</tr>
</tbody>
</table>

We have applied our result to a secretary problem with an unequal monotone rule and showed the equilibrium strategy is a critical level strategy. But, as a remark, the asymptotic numerical result is non-interesting. Under the rule (2.13), player 1 behaves as if it were a one-person-game and player 2, 3 are neglected. A modified setting of the secretary problem has been discussed by Presman and Sonin [7] and Sakaguchi [9].

### 3. Infinite Horizon Case

In this section we shall treat an infinite horizon case \( H = \infty \). The class of rules is therefore \( \{ \mathbf{d} \in \mathcal{D}, P \uparrow \mathbf{d} \uparrow \mathbf{d} \leq \infty \} \). The problem is worth studying when the observation cost is non-negative. Thm.3.1 discusses the case of \( \sigma^2 > 0 \) for all \( \mathbf{d} \), in which case the stopping time is finite. When \( \sigma^2 = 0 \), \( \mathbf{d} = 1, \ldots, p \), some trouble occurs in the multi-variate case. Though we have defined \( \mathbf{v}^* = \limsup \mathbf{v}_n^* \) in (1.12) in the analogy of one-dimensional problem, apparently this definition is not natural for all players under some rules. To avoid this, we assume that the equilibrium stopping time is finite. Then, we can establish the continuity from the finite horizon case and compare the expected gains between rules and between players. From the formulation of our model, this assumption is often satisfied because the process is forced to stop by the conflict among players.
Assumption 3.1. (a) Random vectors $X^\iota_1, X^\iota_2, \ldots, X = (X^\iota_1, \ldots, X^\iota_p)$ are independent and identically distributed with $P[X^\iota_1 < x] = \omega$ for all $\iota$. (b) Each element of cost vector $\mathbf{c} = (c^1, \ldots, c^p)$ is strictly positive. (c) $\pi$ is a monotone rule and let $\Pi$ be the corresponding rule of events. (d) The following simultaneous equation of $V = (v^1, \ldots, v^p)$:

\[ \mathbb{E}[\mathbf{c}^T(X^\iota - X^\iota')] = \mathbb{E}[\mathbf{c}^T(X^\iota - X^\iota')|\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1})] = \mathbb{E}[\mathbf{c}^T(X^\iota - X^\iota')|\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1})] = \mathbf{1} \]

where

\[ \mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1}) \]

is a solution. Where $v^\iota = (v^1, \ldots, v^{\iota-1}, v^{\iota+1}, \ldots, v^p) \in \mathbb{R}^{p-1}$

\[ \mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1})|X^\iota] = \mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1})|X^\iota] \]

and $\mathbf{1} = (1, 1, \ldots, 1)$.

Theorem 3.1. Under Assump.3.1, A strategy $\mathbf{d}^\iota = (d^\iota_1, \ldots, d^\iota_p)$ determined by

\[ d^\iota_\ell = \mathbb{E}[\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1})|X^\iota] \]

for each $\ell$, is an equilibrium strategy in the class \{d\in\mathbb{R}^{p-1}|\mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1}) = 1] \}

and

\[ \mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota)\geq \mathbf{1}) = 1] \]

hold where $\mathbf{I} = (\mathbf{1}, \ldots, \mathbf{1})$ is a solution of (3.1).

By (3.4), $\mathbf{I}$ is called an equilibrium expected net gain. Since the proof is similar to that of Theorem 5.3, 5.4 of [5], we omit it here.

For the rest of the section we shall restrict our attention to the case:

(b') $\mathbf{c} = \mathbf{0}$.

Under the assumption (b'), it may happen that the equilibrium stopping time is not finite. But if the assumption (a) should be added, the following corollary must then hold

(e) $\mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) = 1$ where $\mathbf{c}^T$ is defined by (3.2).

It is seen in Ex.3.2 that there are cases which satisfy (e).

