Further properties of null-additive fuzzy measure on metric spaces

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Objective:

Properties of null-additive fuzzy measure on metric spaces
In this paper, under

- the null-additivity,
- weekly null-additivity and
- converse null-additivity condition,

we shall discuss the relation among

- the inner regularity,
- the outer regularity and
- the regularity of fuzzy measure.

Regularity:

Various regularities of set function:

- E. Pap, *Null-additive Set Functions*, Kluwer, Dordrecht, 1995.
- Q. Jiang, H. Suzuki, Fuzzy measures on metric spaces, *Fuzzy Sets and Systems* 83(1996) 99–106.
- J. Wu, C. Wu, Fuzzy regular measures on topological spaces, *Fuzzy Sets and Systems* 119(2001) 529–533.

References(continued)

- J. Song, J. Li, Regularity of null-additive fuzzy measure on metric spaces, *Int. J. General Systems* 32(2003) 271–279.
- J. Li, Order continuous of monotone set function and convergence of measurable functions sequence, *Applied Mathematics and Computation* 135(2003) 211–218.

References(continued)

- J. Song, J. Li, Regularity of null-additive fuzzy measure on metric spaces, *Int. J. General Systems* 32(2003) 271–279.
- J. Li, Order continuous of monotone set function and convergence of measurable functions sequence, *Applied Mathematics and Computation* 135(2003) 211–218.

References(continued)

- J. Li, M. Yasuda, Lusin's theorem on fuzzy measure spaces, *Fuzzy Sets and Systems* 146(2004) 121–133.

The regularity of null-additive fuzzy measure and proved Egoroff's theorem and Lusin's theorem for fuzzy measure on metric space.

References(continued)

- Y. Narukawa, T. Murofushi, M. Sugeno, Regular fuzzy measure and representation of comonotonically additive functional, *Fuzzy Sets and Systems* 112(2000) 177–186.
- Y. Narukawa, T. Murofushi, Conditions for Choquet integral representation of the comonotonically additive and monotone functional, *J. Math. Anal. Appl.* 282(2003) 201–211.
- Y. Narukawa, T. Murofushi, Regular null-additive measure and Choquet integral, *Fuzzy Sets and Systems* 143(2004) 487–492.

General references:

- I. Dobrakov, On submeasures I, *Dissertations Math.* 112(1974) 1-35.
- Z. Wang, G. J. Klir, *Fuzzy Measure Theory*, Plenum, New York, 1992.

Notation:

(X, d) : a metric space,
$\mathcal{O}$: the classes of all open sets in (X, d)
$\mathcal{C}$: the classes of all closed sets in (X, d)
$\mathcal{K}$: the classes of all compact sets in (X, d)
$\mathcal{B}$ denotes Borel σ-algebra on X, i.e., it is the smallest σ-algebra containing $\mathcal{O}$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{B}$

Definition: continuity

A set function $\mu : \mathcal{B} \to [0, +\infty]$ is said to be *continuous from below*, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$;
continuous from above, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \searrow A$;
strongly order continuous, if $\lim_{n\to\infty}\mu(A_n) = 0$ whenever $A_n \searrow\ B$ and $\mu(B) = 0$;

Definition: additivities

null-additive, if $\mu(E \cup F) = \mu(E)$ for any E whenever $\mu(F) = 0$;
weakly null-additive, if $\mu(E \cup F) = 0$ whenever $\mu(E) = \mu(F) = 0$;
converse-null-additive, if $\mu(E - F) = 0$ whenever $F \subset E$ and $\mu(F) = \mu(E) < +\infty$;

Obviously, the null-additivity of μ implies weakly null-additivity.

Definition: a fuzzy measure

A fuzzy measure on $(X, \mathcal{B})$ is an extended real valued set function $\mu : \mathcal{F} \longrightarrow [0, +\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$;

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$ (monotonicity).

Definition: Wu/Wu *FSS*(2001)

A fuzzy measure μ is called *outer regular* if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exists a set $G \in \mathcal{O}$ such that $A \subset G, \ \mu(G - A) < \epsilon$

A fuzzy measure μ is called *inner regular*, if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exists a set $F \in \mathcal{C}$ such that $F \subset A, \ \mu(A - F) < \epsilon$

A fuzzy measure μ is called *regular*, if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exist a closed set $F \in \mathcal{C}$ and an open set $G \in \mathcal{O}$ such that $F \subset A \subset G$ and $\mu(G - F) < \epsilon$.

Proposition: Li/Yasuda *FSS*(2004)

If μ is weekly null-additive and continuous, then it is regular.

Furthermore, if μ is null-additive, then for any $A \in \mathcal{B}$,

$$\begin{aligned} \mu(A) &= \sup\{\, \mu(F) \mid F \subset A,\ F \in \mathcal{C} \,\} \\ &= \inf\{\, \mu(G) \mid G \supset A,\ G \in \mathcal{O} \,\} \end{aligned}$$

Proposition:

If μ is weekly null-additive and strongly order continuous, then both outer regularity and inner regularity imply regularity.

