Define the stopping region for Player I, II by

$$B_1^* = \{x \in S; d^*(x) \le 0\},$$
$$B_2^* = \{x \in S; e^*(x) \ge 0\}, \tag{4.9}$$

respectively, and let C^* be the complement of $B_1^* \cup B_2^*$. *The infinity-SLA rule* of the Dynkin stopping problem is a stopping rule based on the first hitting time of set B_1^* or B_2^*.

ASSUMPTION 4.2.

(1) *Either of these sets is nonempty and each set B_i^*, $i = 1,2$ is closed with respect to P; that is,*

$$P(x, B_i^*) = 1, \qquad x \in B_i^*. \tag{4.10}$$

(2) *The process eventually hits these sets; that is,*

$$\nu(B_1^* \cup B_2^*) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{4.11}$$

(3) *We assume that*

$$\liminf_n E^x[\psi(X_n)] \le \varphi(x), \qquad x \in B_1^*, \tag{4.12}$$

$$\limsup_n E^x[\varphi(X_n)] \ge \psi(x), \qquad x \in B_2^*. \tag{4.13}$$

THEOREM 4.2. *Under Assumptions 2.1 and 4.2, the sets B_1^* and B_2^* are disjoint and the infinity-SLA rule is optimal. Its optimal value $v(x)$, $x \in S$ of the game variant problem is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1^*, \\ \mathbb{N}_{C^*}\left[P_{B_1^*}\varphi + P_{B_2^*}\psi\right](x), & x \in C^*, \\ \psi(x), & x \in B_2^*. \end{cases} \tag{4.14}$$

PROOF. The proof that the set B_i^*, $i = 1,2$ is disjoint can be obtained similarly to Lemma 2.1(1). Also, the rest of the proof is easily obtained by combining the results in Sections 3 and 4. ∎

Let $C_1^* = \{x \in C^*; \nu(B_1^*) < \infty \text{ a.e. } P^x\}$ and $C_2^* = \{x \in C^*; \nu(B_2^*) < \infty \text{ a.e. } P^x\}$. Then we have, by similar discussion in Section 2,

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) + (d^*)^+(x), & x \in B_1^* \cup C_1^*, \\ \limsup_n E^x\left[\varphi(X_{n\wedge\nu(B_1^*)})\right], & \text{otherwise}, \end{cases} \tag{4.15}$$

and

$$\underline{\psi}(x) = \begin{cases} \psi(x) - (e^*)^-(x), & x \in B_2^* \cup C_2^*, \\ \liminf_n E^x\left[\psi(X_{n\wedge\nu(B_2^*)})\right], & \text{otherwise}. \end{cases} \tag{4.16}$$

Define the following two functions, similar to Section 2.

$$v_1(x) = \begin{cases} \varphi(x), & x \in B_1^*, \\ \mathbb{N}_C\left[P_{B_1^*}\varphi\right](x), & x \in C^*, \\ 0, & x \in B_2^*, \end{cases} \tag{4.17}$$

$$v_2(x) = \begin{cases} \psi(x), & x \in B_2^*, \\ \mathbb{N}_C\left[P_{B_2^*}\psi\right](x), & x \in C^*, \\ 0, & x \in B_1^*. \end{cases} \tag{4.18}$$

An alternative form of (4.14) can be written, by the result of Theorem 3.6, as follows.

COROLLARY 4.3. *Under the same assumptions,*

$$v(x) = v_1(x) + v_2(x), \qquad x \in S, \tag{4.19}$$

where $v_i(x)$, $i = 1,2$ are defined by (4.17) and (4.18), respectively.

What we want to claim is that the game value of Dynkin's problem decomposed into the sum of the two functions under the infinity rule.