Corollary 3.1. Assume the cases (a), (b'), (c), (d), and (e). If $\mathbf{c}$ is bounded with prob. 1, then $\mathbf{d}^\iota$ is an equilibrium strategy in the restricted class \{d\in\mathbb{R}^{p-1}|\mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) = 1] \} and (3.4) holds.

Proof: The proof is immediate by Theorem 5.3, 5.4 of [5]. Hereafter we assume that

\[ a' \]

(a) and components of $\mathbf{x}^\iota$ are independent.

Corollary 3.2. Under the assumptions (a'), (b'), (c), (d), and (e), if $P(X^\iota = y) = 0$ where $y = \sup \{ y : P(X^\iota > y) > 0 \}$, then $\mathbf{d}^\iota$ is an equilibrium strategy in the class \{d\in\mathbb{R}^{p-1}|\mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) = 1] \} and (3.4) holds.

Proof: By Asump. (e), $\mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) = 0$ if $P(X^\iota = \mathbf{1}) = 1$ and (3.4) holds.

\[ \mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) > 0 \text{ from the monotonicity of the rule. From (3.1), it follows that} (a''), \ P(X^\iota = \mathbf{1}) = 0 \text{ and } \mathbb{P}[\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) > 0 \text{ imply } (X^\iota - \mathbf{d}^\iota)^+ = 0 \text{ a.e., that is, } \mathbf{d}^\iota = \mathbf{y} \text{. This means } \mathbf{d}^\iota = \mathbf{y} \text{ a.s. by the assumption. We have} \]

\[ P(X^\iota = \mathbf{1}) = P(X^\iota = \mathbf{1}) \]

This is a contradiction because the left hand side is " > 0" but the right hand side equals zero. Hence we obtain $P(X^\iota = \mathbf{1}) = 0$. For the strategy $\mathbf{d}^\iota = (d^\iota_1, \ldots, d^\iota_p)$ where $d^\iota_\ell$ is any individual strategy, it is seen that $P(X^\iota = \mathbf{1}) = 1$. Hence the proof is immediately completed from Thm.3.1.

Q.E.D.

For a rule $\tau$ with $P(\mathbf{I}(\mathbf{c}^T(X^\iota) = \mathbf{1}) = 1$, there is a player $\ell$ such that

\[ a^\ell = \sup \{ y : P(X^\iota > y) > 0 \} \]

Clearly (3.4) is satisfied for player $\ell$ by (1.12). But for another player $j \neq \ell$, $a^j$ does not necessarily satisfy (3.4). Therefore the solution of (3.1) does not always consist with the equilibrium expected gain in this case. In order to discuss the associated gain including this case, we simply call an expected gain (omitting "equilibrium") by the solution $\mathbf{v}^\iota$, which is the limiting value as $\mathbf{c}^T(X^\iota)$ in the finite horizon case. For this see Figure 4.1 in [5] and Table 3.1.

Now we shall derive a bound of the expected gain by varying $\pi$'s. The expected gain $\mathbf{v}^\iota = \mathbf{v}^\iota(\pi)$ associated with a monotone rule $\pi$ satisfies that

\[ \mathbb{E}[X^\iota - \mathbf{d}^\iota] \leq \mathbb{E}[\mathbf{v}^\iota] \leq \sup \{ y : (X^\iota > y) > 0 \} \]

This is proven by using a result as follows. By (a') and (b'), equation (3.1) implies

\[ \mathbb{E}[\mathbf{v}^\iota(X^\iota - \mathbf{d}^\iota)] = P(X^\iota = \mathbf{1}) \]

where

\[ \mathbf{d}^\iota = (d^\iota_1, \ldots, d^\iota_p) \]
(3.8) \[
\hat{v}_n^j(\nu(x)) = q_n^j(\nu(x)) \frac{p_n^j(\nu(x))}{p_n^j(\nu(x))} \\
= P(\nu(x^1, \ldots, \nu^{d^n})) / P(\nu(x^1, \ldots, \nu^{d^n})),
\]
providing that the denominator is non-zero. Since \( \nu \) is monotone,\(^n\)
\[
0 \leq p_n^j(\nu(x)) \leq 1
\]
holds. Therefore (3.9) implies (3.6) immediately.

