A Version of Lebesgue decomposition theorem for non-additive measure ⋆

Jun LI[1], Masami YASUDA[2] and Jianzeng LI[3]

[1] School of Science, Communication University of China, Beijing 100024, China,
lijun@cuc.edu.cn
[2] Faculty of Science, Chiba University Chiba 263-8522 Japan,
yasuda@math.s.chiba-u.ac.jp
[3] School of Information Engineering, Communication University of China,
Beijing 100024, China,
jzli@cuc.edu.cn

Abstract. In this paper, Lebesgue decomposition type theorems for non-additive measure are shown under the conditions of null-additivity, converse null-additivity, weak null-additivity and σ-null-additivity, etc.. In our discussion, the monotone continuity of set function is not required.

Keywords: Non-additive measure; null-additivity; absolute continuity; Lebesgue decomposition theorem;

1 Introduction

Lebesgue decomposition of a set function μ is stated as: for another given set function ν, μ is represented as $\mu = \mu_c + \mu_s$, where μ_c and μ_s are absolutely continuous and singular with respect to ν, respectively. In measure theory this decomposition is a well-known fact, which is referred to as Lebesgue decomposition theorem [2].

For the case of non-additive measure theory, the situation is not so simple. There are many discussions on Lebesgue decomposition type theorems such as, a version on submeasure (cf. [1]), a version on $\perp$-decomposition measure (cf. [5]), a version on σ-finite fuzzy measure (cf. [3]), and a version on signed fuzzy measure (cf. [10]), and so on. In those discussion, the monotone continuity or autocontinuity of set function are required. However we will try to weaken this condition.

In this paper, we shall show several versions of Lebesgue decomposition theorems of non-additive measure μ for another given non-additive measure ν, where μ is converse null-additive or exhaustive, superadditive or order continuous, and ν is either weakly null-additive and continuous from below or σ-null-additive. In our discussion the monotone continuity of set function is not required, so Lebesgue decomposition theorem is formulated in generality.

⋆ This work was partially supported by the NSFC Grant No.70471063, as well as by the Subject Building Foundation of Communication University of China (No. XK060106).

2 Preliminaries

Let X be a non-empty set, $\mathcal{R}$ a σ-ring of subsets of X, and $(X, \mathcal{R})$ denotes the measurable space.

Definition 2.1. A non-additive measure on $\mathcal{R}$ is an real valued set function $\mu : \mathcal{R} \to [0, +\infty)$ satisfying the following two conditions:

(1) $\mu(\emptyset) = 0$;
(2) $A \subset B$ and $A, B \in \mathcal{R} \;\Rightarrow\; \mu(A) \leq \mu(B)$. (monotonicity)

When a non-additive measure μ is continuous from below, it is called a lower semicontinuous fuzzy measure (cf. [8]). In some literature, a set function μ satisfying the conditions (1) and (2) of Definition 2.1 is called a fuzzy measure.

A set function $\mu : \mathcal{R} \to [0, +\infty)$ is said to be (i) *exhaustive* (cf. [6]), if $\lim\limits_{n \to +\infty} \mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_n$; (ii) *order-continuous* (cf. [2]), if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \in \mathcal{R}$ and $A_n \searrow \emptyset$ $(n \to \infty)$; (iii) *strongly order-continuous* (cf. [4]), if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n, A \in \mathcal{R}$, $\searrow A$ and $\mu(A) = 0$; (iv) to have *property* (S) (cf. [7]), if for any $\{A_n\}_{n=1,2,\cdots} \subset \mathcal{R}$ with $\lim_{n\to+\infty} \mu(A_n) = 0$, there exists a subsequence $\{A_{n_i}\}_{i=1,2,\cdots}$ of $\{A_n\}_{n=1,2,\cdots}$ such that $\mu(\limsup A_{n_i}) = 0$.

