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MULTI-VARIATE STOPPING PROBLEM WITH A MAJORITY RULE

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Abstract This paper studies the stopping problem for random vectors of p components which correspond to the payoffs to a group of p players. The observation process is stopped at the first time when no less than $r(1 \leq r \leq p)$ players declare to stop. We call it a majority rule. The object of this paper is to find out a reasonable stopping strategy under a class of these rules, in both cases of finite and infinite decision horizons. We solve our stopping problem by introducing the concept of an equilibrium point in the non-cooperative game theory. Several examples including a variant of the secretary problem are given.

1. Introduction

Random vectors X_n, n=1,2,$\cdots$, are observed sequentially by many players with a common known joint distribution. Cost vectors c_n, n=1,2,$\cdots$, are incurred at each stage. Let X_n, c_n be vectors of p components

$$X_n = \begin{pmatrix} X_n^1 \\ \vdots \\ X_n^p \end{pmatrix} = (X_n^1, \ldots, X_n^p)^T, \qquad c_n = \begin{pmatrix} c_n^1 \\ \vdots \\ c_n^p \end{pmatrix} = (c_n^1, \ldots, c_n^p)^T$$

where T denotes the transposition of vectors.

If the group of p players agree to stop the observation process at the n-th stage, each player $i(1 \leq i \leq p)$ gets the reward X_n^i but incurs the observation cost c_n^i accumulated upto the n-th stage. The net gain of player i would then be $X_n^i - c_n^i$. Each player wishes to make his expected net gain as large as possible. The problem is to find out the timing when to stop the process under a reasonable class of rules.

The class of stopping rules we consider in this paper is as follows: (i) Each player can declare to stop at any stage. (ii) At the beginning of this game, the majority level $r(1 \leqq r \leqq p)$ is admitted by the whole players. (iii) In the course of the observation process, if the number of players declaring to stop is greater than or equal to the level r, the process must be stopped. This generalizes the so-called majority rule. Under the class of these rules we shall derive a "good" one in the sense of the equilibrium. Roughly speaking, it is a p collection of the individual stopping strategies, such that there should be no other individual one which yields a greater gain for each player, if other players do not alter their strategies.

In Section 2, we formulate the multi-variate stopping problem with this rule precisely by the concept of equilibrium points in the non-cooperative game theory(Nash[3]). The finite horizon case is treated in Section 3. We derive a recursive relation of expected net gains and an equilibrium stopping strategy for each player. Sakaguchi[6] has already obtained the result in the case of the unanimity stopping rule, that is, p=r. In Section 4, we compute some examples of the majority stopping rule(p>r) for uniform and normal distribution cases in order to illustrate the equilibrium expected net gains. A bivariate version of the secretary problem is investigated where each applicant has two abilities so that they are not totally ordered. Presman and Sonin[4] treat this with another stopping rule. They consider the model in which each player's decision does not affect the stopping of the process but his reward only. In the final section the infinite horizon case is considered, and we obtain an equilibrium equation which determines the equilibrium stopping strategy. An example of a uniform distribution case is given to get the majority level which maximizes the equilibrium expected net gain.

After writing the first draft of this paper, we received details of the similar work by Sakaguchi[7] and Kadane[2]. The structure of their problem is a multi-lateral sequential process in which decisions when to stop are made by the players alternately, instead of the simultaneous decision under a majority rule. Although our main result of the equilibrium stopping strategy is essentially identical to theirs of the optimal procedure, the formulation and analysis in this paper are different from [7] and [2].

2. Statement of the problem

In this section we shall describe our stopping problem in terms of notations used in the non-cooperative game theory.

Whether player i declare to stop the process at the n-th stage or not, depends on the observed value of the random vector X_n. If he agrees to stop, $\sigma^i_n=1$, and otherwise 0. Assume $\sigma^i_n \in \{0,1\}$ is measurable with respect to $\mathcal{B}(X_n)$, i.e., the σ-algebra generated by X_n. For each $i(1 \leqq i \leqq p)$, a sequence of random variables $\sigma^i=(\sigma^i_1,\sigma^i_2,\cdots)$ is called <u>an individual stopping strategy</u>(abr. by ISS) for player i. Let S^i be the class of the ISS's for player i and let $S=(S^1,S^2,\cdots,S^p)^T$. A matrix $\sigma=(\sigma^1,\sigma^2,\cdots,\sigma^p)^T$, an element of S, is called <u>a stopping strategy</u>(abr. by SS), which is a collection of all p players' ISS. So, columns and rows of a matrix σ correspond to the individual players and the sequential stages, respectively.

Given any fixed integer $r(1 \leqq r \leqq p)$, define

$$t(\sigma)= \text{first } n \geqq 1 \text{ such that } \sum_{i=1}^{p} \sigma^i_n \geqq r \tag{2.1}$$

for each $\sigma \in S$, and we call it <u>a stopping time</u> generated by the SS σ. The stopping time $t(\sigma)$ shows that the observation process must be stopped at the first time such that the number of players who declare to stop, i.e., the column sum of a matrix σ, is greater than or equal to the majority level r. The expected net gain for player i is

$$E[\ Y^i_{t(\sigma)}\ ] \tag{2.2}$$

where $Y^i_n=X^i_n-c^i_n$ $(1 \leqq i \leqq p)(n=1,2,\cdots)$.

Throughout the paper we shall use the following mathematical notations: For any set A, $\bar{A}$ is the complement of A and I_A is the indicator of A. E[X] is the expectation of a random variable X. $E_A[X]$ means $E[I_A \cdot X]$, and $E[\cdot|A]$ or $E[\cdot|X]$ is the conditional expectation. Also we use $X^+= \max(X,0)$.