From the above argument, \( p_n^j(\nu(x)) = 1 \) implies \( \nu^j = \nu^x \), and
\[
p_n^j(\nu(x)) = 0 \] implies \( \nu^j = \sup y : P(y, x) > 0 \). The second assertion

For every \( \nu(x) \rightarrow \nu(x^k) = \nu \) as remarked at (3.5). Here these two extreme

cases are interpreted as follows.

First, \( p_n^j(\nu(x)) = 0 \) is equivalent to \( P(\nu(x), \ldots, \nu^{d^n}) = 0 \), e.g.
and also to \( \pi(\nu(x), \ldots, \nu^{d^n}) = 0 \) with prob. 1. This means that whenever

Theorem 3.2. Let \( \nu_n = (\nu_n^1, \ldots, \nu_n^p) \) and \( \nu_n = (\nu_n^1, \ldots, \nu_n^p) \) be
the expected gains corresponding to \( \Pi \) and \( \Pi \) respectively. For player \( \nu \) and \( \nu \),

Theorem 3.3. Under a fixed \( \Pi \), if \( \nu^x \) and \( \nu^x \) are identically
distributed and if

Example 3.1. Consider a majority rule \( \Pi = (p, r) \) of players, where

In fact, since the rule \( \Pi = (\nu) \) is symmetric, we can set the players' gains being equal:

\[
\nu_n^x = \nu_n^x, \quad \xi = 1, \ldots, r.
\]

Hence

\[
\nu_n^x = \nu_n^x = 0, \quad \xi = 1, \ldots, r.
\]

and \( \eta(\nu, \nu) = P(\nu, \nu) / \sum \nu(\nu, \nu) \nu(\nu, \nu) \)

Theorem 3.4. If, for player \( \nu \),

For \( \nu = 1, \ldots, p-1, \Pi = (\nu) \) has an equilibrium strategy and \( \nu_n \) is an

Figure 4.1 in [5] shows each expected gain of (3.15) for \( p = 5 \) players.

Example 3.2. Let components of random vectors be independent and

Identical distribution with a common uniform distribution \( U(0,1) \). Table 3.1
shows a numerical example with \( p = 3 \) for non-trivial monotone rules. In the

The four rules \( P(\nu, x) = 1 \), but this does not hold in other cases.
From (3.5), there exist players who attain the maximum expected gain "unity" in the last four rules. Each expected gain is the limiting value of the finite horizon case. Except for the 5-th, 6-th and 7-th rule, the value is an equilibrium one by Cor. 3.2.

Table 3.1 Monotone rules with \( p = 3 \).

<table>
<thead>
<tr>
<th>Monotone rule</th>
<th>1 ( \times 2 \times 3 )</th>
<th>1 ( \times 2 \times 2 )</th>
<th>1 ( \times 2 \times 3 )</th>
<th>1 ( \times 2 \times 2 \times 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(1,2,3) )</td>
<td>( (1,2,3) )</td>
<td>( (1,2,2) )</td>
<td>( (1,2,3) )</td>
<td>( (1,2,2,3) )</td>
</tr>
<tr>
<td>Comments</td>
<td>majority</td>
<td>pl.3 is</td>
<td>asymmetric</td>
<td>majority</td>
</tr>
<tr>
<td></td>
<td>for the rule</td>
<td>pl.3 is</td>
<td>case</td>
<td>for the rule</td>
</tr>
<tr>
<td></td>
<td>(p,r)=(3,1)</td>
<td>outsider</td>
<td>asymmetric</td>
<td>(p,r)=(3,2)</td>
</tr>
</tbody>
</table>