Definition 2.2. ([9]) Let μ and ν be two non-additive measures. We say that

(1) μ is absolutely continuous of Type I with respect to ν, denoted by $\mu \ll_I \nu$, if $\mu(A) = 0$ whenever $A \in \mathcal{R}$, $\nu(A) = 0$;
(2) μ is absolutely continuous of Type VI with respect to ν, denoted by $\mu \ll_{VI} \nu$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(A) < \epsilon$ whenever $A \in \mathcal{R}$, $\nu(A) < \delta$.

Definition 2.3. ([1]) We say that μ is singular with respect to ν, and denote $\mu \perp \nu$, if there is a set $Q \in \mathcal{R}$ such that $\nu(Q) = 0$ and $\mu(E - Q) = 0$ for any $E \in \mathcal{R}$.

For non-additive measures μ and ν, if $\mu \ll_{VI} \nu$, then $\mu \ll_I \nu$. The inverse statement may not be true.

Proposition 2.1. *Let μ and ν be two non-additive measures. If μ is strongly order-continuous, ν have property (S), then $\mu \ll_{VI} \nu$ if and only if $\mu \ll_I \nu$.*

3 Null-Additivity of Set Function

The several kinds of null-additivity of non-additive measure play important role in establishing Lebesgue decomposition type theorems of non-additive measure. We recall them in the following.

Definition 3.1. ([8, 9]) A set function $\mu : \mathcal{R} \to [0, +\infty)$ is said to be (i) *null-additive*, if $\mu(E \cup F) = \mu(E)$ whenever $E, F \in \mathcal{R}$ and $\mu(F) = 0$; (ii) *weakly null-additive*, if $\mu(E \cup F) = 0$ whenever $E, F \in \mathcal{R}$, $\mu(E) = \mu(F) = 0$; (iii) *converse null-additive*, if $\mu(A - B) = 0$ whenever $A \in \mathcal{R}$, $B \in A \cap \mathcal{R}$, $\mu(B) = \mu(A)$; (iv) *pseudo-null-additive*, if $\mu(B \cup C) = \mu(C)$, whenever $A \in \mathcal{R}, B \in A \cap \mathcal{R}, C \in A \cap \mathcal{R}, \mu(A - B) = \mu(A)$.

Proposition 3.1. *If μ is null-additive, then it is weakly null-additive. If μ is pseudo-null-additive, then it is converse null-additive.*

Definition 3.2. ([6]) A non-additive measure μ is said to be *σ-null-additive*, if for every sequence $\{B_i\}_{i=1,2,\cdots}$ of pairwise disjoint sets from $\mathcal{R}$ and $\mu(B_i) = 0$ we have

$$\mu\left(A \cup \bigcup_{i=1}^{+\infty} B_i\right) = \mu(A) \quad \forall A \in \mathcal{R}.$$

Proposition 3.2. *μ is σ-null-additive if and only if μ is null-additive and $\mu(B_i) = 0$ $(i = 1, 2, \cdots)$ implies $\mu(\cup_{i=k}^{+\infty} B_i) = 0$ for every sequence $\{B_i\}_{i=1,2,\cdots}$ of pairwise disjoint sets from $\mathcal{R}$.*

4 Lebesgue Decomposition Theorems of Non-Additive Measure

In this section we show the Lebesgue decomposition type theorems of non-additive measure. Unless stated, in the following we always assume that all set functions are non-additive measure.

Lemma 4.1. *Let μ be converse null-additive non-additive measure on $\mathcal{R}$. Then, there is a set $Q \in \mathcal{R}$ such that $\mu(E - Q) = 0$; and further, if μ is null-additive, then $\mu(E) = \mu(Q \cap E)$ for any $E \in \mathcal{R}$.*

Proof. Let $\alpha = \sup\{\mu(E) \ : \ E \in \mathcal{R}\}$. By the definition of α, we can choose a sequence $\{E_n^{(1)}\}_{n=1,2,\cdots}$ from $\mathcal{R}$ such that for every $n = 1, 2, \cdots$,

$$\alpha - \frac{1}{n} < \mu(E_n^{(1)}) \leq \alpha.$$

Denote $Q_1 = \bigcup_{n=1}^{+\infty} E_n^{(1)}$, then $Q_1 \in \mathcal{R}$. Thus we have

$$\alpha - \frac{1}{n} < \mu(E_n^{(1)}) \leq \mu(Q_1) \leq \alpha,$$

$n = 1, 2, \cdots$. Let $n \to +\infty$, we have $\mu(Q_1) = \alpha$.