Definition 2.1. A SS $*\sigma= (*\sigma^1,*\sigma^2,\cdots,*\sigma^p)^T$ is called an equilibrium SS if

$$E[\ Y^i_{t(*\sigma(i))}] \leqq E[\ Y^i_{t(*\sigma)}] \quad \text{for all } \sigma^i \in S^i \quad (i=1,2,\cdots,p), \tag{2.3}$$

where $*\sigma(i)=(*\sigma^1,\cdots,*\sigma^{i-1},\ \sigma^i\ ,*\sigma^{i+1},\cdots,*\sigma^p)^T$.

The definition of the SS $*\sigma$ means that there exists no other ISS which yields a greater gain for each player if other players do not alter their ISS's. This concept was first introduced by Nash[3] in non-cooperative game theory.

Now we show a fundamental lemma to be used for deriving the equilibrium SS in the following section. Consider a simple model: Each player i observes a random variable $X=(X^1,\cdots,X^p)^T$ with a cost c^i and decides whether to stop or not. If no less than r players agree to stop, player i's net gain is X^i-c^i. Otherwise, he receives a given constant value v^i-c^i. Let $A^i \varepsilon \mathcal{B}(X)$ be an individual stopping event for player i, that is, if A^i occurs then player i declares to stop. The event $g^{\{i\}}(r)$ is the union of exactly r events among $\{A^1,\cdots,A^{i-1},A^{i+1},\cdots,A^p\}$. It occurs if and only if the number of players declaring to stop except player i is just equal to r. Let $G^{\{i\}}(r) = \sum_{k=r}^{p} g^{\{i\}}(k)$. Then a stopping event B of the process with respect to $A^1,\cdots,A^p$ is represented for player i by

(2.4) $$B = A^i \cap G^{\{i\}}(r-1) + \overline{A}^i \cap G^{\{i\}}(r) \qquad (i=1,\cdots,p).$$

Lemma 2.2. The maximum of $E_B[X^i-c^i] + P(\overline{B})(v^i-c^i)$ subject to $A^i \varepsilon \mathcal{B}(X)$ for any fixed $A^1,\cdots,A^{i-1},A^{i+1},\cdots,A^p$ is attained by $*A^i = \{X^i \geqq v^i\}$, which depends only on the observed value of X^i. That is,

(2.5) $$E_B[X^i-c^i] + P(\overline{B})(v^i-c^i) \leqq u^i-c^i$$

holds, where $u^i = E[(X^i-v^i)^+ I_{g^{\{i\}}(r-1)}] + E[(X^i-v^i)P(G^{\{i\}}(r)|X^i)] + v^i$.

Proof: By the elementary calculation,

$$E_B[X^i-c^i] + P(\overline{B})(v^i-c^i) = E[(X^i-v^i)I_B] + v^i - c^i .$$

From the definition of (2.4), $B=A^i \cap g^{\{i\}}(r-1)+G^{\{i\}}(r)$. Substituting this, it is proved immediately.

Cleary the left-hand side of (2.5) is equal to player i's net gain. By Lemma 2.2, he will declare to stop when $\{X^i \geqq v^i\}$ occurs according to his own profit X^i only, independently of the stopping events of other players, so as to maximize his expected net gain.

3. The stopping problem with a finite horizon

In this section, we study the stopping problem where the number of observations is limited by N. Assume that (a) $\sigma_N^i=1$ for all i, that is, the observation process must be stopped at the N-th stage. (b) The rewards $X_1,X_2,\cdots,X_N$ are independent random vectors with $E|X_n^i|<\infty$ for each i,n. And (c) the observation cost $c_n = nc$ for $n=1,\cdots,N$, where $c=(c^1,\cdots,c^p)^T$ is a constant vector.

Let $v_n = (v_n^1,v_n^2,\cdots,v_n^p)^T$ $(n=1,2,\cdots,N)$ be a sequence of p dimensional vectors defined by

(3.1) $$v_{n+1} = \begin{pmatrix} E[(X_{N-n}^1-v_n^1)^+P(*g_{N-n}^{\{1\}}(r-1)|X_{N-n}^1)] + E[(X_{N-n}^1-v_n^1)P(*G_{N-n}^{\{1\}}(r)|X_{N-n}^1)] \\ \vdots \\ E[(X_{N-n}^p-v_n^p)^+P(*g_{N-n}^{\{p\}}(r-1)|X_{N-n}^p)] + E[(X_{N-n}^p-v_n^p)P(*G_{N-n}^{\{p\}}(r)|X_{N-n}^p)] \end{pmatrix} + v_n - c, \quad (n=1,2,\cdots,N-1)$$

(3.2) $$v_1 = \begin{pmatrix} E[X_N^1] \\ \vdots \\ E[X_N^p] \end{pmatrix} - c .$$

Here each $*g_{N-n}^{\{i\}}(k)$ is defined, similarly as in Section 2, by the events $*A_{N-n}^1=\{X_{N-n}^1 \geqq v_n^1\},\cdots,*A_{N-n}^p=\{X_{N-n}^p \geqq v_n^p\}$ at the (N-n)-th stage.

For each $i=1,2,\cdots,p$, define the ISS $*\sigma^i=(*\sigma_1^i,*\sigma_2^i,\cdots,*\sigma_N^i)$ for player i by

(3.3) $$\begin{cases} *\sigma_n^i = 1 \text{ if } *A_n^i \text{ occurs, i.e., } X_n^i \geqq v_{N-n}^i \quad (n=1,2,\cdots,N-1) \\ \quad = 0 \text{ otherwise and} \\ *\sigma_N^i = 1. \end{cases}$$

Then we can state the following theorem.

Theorem 3.1. The above SS $*\sigma=(*\sigma^1,*\sigma^2,\cdots,*\sigma^p)^T$ is an equilibrium SS and $E[Y_{t(*\sigma)}] = v_N$.