\( \nu^1 \) | 0.5437 | \( \sqrt{2}/2 \) | \( \sqrt{2}/2 \) | \( \sqrt{2}/2 \) |

\( \nu^2 \) | 0.5437 | \( \sqrt{2}/2 \) | \( \sqrt{2}/2 \) | \( \sqrt{2}/2 \) |

\( \nu^3 \) | 0.5437 | 0.5 | \( \sqrt{2}/2 \) | \( \sqrt{2}/2 \) |

<table>
<thead>
<tr>
<th>( x^1 )</th>
<th>( x^2 )</th>
<th>( x^3 )</th>
<th>( x^1 \times x^2 \times x^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>pl.1 is</td>
<td>pl.1 has</td>
<td>pl.3 is</td>
<td>unanimity</td>
</tr>
<tr>
<td>a dictator</td>
<td>a veto</td>
<td>pl.3 is</td>
<td>(p,r)=(3,3)</td>
</tr>
<tr>
<td>power</td>
<td>outsider</td>
<td>an</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>( \sqrt{2}/2 )</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>( \sqrt{2}/2 )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Acknowledgement

The authors wish to express their thanks to the referees for helpful comments and suggestions.

References


Masami Yasuda
Statistics Laboratory
College of General Education
Chiba University
Yayoi-cho, Chiba, 260, Japan

Junichi Nakagami
Department of Mathematics
Faculty of Science
Chiba University
Yayoi-cho, Chiba, 260, Japan

Masami Kurano
Department of Mathematics
Faculty of Education
Chiba University
Yayoi-cho, Chiba, 260, Japan
単調ルールによる多変量停止問題

千葉大学 安田正実
中神潤一
関野正美

P変数確率変数 $X_n = (X_1^n, \ldots, X_P^n)$, \( n \geq 1 \) が P 人の集団（各人をプレイヤーとよぶ）によって次々に観測され、集団全体の決定のまちがいの観測過程を停止できるとする。もし t 期で停止すると、プレイヤー i (i = 1, \ldots, P) は $Y_i^t = X_i^t - tc_i^t$ の利得を受けとる。ただし、$C = (c^1, \ldots, c^P)$ は、k 期間当りの観測費用である。各プレイヤーは停止時に自分の期待利得を最大にしたいと思っている。

プレイヤー i が $X_n$ の実現値を観測したとき、n 期での過程の停止宣言を $d_i^n = 1$, 続続宣言を $d_i^n = 0$ で表す。この系列 $d = (d_1^1, d_2^1, \ldots)$ をプレイヤー i の個人停止戦略とよび、行列 $d = (d_1^1, d_2^1, \ldots, d_P^n)$ を停止戦略とよぶ。このとき、各プレイヤーの意見を集約する集団の決定ルールが必要になる。我々は決定ルールを表すために、$[0, 1]$ 上の P 変数線形関数 $f: [0, 1]^P \rightarrow [0, 1]$ を用いる。線形関数が単調で $f(1, \ldots, 1) = 1$ であるとき、単調ルールをとる。このルールは Kadane の確率論的選択問題で導入した Winning class と本質的に同じものである。

本論文は集団の意志決定ルールとして単調ルールを用い、多人数停止問題を非協力ゲームとして定式化した。さらに Nash の概念による均衡停止戦略 *d を定義し、その存在性の証明と解析を行なった。これは既存の Multi-Variate Stopping Problem with a Majority Rule の拡張である。

有限期間 (N < ∞) では、漸化式で定まるベクトル列 $V_n = (v_1^n, \ldots, v_P^n)$ に対して、プレイヤー i (i = 1, \ldots, P) が $X_n = v_i^n$ なる n で停止宣言することが均衡停止戦略になる。また均衡期待利得は $b_i^n$ である。例として不平等ルールでの総合問題を扱った。

無限期間 (N = ∞) については、連立方程式の解 $V = (u_1, \ldots, u_P)$ によって同様な均衡停止戦略が求められる。特に $C = 0$ で、$X_n = (X_1^n, \ldots, X_P^n)$ が要素についても i.e.d. の場合に、各プレイヤーの単調ルールに付随した集団に対する "パワー" を表す $\rho$ を定義し、これと均衡期待利得 $\rho V$ との比較を行なった。