Similarly, there exists a sequence $\{E_n^{(2)}\}_{n=1,2,\cdots}$ from $\mathcal{R}$ such that

$$E_n^{(2)} \subset X - Q_1, \quad \forall n \geq 1,$$

and

$$\mu(Q_2) = \sup\{\mu(E) \ : \ E \in \mathcal{R}, \quad E \subset X - Q_1\}$$

where $Q_2 = \bigcup_{n=1}^{+\infty} E_n^{(2)}$.

Let us denote $Q = Q_1 \cup Q_2$. Then $Q \in \mathcal{R}$, and

$$\alpha = \mu(Q_1) \leq \mu(Q) \leq \sup\{\mu(E) \ : \ E \in \mathcal{R}\}.$$

Therefore $\mu(Q) = \mu(Q_1) = \alpha$. By the converse null-additivity of μ, and noting $Q_1 \cap Q_2 = \emptyset$, we have $\mu(Q_2) = \mu(Q - Q_1) = 0$. Therefore, for any $E \in \mathcal{R}$, it is follows from

$$E - Q \subset E - Q_1 \subset X - Q_1$$

that

$$\mu(E - Q) \leq \mu(Q_2) = 0.$$

Thus, for any $E \in \mathcal{R}$, we have $\mu(E - Q) = 0$. When μ is null-additive, then $\mu(E) = \mu((E-Q) \cup (E \cap Q) = \mu(Q \cap E)$ for any $E \in \mathcal{R}$. The proof of the lemma is now complete. □

Lemma 4.2. *Let μ be exhaustive non-additive measure on $\mathcal{R}$. Then, there is a set $Q \in \mathcal{R}$ such that $\mu(E - Q) = 0$; and further, if μ is null-additive, then $\mu(E) = \mu(Q \cap E)$ for any $E \in \mathcal{R}$.*

Proof. It is similar to the proof of Theorem 2 in [3] and Lemma 4.1 above.

The following theorems with their corollaries are several versions of Lebesgue decomposition theorem for non-additive measure.

Theorem 4.1. (Lebesgue decomposition theorem) *Let μ and ν be non-additive measures on $\mathcal{R}$. If μ is either converse null-additive or exhaustive, ν is weakly null-additive and continuous from below, then there exists a set $Q \in \mathcal{R}$ such that those non-additive measures μ_c and μ_s defined by*

$$\mu_c(E) = \mu(E - Q) \quad \text{and} \quad \mu_s(E) = \mu(E \cap Q), \quad \forall E \in \mathcal{R}$$

satisfy $\mu_c \ll_I \nu$ and $\mu_s \perp \nu$, respectively.

Proof. Put $\mathcal{R}_1 = \{A \in \mathcal{R} \ : \ \nu(A) = 0\}$. Then, $\mathcal{R}_1$ is a σ-subring of $\mathcal{R}$ by the weak null-additivity and continuity from below of ν. By using Lemma 4.1 and 4.2, we may take a set $Q \in \mathcal{R}_1$ such that

$$\mu(Q) = \sup\{\mu(A) \ : \ A \in \mathcal{R}_1\} \quad \text{and} \quad \mu(E - Q) = 0, \quad \forall E \in \mathcal{R}_1.$$

Now we take that $\mu_c(E) = \mu(E - Q)$ and $\mu_s(E) = \mu(E \cap Q)$, $E \in \mathcal{R}$, then μ_c and μ_s satisfy the required properties:

$$\mu_c(E) = 0 \ \text{ if } \ \nu(E) = 0$$

and

$$\mu_s(E - Q) = \mu((E - Q) \cap Q) = \mu(\emptyset) = 0$$

for any $E \in \mathcal{R}$. □

From Proposition 3.1 and Theorem 4.1, we have the following corollary.