Proof: Denote $t^*=t(*\sigma)$ for simplicity. We use backward induction on n. Suppose that

(3.4) $$E[Y_{t^*}|t^* \geqq n+1] = v_{N-n} - nc .$$

Then, putting $*B_n=\{\sum_{i=1}^p *\sigma_n \geqq r\}$, we have

$$\begin{aligned} E[Y_{t^*}|t^* \geqq n] &= E_{*B_n}[Y_n|t^* \geqq n] + P(*\overline{B}_n)E[Y_{t^*}|t^* \geqq n,*B_n] \\ &= E_{*B_n}[Y_n] + P(*\overline{B}_n)E[Y_{t^*}|t^* \geqq n+1] \\ &= E_{*B_n}[X_n-nc] + P(*\overline{B}_n)(v_{N-n}-nc) \text{ (by inductive hypothesis)} \\ &= E_{*B_n}[X_n-c] + P(*\overline{B}_n)(v_{N-n}-c) - (n-1)c . \end{aligned}$$

Since $*B_n=*A_n^i \cap *G_n^{\{i\}}(r-1)+*\overline{A}_n^i \cap *G_n^{\{i\}}(r)$ and by Lemma 2.2,

$$E[Y_{t^*}^i|t^* \geqq n] = E[(X_n^i-v_{N-n}^i)^+P(*g_n^{\{i\}}(r-1)|X_n^i)]$$

$$+ E[(X_n^i - v_{N-n}^i)P(*G_n^{\{i\}}(r)|X_n^i)] + v_{N-n}^i - c^i - (n-1)c^i$$
$$= v_{N-n+1}^i - (n-1)c^i ,$$

showing that (3.4) is true for n-1. Since (3.4) is trivially true for n=0, the latter part of the theorem is proved.

Nextly we shall show $*\sigma$ is an equilibrium SS. It suffices to show that for any given i, say i=1,

(3.5) $$E[Y^1_{t(\sigma_{\{n\}})}] \leqq E[Y^1_{t(\sigma_{\{n-1\}})}] \text{ for any } \sigma^1=(\sigma_1^1,\cdots,\sigma_N^1)\varepsilon S^1,\ 1\leqq n\leqq N$$

where $\sigma_{\{n\}}=(\sigma^1_{\{n\}},*\sigma^2,\cdots,*\sigma^p)^T$, $\sigma^1_{\{n\}}=(\sigma_1^1,\cdots,\sigma_n^1,*\sigma_{n+1}^1,\cdots,*\sigma_N^1)$ and $\sigma^1_{\{0\}}=*\sigma^1$, $\sigma^1_{\{N\}}=\sigma^1$. An ISS for player 1, $\sigma^1_{\{n\}}$ means that player 1 uses any fixed σ^1 from the stage 1 to n and uses the equilibrium $*\sigma^1$ from the stage n+1 to N. To show (3.5), putting $t\{n\}=t(\sigma_{\{n\}})$,

$$E[Y^1_{t\{n\}}] = E_{t\{n\}<n}[Y^1_{t\{n\}}] + P(t\{n\}\geqq n)E[Y^1_{t\{n\}}|t\{n\}\geqq n]$$
$$= E_{t\{n-1\}<n}[Y^1_{t\{n-1\}}] + P(t\{n-1\}\geqq n)E[Y^1_{t\{n\}}|t\{n\}\geqq n]$$

from the definition of $\sigma^1_{\{n\}}$. Let $A_n^1=\{\sigma_n^1=1\}$ and $*A_n^i=\{*\sigma_n^i=1\}$ for $i\neq 1$. A stopping event B_n of the process with respect to $A_n^1,*A_n^2,\cdots,*A_n^p$ for player 1 is represented by $B_n = A_n^1\cap *G_n^{\{1\}}(r-1) + \bar{A}_n^1\cap *G_n^{\{1\}}(r) = \{\sigma_n^1 + \sum_{i=2}^p *\sigma_n^i \geqq r\}$. So

$$E[Y^1_{t\{n\}}|t\{n\}\geqq n] = E_{B_n}[Y_n^1] + P(\bar{B}_n)E[Y^1_{t\{n\}}|t\{n\}\geqq n+1]$$
$$= E_{B_n}[Y_n^1] + P(\bar{B}_n)E[Y^1_{t*}|t*\geqq n+1]$$
$$= E_{B_n}[X_n^1 - nc^1] + P(\bar{B}_n)(v^1_{N-n} - nc^1)$$
$$= E_{B_n}[X_n^1 - c^1] + P(\bar{B}_n)(v^1_{N-n} - c^1) - (n-1)c^1$$
$$\leqq v^1_{N-n+1} - (n-1)c^1$$
$$= E[Y_{t*}|t*\geqq n]$$

by (3.4), lemma 2.2 and (3.1). The above equality holds when $\sigma_n^1 = *\sigma_n^1$. Therefore, from the definition of $\sigma^1_{\{n-1\}}$, $t*=t\{n-1\}$ on $\{t*\geqq n\}$, we have

$$E[Y^1_{t\{n\}}] \leqq E_{t\{n-1\}<n}[Y^1_{t\{n-1\}}] + P(t\{n-1\}\geqq n)E[Y^1_{t\{n-1\}}|t\{n-1\}\geqq n]$$
$$= E[Y^1_{t\{n-1\}}].$$

Q.E.D.

The assumption (b) is inessential. Namely, the reward X_n at the n-th stage may depend on the previous $X_1,\cdots,X_{n-1}$. A similar recursive relation as (3.1) is derived and the existence of the equilibrium SS such as Theorem 3.1

can be shown.