Proposition:

Let μ be a null-additive fuzzy measure.
(1) If μ is continuous from below, then inner regularity implies

$$\mu(A) = \sup\{\ \mu(F) \mid F \subset A,\ \ F \in \mathcal{C}\ \}$$

for all $A \in \mathcal{B}$;
(2) If μ is continuous from above, then outer regularity implies

$$\mu(A) = \inf\{\ \mu(G) \mid A \subset G,\ \ G \in \mathcal{O}\ \}$$

for all $A \in \mathcal{B}$.

Proposition:

Let μ be a converse-null-additive fuzzy measure.
(1) If μ is continuous from below and strongly order continuous, and for any $A \in \mathcal{B}$,

$$\mu(A) = \sup\{\ \mu(F) \mid F \subset A,\ F \in \mathcal{C}\ \},$$

then μ is inner regular.
(2) If μ is continuous from above, and for any $A \in \mathcal{B}$,

$$\mu(A) = \inf\{\ \mu(G) \mid A \subset G,\ G \in \mathcal{O}\ \},$$

then μ is outer regular.

Definition: strongly regular

A fuzzy measure μ is called *strongly regular*, if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exist a compact set $K \in \mathcal{K}$ and an open set $G \in \mathcal{O}$ such that $K \subset A \subset G$ and $\mu(G - K) < \epsilon$.

Proposition:

Let μ be null-additive and continuous from below. If μ is strongly regular, then for any $A \in \mathcal{B}$,

$$\mu(A) = \sup\{\, \mu(K) \mid K \subset A,\ K \in \mathcal{K} \,\}.$$

Proposition:

Let μ be null-additive and order continuous. If for any $A \in \mathcal{B}$,

$$\mu(A) = \sup\{\ \mu(K) \mid K \subset A,\ K \in \mathcal{K}\ \},$$

then μ is strongly regular.

In the rest of the paper, we assume that $(X,\ d)$ is complete and separable metric space, and that μ is finite continuous fuzzy measure.

Theorem:

If μ is null-additive, then μ is strongly regular.

Lemma:

Let μ be a finite continuous fuzzy measure. Then for any $\epsilon > 0$ and any double sequence $\{A_n^{(k)} \mid n \geq 1, k \geq 1\} \subset \mathcal{B}$ satisfying $A_n^{(k)} \searrow \emptyset \ (k \to \infty), \ n = 1, 2, \ldots$, there exists a subsequence $\{A_n^{(k_n)}\}$ of $\{A_n^{(k)} \mid n \geq 1, k \geq 1\}$ such that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n^{(k_n)}\right) < \epsilon \quad (k_1 < k_2 < \ldots)$$

Lemma:

If μ be continuous fuzzy measure, then for each $\epsilon > 0$, there exists a compact set $K_\epsilon \in \mathcal{K}$ such that $\mu(X - K_\epsilon) < \epsilon$.

Corollary:

If μ is null-additive, then for any $A \in \mathcal{B}$ the following statements hold:

(1) For each $\epsilon > 0$, there exist a compact set $K_\epsilon \in \mathcal{K}$ such that $K_\epsilon \subset A$ and $\mu(A - K_\epsilon) < \epsilon$;

(2) $\mu(A) = \sup\{\, \mu(K) \mid K \subset A,\ K \in \mathcal{K}\,\}$.

Theorem: Egoroff's theorem

Let μ be null-additive continuous fuzzy measure. If $\{f_n\}$ converges to f almost everywhere on X, then for any $\epsilon > 0$, there exists a compact subset $K_\epsilon \in \mathcal{K}$ such that $\mu(X - K_\epsilon) < \epsilon$ and $\{f_n\}_n$ converges to f uniformly on K_ϵ.

Theorem: Lusin's theorem

Let μ be null-additive continuous fuzzy measure. If f is a real measurable function on X, then, for each $\epsilon > 0$, there exists a compact subset $K_\epsilon \in \mathcal{K}$ such that f is continuous on K_ϵ and $\mu(X - K_\epsilon) \leq \epsilon$.

Definition: Jiang/Suzuki *FSS*(1996)

A set $A \in \mathcal{B}$ with $\mu(A) > 0$ is call an atom if $B \subset A$ then

(i) $\mu(B) = 0$, or

(ii) $\mu(A) = \mu(B)$ and $\mu(A - B) = 0$.

Fuzzy Integral:

Consider a nonnegative real-valued measurable function f on A. The *fuzzy integral* of f on A with respect to μ, denoted by $(S)\int_A f\,d\mu$, is defined by

$$(S)\int_A f\,d\mu = \sup_{0\leq\alpha<+\infty} [\alpha \wedge \mu(\{x : f(x) \geq \alpha\} \cap A)]$$

Theorem:

Let μ be null-additive and continuous. If A is an atom of μ, then there exists a point $a \in A$ such that the fuzzy integral satisfies

$$(S) \int_A f d\mu = f(a) \wedge \mu(\{a\})$$

for any non-negative measurable function f on A.