Corollary 4.1. *If μ is pseudo-null-additive, ν is null-additive and continuous from below, then there exists a set $Q \in \mathcal{R}$ such that those non-additive measures μ_c and μ_s defined by*

$$\mu_c(E) = \mu(E - Q) \quad \textit{and} \quad \mu_s(E) = \mu(E \cap Q), \quad \forall E \in \mathcal{R}$$

satisfy $\mu_c \ll_I \nu$ and $\mu_s \perp \nu$, respectively.

Definition 4.1. ([6]) A set function $\mu : \mathcal{R} \to [0, +\infty)$ is called *superadditive*, if for every $A, B \in \mathcal{R}$ and $A \cap B = \emptyset$,

$$\mu(A \cup B) \geq \mu(A) + \mu(B).$$

Proposition 4.1. *If μ is superadditive, then it is converse null-additive and exhaustive.*

Note that a condition of superadditivity do not imply the order continuity. There is a counterexample: A superadditive non-additive measure is not necessarily order continuous. Counterexample: Let μ be the non-additive measure on 2^N, the power set of the set N of all the positive integers, defined by $\mu(E) = 1$ if E is cofinite (i.e., the complement of a finite set), 0 otherwise. Then obviously μ is superadditive non-additive measure. However $A_n = \{n, n+1, \cdots\}$ for $n = 1, 2, \cdots$ decreasingly converges to the empty set, but the limit of $\{\mu(A_n)\}$ is equal to 1, not zero. This counterexample is noticed by the anonymous referee.

As a direct result of Proposition 4.1 and Theorem 4.1, we can obtain the following theorem immediately.

Theorem 4.2. *Let μ and ν be non-additive measures on $\mathcal{R}$. If μ is superadditive, ν is weakly null-additive and continuous from below, then there exist non-additive measures μ_c and μ_s on $\mathcal{R}$ such that $\mu_c \ll_I \nu$, $\mu_s \perp \nu$, and $\mu \geq \mu_c + \mu_s$.*

By using Proposition 3.2, similar to the proof of Theorem 4.1, we can prove the following theorem.

Theorem 4.3. *Let μ and ν be non-additive measures on $\mathcal{R}$. If μ is either converse null-additive or exhaustive, ν is σ-null-additive, then there exist non-additive measures μ_c and μ_s on $\mathcal{R}$ such that $\mu_c \ll_I \nu$ and $\mu_s \perp \nu$.*

Corollary 4.2. *Let μ and ν be non-additive measures on $\mathcal{R}$. If μ is superadditive, ν is σ-null-additive, then there exist non-additive measures μ_c and μ_s on $\mathcal{R}$ such that $\mu_c \ll_I \nu$, $\mu_s \perp \nu$, and $\mu \geq \mu_c + \mu_s$.*

Note that it is not required that ν has the continuity from below in Theorem 4.3 and Corollary 4.2.

Combining Theorem 4.1 and Proposition 2.1, we can obtain the following result.

Theorem 4.4. *Let μ and ν be non-additive measures on $\mathcal{R}$. If μ is strongly order continuous, ν is weakly null-additive and continuous from below, and have property (S), then there exists a set $Q \in \mathcal{R}$ such that those non-additive measures μ_c and μ_s defined by*

$$\mu_c(E) = \mu(E - Q) \quad \textit{and} \quad \mu_s(E) = \mu(E \cap Q), \quad \forall E \in \mathcal{R}$$

satisfy $\mu_c \ll_{VI} \nu$ and $\mu_s \perp \nu$, respectively.

Conclusions: There are several versions of Lebesgue decomposition theorem as noted in the first section. Here we proved that Lebesgue decomposition type theorems for non-additive measure are shown under the conditions of null-additivity, converse null-additivity, weak null-additivity and σ-null-additivity. It should be clarified these relations and also considered applications to the various fields.

Acknowledgements: The authors would like to thank anonymous referees a useful comment. We impose the exhaustivity for Lebesgue decomposition theorem of Theorem 4.2 in stead of the order continuity. By this comment, the assertion of Theorem 4.2 is corrected. The research of the first author was supported by the Postdoctoral Research Finance of Communication University of China.

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