Remark The second terms in the bracket of (3.1) are zero for the unanimity case(p=r), and hence (3.1) coincides with (11) of Sakaguchi[5]. We also find that if $X_n=(X_n^1,\cdots,X_n^p)^T$, $n\geqq 1$, are i.i.d. with a common random variable $X=(X^1,\cdots,X^p)^T$ and c=0, (3.1) and (3.2) reduce to

$$v^i_{n+1} = v_n^i + E_{*B_n}[X^i - v_n^i], \quad n \geqq 1,$$
$$v_1^i = E[X^i],$$

i=1,···,p, where $*B_n$={At least r, among p inequalities $X^i \geqq v^i_{n-1}$ i=1,···,p, hold}. See Theorem 2 in Sakaguchi[7]. Also see Remark 5 in [7] and Kadane[2].

Sakaguchi[7] formulated the multi-variate stopping problem by a different approach in which sequential decisions when to stop are made by the players alternately. That is, the decision is made firstly by player 1 and secondly by player 2 after being informed of player 1's choice and so on. By using dynamic programming technique, he obtained the recursive relations which determine the optimal values and strategies in his several-choice problem. Moreover he showed that the optimal procedure is independent of the order of decision-making by the players(This fact is called "reversibility"). If only one choice is permitted, his recursive relations (2.10) and (2.11) are identical to our (3.1) which determines the equilibrium SS.

4. Examples

Example 4.1. Let $X_n=(X_n^1,\cdots,X_n^p)^T$ n=1,···,N be independent, identically distributed and c = 0. Sakaguchi[6] gave examples of the unanimity rule(p=r=2) for bivariate uniform and normal distributions. Here we shall discuss the majority case($p\geqq r$).

(i) If $X_n^1,\cdots,X_n^p$ are independent and identically uniformly distributed on (0,1) for each n, then the values for the players are equal, so we set $v_n = v_n^1=\cdots=v_n^p$. Since

$$P(g^{\{1\}}_{N-n}(r-1)|X^1_{N-n}) = P(g^{\{1\}}_{N-n}(r-1)) = \binom{p-1}{r-1}(1-v_n)^{r-1}v_n^{p-r},$$
$$P(G^{\{1\}}_{N-n}(r)|X^1_{N-n}) = \sum_{k=r}^{p-1}\binom{p-1}{k}(1-v_n)^k v_n^{p-k-1},$$

(3.1) and (3.2) give

(4.1) $$v_{n+1} = \frac{(1-v_n)^2}{2}\binom{p-1}{r-1}(1-v_n)^{r-1}v_n^{p-r} + (\tfrac{1}{2} - v_n)\sum_{k=r}^{p-1}\binom{p-1}{k}$$

$$\times (1-v_n)^k v_n^{p-k-1} + v_n, \quad (n \geq 1)$$

(4.2) $v_1 = 1/2$.

The graphs of the values v_n for p=5 and r=1,2,•••,5 are shown in Figure 4.1. If the majority level r is low (i.e., r=1,2), the players get small values of v_n. Since each player is strongly affected by the "forced stopping" in the low majority level, he must stop so early that his value is small. If the level r is high (i.e., r=4,5), this affect is weakened and therefore he gets a large value. When the number n of observations is sufficiently large, the unanimity case(i.e.,r=5) has the highest value. The rule of unanimity perhaps brings an inevitable "cooperative nature" to the non-cooperative problem. When n is less than about 10^3, the majority case(r=3,4) is more desirable than other cases. Probably there is no enough time to wait until the unanimous agreement is reached. The number inside the parenthese ( ) beside each curve represents the limiting value of v_n. Each value is the unique root in the closed interval [1/2,1] of the following five equations respectively: $v-1=0(r=5)$, $4v^2-2v-1=0(r=4)$, $6v^3-v-1=0(r=3)$, $4v^4+2v^3+v^2-1=0(r=2)$, $v^5+v^4+v^3+v^2+v-1=0(r=1)$. These equations are derived from (4.1) by $v=v_{n+1}=v_n$, which will be called an equilibrium equation in the next section.

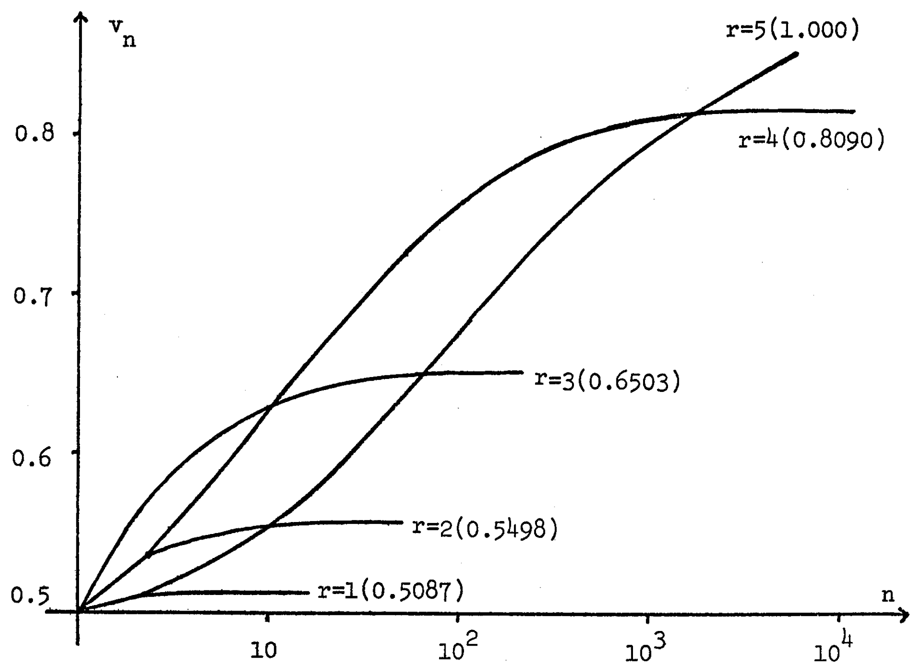

Figure 4.1. The graph of v_n for p-fold U(0,1), p=5

(ii) Let p=2, r=1 and let X_n be normally distributed with $N(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix})$ for each n. By putting $w_n^i=(v_n^i-\mu_i)/\sigma_i$ (i=1,2), (3.1) becomes $w_n^1 = w_n^2 = w_n$ and

$$(4.3) \quad w_{n+1} = \int_{-\infty}^{\infty}(x-w_n)^+\Phi(\frac{w_n-\rho x}{\sqrt{(1-\rho^2)}})\phi(x)dx + \int_{-\infty}^{\infty}(x-w_n)\overline{\Phi}(\frac{w_n-\rho x}{\sqrt{(1-\rho^2)}})\phi(x)dx + w_n$$
$$= \int_{-\infty}^{w_n}(w_n-x)\Phi(\frac{w_n-\rho x}{\sqrt{(1-\rho^2)}})\phi(x)dx$$

$$(4.4) \quad w_1 = 0$$

where $\phi(x)=\sqrt{(2\pi)}^{-1}\exp(-x^2/2)$, $\Phi(x) = \int_{-\infty}^{x}\phi(y)dy$ and $\overline{\Phi}(x) = 1 - \Phi(x)$.

Table 4.2 Results for bivariate normal distributions*

n \ ρ	-0.7	-0.4	-0.2	0.0	0.2	0.4	0.7
2	.0598	.1197	.1596	.1995	.2394	.2793	.3391
	.0598	.1197	.1596	.1995	.2394	.2793	.3391
3	.0756	.1643	.2277	.2933	.3606	.4286	.5306
	.1066	.2045	.2673	.3288	.2893	.4493	.5389
4	.0802	.1836	.2620	.3462	.4345	.5251	.6612
	.1447	.2697	.3480	.4237	.4977	.5706	.6792
5	.0816	.1924	.2806	.3785	.4838	.5935	.7588
	.1769	.3225	.4123	.4984	.5821	.6642	.7866
6	.0820	.1965	.2910	.3991	.5184	.6447	.8358
	.2045	.3667	.4654	.5597	.6509	.7402	.8731
7	.0821	.1985	.2970	.4126	.5434	.6843	.8989
	.2288	.4046	.5107	.6115	.7088	.8037	.9453
8	.0822	.1994	.3004	.4216	.5619	.7157	.9519
	.2503	.4377	.5500	.6563	.7587	.8586	1.0071
9	.0822	.1999	.3024	.4277	.5758	.7412	.9974
	.2697	.4670	.5846	.6956	.8023	.9064	1.0610
10	.0822	.2001	.3036	.4317	.5863	.7622	1.0370
	.2872	.4933	.6155	.7307	.8411	.9488	1.1086

* The upper(lower) numbers relate to the majority(unanimity) case

Table 4.2 shows some computed values of w_n given by (4.3) in the majority case, which is compared with those of μ_n in the unanimity case (see Remark in Section 3 or Sakaguchi[6]) determined by

$$\mu_{n+1} = \mu_n + \sqrt{(1-\rho^2)}\int_{\mu_n}^{\infty} \Psi\left(\frac{\mu_n-\rho x}{\sqrt{(1-\rho^2)}}\right)\phi(x)dx \quad (n \geqq 1),\ \Psi(x) = \phi(x) - x\bar{\Phi}(x),$$

$$\mu_1 = 0 .$$

Contrary to the previous Example(i), the components of X_n are mutually dependent in this example. From Table 4.2, the value w_n increases as the correlation coefficient ρ increases for a fixed n. If ρ is highly negative (ρ=-0.7), the value of the majority case becomes very small. Intuitively speaking, the process is forced to stop so early because of the conflict of interests between the two players.

Example 4.2. This example is a variant of the so-called secretary problem. In a stopping problem of Chow, Robbins and Siegmund[1] *et al.*, a girl usually corresponds only one ability and so a group of girls is totally ordered. We consider the situation where an executive interviews a group of girls one by one with respect to two kind of abilities. For example, one is a skill of type-writing(ability 1) and the other is a proficiency in a certain foreign language(ability 2). He considers both abilities equitably and would like to choose the girl who is best in either ability. That is, there exist two objectives in himself: One is that the accepted girl should be best in ability 1(say, player 1) and the other is that she should be best in ability 2(say, player 2). The above situation is also equivalent to the next. Two professors want to choose one secretary from N girls. Professor 1, 2 wants to choose a best girl in ability 1, 2 respectively. They interview girls one by one and immediately decide whether to accept or not. If one of two professors say "yes", then the girl is accepted. Each professor wishes to maximize the probability of his win, that is, to choose the best girl referring to the corresponding ability. These two situations are formulated as our non-cooperative stopping problem with a majority rule (p,r)=(2,1).

We assume that girls appear one by one at random without recall as in the ordinary problem, and that there is no dependency between the two abilities. Let absolute ranks of the n-th girl be denoted by $(a_n^1,a_n^2)^T$, where $(a_1^i,a_2^i,\cdots,a_N^i)$ is a permutation of integers 1,2,$\cdots$,N (i=1,2). All permutations are equally likely for each i and independent for different i. Let y_n^i={number of terms $a_1^i,\cdots,a_n^i$ which are $\leqq a_n^i$}, that is, relative rank of the n-th girl in ability i and

$$P\left\{\begin{pmatrix}Y_n^1\\Y_n^2\end{pmatrix} = \begin{pmatrix}y^1\\y^2\end{pmatrix}\right\} = \begin{cases} n^{-2}, & \text{if } 1 \leqq y^1,\ y^2 \leqq n,\\ 0, & \text{otherwise.}\end{cases}$$

Put $\mathcal{F}_n^i = \mathcal{B}(y_1^i,\cdots,y_n^i)$, $X_n^i = P(a_n^i=1|\mathcal{F}_n^i)$, i=1,2. Then this secretary problem is reduced to our stopping problem for random vectors $X_n=(X_n^1,X_n^2)^T$, n=1,$\cdots$,N with the majority rule (p,r)=(2,1).

First note that X_n, n=1,$\cdots$,N are independent and

(4.5) $$X_n^i = P(a_n^i=1|y_n^i) = \begin{cases} n/N, & \text{if } y_n^i=1,\\ 0, & \text{if } y_n^i\neq 1.\end{cases}$$

Hence $P(X_n^i= n/N) = 1/n$, $P(X_n^i=0) = 1 - 1/n$ for every i. Because of symmetry the values for the players are equal and the equilibrium SS $*\sigma=(*\sigma^1,*\sigma^2)^T$ is given by

$$*\sigma_n^i = 1 \quad \text{if} \quad X_n^i \geqq v_{N-n}$$

and $*\sigma_n^i$=0, otherwise (i=1,2). Where v_n's satisfy, from (3.1) and (3.2),

(4.6) $$v_{n+1} = E[(X_{N-n}^1-v_n)^+]P(X_{N-n}^2 < v_n) + E[X_{N-n}^1-v_n]P(X_{N-n}^2 \geqq v_n) + v_n, \quad n \geqq 1$$

and v_1=1/N.

By easy calculations, (4.6) becomes that, when $v_n < (N-n)/N$,

(4.7) $$v_{n+1} = N^{-1} + \frac{(N-n-1)^2}{(N-n)^2} v_n = \frac{(N-n-1)^2}{N} \sum_{i=1}^{n+1} (N-i)^{-2}$$

and, when $v_n \geqq (N-n)/N$,

(4.8) $$v_{n+1} = v_n.$$

So from (4.7)

(4.9) $$(n-1)/N \geqq v_{N-n+1} \quad \text{implies} \quad n/N \geqq v_{N-n}.$$

Therefore, if $X_m^i \geqq v_{N-m}$ for some m, then $X_n^i \geqq v_{N-n}$, $n\geqq m$. It is equivalent to y_n^i=1 for $n\geqq m$. From the above arguments, the equilibrium SS $*\sigma$ can be described as follows: There exists a number m* (independent of player i) which satisfies

(4.10) $$\begin{cases} *\sigma_n^i = 0, & n \leqq m^*-1,\\ \quad = \begin{Bmatrix}0\\1\end{Bmatrix}, & m^* \leqq n, \text{ if } y_n^i \begin{Bmatrix}\neq\\=\end{Bmatrix} 1.\end{cases}$$

That is, $*\sigma^i$ is a rule which observes until the (m*-1)th stage and accepts the first girl who is the best in ability i among all girls previously appeared.

The number m^* is determined by the smallest m such that

(4.11) $m/N \geqq v_{N-m}$.

Thus we have, from (4.7),

(4.12) $m^* = \inf \{m;\ m^{-2}+(m+1)^{-2}+\cdots+(N-1)^{-2} \leqq 1/m\}$.

Taking a limit as $N \to \infty$, it is seen that $\lim_{N\to\infty} m^*/N$ exists and equals 0. Also its expected equilibrium value equals 0. A numerical example shows Table 4.3. Refering to this result, the limiting expected value is degenerated . Indeed, between the conflict of two players, once an opponent declares to stop, the other has no chance to win even if it appears the best girl at a later time. In the Presman and Sonin's model[4], the process continues to the last and there is a chance to win after an opponent declares to stop. So the observation-interval is longer and the limiting expected value is a strict positive number.

For comparison, consider the unanimity rule (p,r)=(2,2) in which two players must agree to stop. The recurrence relation (3.1) becomes

(4.13) $v_{n+1} = E(X^1_{N-n}-v_n)^+P(X^2_{N-n} \geqq v_n) + v_n$.

From this, we have

(4.14) $v_{n+1} = (N-n-1)(N-n)^{-1}\{(N-1)^{-1}+N^{-1}((N-2)^{-1}+\cdots+(N-n-1)^{-1})\}$

when $v_n < (N-n)/N$. Thus

(4.15) $m^* = \inf \{m\ ;\ m^{-1}+\cdots+(N-2)^{-1}+N(N-1)^{-1} \leqq 1/(m+1)\}$.

In this rule,it must appear the best girl for both players at a same time in order to win the game. The probability of this event is so small that the expected value tends to degenerate. See Table 4.3.

Remark: A one-stage look ahead (OLA) policy by Ross[5] derives explicit optimal policies in a stopping problem. We see that, in this equilibrium case, the OLA policy can be applied.

Table 4.3 The best choice problem with a rule (p,r)=(2.1) (the upper numbers) and a rule (p,r)=(2,2) (the lower numbers)

(Next page)

N	m*	Expected Equilibrium Value
5	2 2	0.2847 0.3083
10	3 2	0.2159 0.1914
20	4 3	0.1546 0.1183
100	8 4	0.0703 0.0351
200	11 5	0.0500 0.0202
1000	22 5	0.0220 0.0053

5. The stopping problem with an infinite horizon

In this section we study the stopping problem where the number of observations is not limited, i.e., $N=\infty$. Let $X=(X^1,\cdots,X^p)^T$ be a random vector with $E[|X^i|] < \infty$ for $i=1,\cdots,p$. Assume that (a) the rewards $X_1,X_2,\cdots$ are independently and identically distributed random vectors with the common distribution of X and (b) the observation cost $c_n = nc$ for $n=1,2,\cdots$ where $c = (c^1,c^2,\cdots,c^p)^T$ is a constant vector with positive elements.

Consider the following equilibrium equation with respect to $v = (v^1,v^2,\cdots,v^p)^T$:

$$(5.1)\qquad \begin{pmatrix} E[(X^1-v^1)^+P(g^{\{1\}}(r-1)|X^1)] + E[(X^1-v^1)P(G^{\{1\}}(r)|X^1)] \\ \vdots \\ E[(X^p-v^p)^+P(g^{\{p\}}(r-1)|X^p)] + E[(X^p-v^p)P(G^{\{p\}}(r)|X^p)] \end{pmatrix} = c\ ,$$

where $g^{\{i\}}(r-1)$ and $G^{\{i\}}(r)$ are defined, similarly as in Section 2, by the events $A^i = \{X^i \geqq v^i\}$, $i=1,\cdots,p$. We further assume that (c) the equation (5.1) has at least one root if c^i is smaller than the upper limit of X^i, i.e., $\sup\{\alpha; P(X^i > \alpha) < 1\}$ and denote it by $*v = (*v^1,*v^2,\cdots,*v^p)^T$.

Remark 1: Trivially, if c^i is no less than the upper limit of X^i, then player i always declares to stop, that is, his ISS is $\sigma^i=(1,1,\cdots)$. Neglecting player i, the stopping problem with (p,r) reduces to the one with $(p-1,r-1)$.

Remark 2: Even if it is in the unanimity case, the assumption (c) does not necessarily hold. For example, let $p=r=2$ and let X^1,X^2 be independently exponentially distributed with parameters λ_1,λ_2 respectively. If $\lambda_1 c^1 \neq \lambda_2 c^2$, then (5.1) has no roots. If $\lambda_1 c^1=\lambda_2 c^2<1$, then it has a continum of roots.

For each $i=1,2,\cdots,p$, define the ISS $*\sigma^i=(*\sigma^i,*\sigma^i,\cdots)$ for player i by

$$\begin{cases} *\sigma^i_n = 1 & \text{if } X_n \geqq *v^i \\ \quad\;\; = 0 & \text{otherwise} \qquad (n=1,2,\cdots). \end{cases}$$

To show that the SS $*\sigma$ is an equilibrium SS, we will prepare the following truncated game.

For a sequence of X_n and fixed N, define

$$(5.2) \qquad \begin{aligned} {}^N W_n &= Y_n = X_n - nc && \text{if } n < N \\ &= *v - (N-1)c && \text{if } n \geqq N. \end{aligned}$$

The stopping problem for $\{{}^N W_n\}$ is called an N-truncated game with the terminal reward $*v$.

Lemma 5.1. In the above N-truncated game,

$$(5.3) \qquad E[{}^N W_{t(*\sigma)}] = *v \qquad \text{for all N.}$$

Proof: The proof proceeds by induction on N. Put $t^*=t(*\sigma)$. By the inductive assumption,

$$E[{}^{N+1}W_{t^*}|t^*>1] = E[{}^N W_{t^*}-c] = *v - c.$$

Therefore, for the (N+1)-truncated game and i=1,

$$\begin{aligned} E[{}^{N+1}W^1_{t^*}] &= E_{t^*=1}[{}^{N+1}W^1_1] + P(t^*>1)E[{}^{N+1}W^1_{t^*}|t^*>1] \\ &= E_{t^*=1}[X^1-c^1] + P(t^*>1)(*v^1-c^1) \\ &= E[(X^1-c^1)P(*g^{\{1\}}(r-1)|X^1) \\ &\quad + E[(X^1-*v^1)P(*G^{\{1\}}(r)|X^1) + *v^1 - c^1 \qquad \text{(from Lemma 2.2)} \\ &= *v^1 . \end{aligned}$$

The next lemma is easily verified by a result of Chow, Robbins and Siegmund [1] since the number p is finite.

Lemma 5.2. There exists V such that $E[Y^i_{t(\sigma)}] \leqq V < \infty$ for all SS σ and $i=1,2,\cdots,p$.

Now we can prove two theorems assuming the above (a),(b) and (c).

Theorem 5.3.

$$(5.4) \qquad E[Y_{t(*\sigma)}] = *v .$$

Proof: Put $t^*=t(*\sigma)$ and i=1. Considering the N-truncated game, we have by Lemma 5.2

$$\begin{aligned} |E[Y^1_{t^*}] - E[{}^N W^1_{t^*}]| &= P(t^*>N)|E[Y^1_{t^*}|t^*>N] - \{*v^1-(N-1)c^1\}| \\ &\leqq P(t^*>N)(V+*v^1) . \end{aligned}$$

From (5.3), the proof is completed by taking a limit as $N\to\infty$. Q.E.D.

Theorem 5.4. The SS $*\sigma = (*\sigma^1,\cdots,*\sigma^p)^T$ is an equilibrium SS.

Proof: It suffices to prove the theorem for i=1. For any ISS for player 1, $\sigma^1=(\sigma^1_1,\sigma^1_2,\cdots)$, let a SS $\sigma\{n\}$ be $\sigma\{n\} = (\sigma^1\{n\},*\sigma^2,\cdots,*\sigma^p)^T$ where $\sigma^1\{n\} = (\sigma^1_1,\cdots,\sigma^1_n,*\sigma^1_{n+1},*\sigma^1_{n+2},\cdots)$ $(n\geqq 1)$ and $\sigma\{0\} = (*\sigma^1,*\sigma^2,\cdots,*\sigma^p)^T$.

First we prove that

$$(5.5) \qquad E[Y^1_{t(\sigma\{n\})}] \leqq E[Y^1_{t(\sigma\{n-1\})}], \quad n\geqq 1 .$$

by the similar arguments as in Theorem 3.1, we have

$$\begin{aligned} E[Y^1_{t(\sigma\{n\})}] &= E_{t(\sigma\{n\})\leqq n-1}[Y^1_{t(\sigma\{n\})}] \\ &\quad + P(t(\sigma\{n\})\geqq n)E[Y^1_{t(\sigma\{n\})}|t(\sigma\{n\})\geqq n] \\ &= E_{t(\sigma\{n-1\})\leqq n-1}[Y^1_{t(\sigma\{n-1\})}] \\ &\quad + P(t(\sigma\{n-1\})\geqq n)E[Y^1_{t(\sigma\{n\})}|t(\sigma\{n\})\geqq n] , \text{ and} \end{aligned}$$

$$E[Y^1_{t(\sigma\{n\})}|t(\sigma\{n\})\geqq n] \leqq E[Y^1_{t(\sigma\{n-1\})}|t(\sigma\{n-1\})\geqq n] .$$

Hence (5.5) is proved.

Next we show that

$$(5.6) \qquad \lim_{n\to\infty}\{E[Y^1_{t(\tilde\sigma)}] - E[Y^1_{t(\sigma\{n\})}]\}\leqq 0$$

where $\tilde\sigma = (\sigma^1,*\sigma^2,\cdots,*\sigma^p)^T$ for any ISS σ^1. Because

$$E[Y^1_{t(\tilde\sigma)}] - E[Y^1_{t(\sigma\{n\})}] = E_{t(\tilde\sigma)>n}[Y^1_{t(\tilde\sigma)}] - E_{t(\tilde\sigma)>n}[Y^1_{t(\sigma\{n\})}]$$

(from the definition of $\sigma\{n\}$)

$$\leqq P(t(\tilde{\sigma}) > n)\{V - {}^*v^1\}$$

(by Lemma 5.2 and Theorem 5.3).

Since we can assume $P(t(\tilde{\sigma})<\infty) = 1$ without loss of generality, (5.6) is shown by letting n to infinity.

Combining (5.5) and (5.6), it is clear that

$$E[Y^1_{t(\tilde{\sigma})}] \leqq E[Y^1_{t({}^*\sigma)}] .$$

Therefore the SS $^*\sigma$ is an equilibrium SS. Q.E.D.

Example 5.1. Let X_n be a random vector of which components $X^1_n, \cdots, X^p_n$ are independently and identically distributed with a common uniform distribution on (0,1). Suppose $c^1=\cdots=c^p=c$. Because of symmetry, we shall consider the equilibrium equation with $v^1=\cdots=v^p=v$ in (5.1), i.e.,

$$(5.7) \qquad \frac{(1-v)^2}{2}\binom{p-1}{r-1}(1-v)^{r-1}v^{p-r} + \left(\frac{1}{2} - v\right)\sum_{k=r}^{p-1}\binom{p-1}{k}(1-v)^k v^{p-k-1} = c.$$

The root of (5.7) gives the expected equilibrium value as in Theorem 5.3.

Nextly we obtain the majority level r* which maximizes the equilibrium expected net gain v for any fixed p. Denote the left-hand side of (5.7) by $f_r(v)$. Then we easily find that, for each r,

$$(5.8) \qquad f_{r+1}(v) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} f_r(v) \qquad \text{if} \quad v \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} r/p.$$

Therefore

$$f^*(v) = \max_{1\leqq k\leqq p} f_k(v) = f_r(v) \quad \text{if} \quad (r-1)/p < v < r/p$$

Since $f_r(v)$ is strictly decreasing in $v \in (0,r/p]$ for each r, so is $f^*(v)$. Let $c_k = f^*(k/p)$ for $k=0,1,\cdots,p$. Clearly, if $c_{r-1} < c < c_r$, the integer r is the seeking majority level r*. The numerical values for p=5 are shown in Figure 5.1, where r*=2, for instance, is the maximal majority level for $0.1097 < c < 0.3000$.

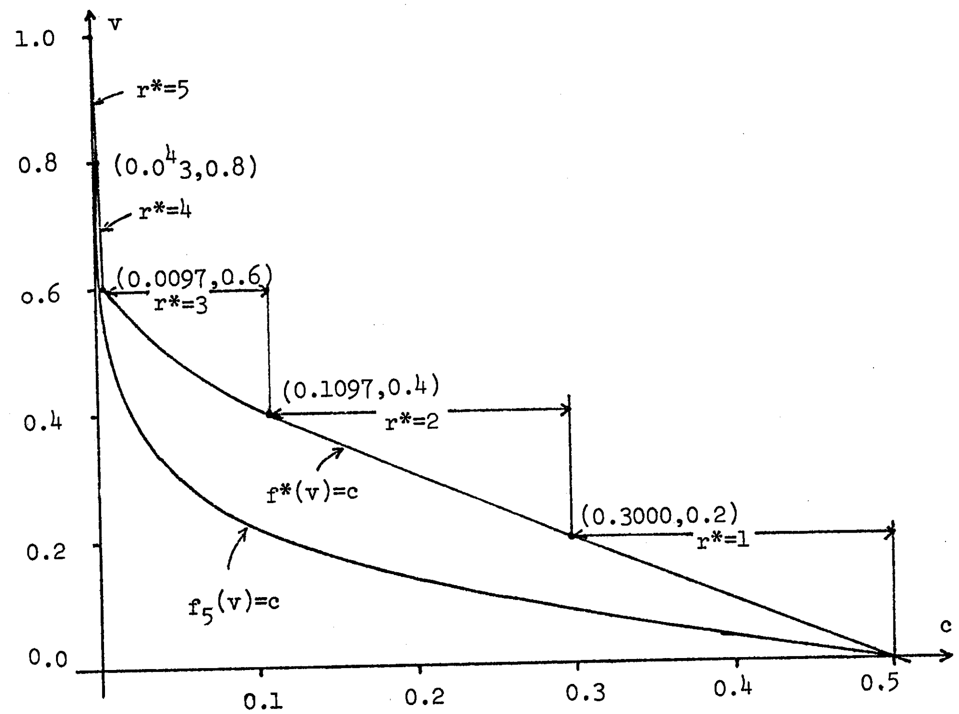

Figure 5.1. The majority level r* for p=5

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